1. Write down the three types of elementary matrices and find their inverses. Show that an \( n \times n \) matrix \( A \) is invertible if and only if it can be written as a product of elementary matrices. Use this method to find the inverse of
\[
\begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 1 \\
0 & 3 & -1
\end{pmatrix}.
\]

2. (Another proof of the row rank column rank equality.) Let \( A \) be an \( m \times n \) matrix of (column) rank \( r \). Show that \( r \) is the least integer for which \( A \) factorises as \( A = BC \) with \( B \in \text{Mat}_m(F) \) and \( C \in \text{Mat}_{r,n}(F) \). Using the fact that \((BC)^T = C^TB^T\), deduce that the (column) rank of \( A^T \) equals \( r \).

3. Let \( V \) be a 4-dimensional vector space over \( \mathbb{R} \), and let \( \{\xi_1, \xi_2, \xi_3, \xi_4\} \) be the basis of \( V^* \) dual to the basis \( \{x_1, x_2, x_3, x_4\} \) for \( V \). Determine, in terms of the \( \xi_i \), the bases dual to each of the following:
   \begin{enumerate}
   \item \( \{x_2, x_1, x_4, x_3\} \);
   \item \( \{x_1, 2x_2, x_3, x_4\} \);
   \item \( \{x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4\} \);
   \item \( \{x_1, x_2 - x_1, x_3 - x_2 + x_1, x_4 - x_3 + x_2 - x_1\} \).
   \end{enumerate}

4. Let \( P_n \) be the space of real polynomials of degree at most \( n \). For \( x \in \mathbb{R} \) define \( \varepsilon_x \in P_n^* \) by \( \varepsilon_x(p) = p(x) \). Show that \( \varepsilon_0, \ldots, \varepsilon_n \) form a basis for \( P_n^* \), and identify the basis of \( P_n \) to which it is dual.

5. \( a) \) Show that if \( x \neq y \) are vectors in the finite dimensional vector space \( V \), then there is a linear functional \( \theta \in V^* \) such that \( \theta(x) \neq \theta(y) \).
   \( b) \) Suppose that \( V \) is finite dimensional. Let \( A, B \leq V \). Prove that \( A \leq B \) if and only if \( A^\circ \geq B^\circ \). Show that \( A = V \) if and only if \( A^\circ = \{0\} \).

6. For \( A \in \text{Mat}_{m,m}(F) \) and \( B \in \text{Mat}_{m,n}(F) \), let \( \tau_A(B) \) denote \( trAB \). Show that, for each fixed \( A \), \( \tau_A : \text{Mat}_{m,n}(F) \to F \) is linear. Show moreover that the mapping \( A \mapsto \tau_A \) defines a linear isomorphism \( \text{Mat}_{m,m}(F) \to \text{Mat}_{m,m}(F)^* \).

7. \( a) \) Let \( V \) be a non-zero finite dimensional real vector space. Show that there are no endomorphisms \( \alpha, \beta \) of \( V \) with \( \alpha \beta - \beta \alpha = \text{id}_V \).
   \( b) \) Let \( V \) be the space of infinitely differentiable functions \( \mathbb{R} \to \mathbb{R} \). Find endomorphisms \( \alpha \) and \( \beta \) of \( V \) such that \( \alpha \beta - \beta \alpha = \text{id}_V \).

8. Suppose that \( \psi : U \times V \to F \) is a bilinear form of rank \( r \) on finite dimensional vector spaces \( U \) and \( V \) over \( F \). Show that there exist bases \( e_1, \ldots, e_m \) for \( U \) and \( f_1, \ldots, f_n \) for \( V \) such that
\[
\psi \left( \sum_{i=1}^m x_i e_i, \sum_{j=1}^n y_j f_j \right) = \sum_{k=1}^r x_k y_k
\]
for all \( x_1, \ldots, x_m, y_1, \ldots, y_n \in F \). What are the dimensions of the left and right kernels of \( \psi \)?

9. Let \( A \) and \( B \) be \( n \times n \) matrices over a field \( F \). Show that the \( 2n \times 2n \) matrix
\[
C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix}
\]
by elementary row operations (which you should specify). By considering the determinants of \( C \) and \( D \), obtain another proof that \( \det AB = \det A \det B \).
10. Let $A$, $B$ be $n \times n$ matrices, where $n \geq 2$. Show that, if $A$ and $B$ are non-singular, then

(i) $\text{adj}(AB) = \text{adj}(B)\text{adj}(A)$,  
(ii) $\det(\text{adj} A) = (\det A)^{n-1}$,  
(iii) $\text{adj}(\text{adj} A) = (\det A)^{n-2}A$.

What happens if $A$ is singular? [Hint: Consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]

11. Show that the dual of the space $P_1$ of real polynomials is isomorphic to the space $\mathbb{R}^n$ of all sequences of real numbers, via the mapping which sends a linear form $\xi : P \rightarrow \mathbb{R}$ to the sequence $(\xi(1), \xi(t), \xi(t^2), \ldots)$.

In terms of this identification, describe the effect on a sequence $(a_0, a_1, a_2, \ldots)$ of the linear maps dual to each of the following linear maps $P \rightarrow P$: 
(a) The map $D$ defined by $D(p)(t) = p'(t)$.
(b) The map $S$ defined by $S(p)(t) = p(t^2)$.
(c) The map $E$ defined by $E(p)(t) = p(t - 1)$.
(d) The composite $DS$.
(e) The composite $SD$.

Verify that $(DS)^* = S^*D^*$ and $(SD)^* = D^*S^*$.

12. Suppose that $\psi : V \times V \rightarrow F$ is a bilinear form on a finite dimensional vector space $V$. Take $U$ a subspace of $V$ with $U = W^\perp$ some subspace $W$ of $V$. Suppose that $\psi|_{U \times U}$ is non-singular. Show that $\psi$ is also non-singular.

13. Let $V$ be a vector space. Suppose that $f_1, \ldots, f_n, g \in V^*$. Show that $g$ is in the span of $f_1, \ldots, f_n$ if and only if $\bigcap_{k=1}^n \ker f_k \subset \ker g$.

14. Let $\alpha : V \rightarrow V$ be an endomorphism of a real finite dimensional vector space $V$ with $\text{tr}(\alpha) = 0$.

(i) Show that, if $\alpha \neq 0$, there is a vector $v$ with $\alpha(v)$ linearly independent. Deduce that there is a basis for $V$ relative to which $\alpha$ is represented by a matrix $A$ with all of its diagonal entries equal to 0.

(ii) Show that there are endomorphisms $\beta, \gamma$ of $V$ with $\alpha = \beta \gamma - \gamma \beta$.

The final question is based on non-examinable material.

15. Let $Y$ and $Z$ be subspaces of the finite dimensional vector spaces $W$ and $V$ respectively. Suppose that $\alpha : V \rightarrow W$ is a linear map such that $\alpha(Y) \subset Z$. Show that $\alpha$ induces linear maps $\alpha|_Y : Y \rightarrow Z$ via $\alpha|_Y(y) = \alpha(y)$ and $\bar{\alpha} : V/Y \rightarrow W/Z$ via $\bar{\alpha}(v + Y) = \alpha(v) + Z$.

Consider a basis $(v_1, \ldots, v_n)$ for $V$ containing a basis $(v_1, \ldots, v_k)$ for $Y$ and a basis $(w_1, \ldots, w_m)$ for $W$ containing a basis $(w_1, \ldots, w_l)$ for $Z$. Show that the matrix representing $\alpha$ with respect to $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ is a block matrix of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. Explain how to determine the matrices representing $\alpha|_Y$ with respect to the bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_l)$ and representing $\bar{\alpha}$ with respect to the bases $(v_{k+1} + Y, \ldots, v_n + Y)$ and $(w_{l+1} + Z, \ldots, w_m + Z)$ from this block matrix.