1. Show that none of the following matrices are similar:

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Is the matrix

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

similar to any of them? If so, which?

2. Find a basis with respect to which \( \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \) is in Jordan normal form. Hence compute \( \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}^n \).

3. (a) Recall that the Jordan normal form of a \( 3 \times 3 \) complex matrix can be deduced from its characteristic and minimal polynomials. Give an example to show that this is not so for \( 4 \times 4 \) complex matrices.

(b) Let \( A \) be a \( 5 \times 5 \) complex matrix with \( A^4 = A^2 \neq A \). What are the possible minimal and characteristic polynomials? If \( A \) is not diagonalisable, how many possible JNFs are there for \( A \)?

4. Let \( \alpha \) be an endomorphism of the finite dimensional vector space \( V \) over \( \mathbb{F} \), with characteristic polynomial \( \chi_\alpha(t) = t^n + c_{n-1}t^{n-1} + \cdots + c_0 \). Show that \( \det(\alpha) = (-1)^nc_0 \) and \( \tr(\alpha) = -c_{n-1} \).

5. Let \( \alpha \) be an endomorphism of the finite-dimensional vector space \( V \), and assume that \( \alpha \) is invertible. Describe the eigenvalues and the characteristic and minimal polynomial of \( \alpha^{-1} \) in terms of those of \( \alpha \).

6. Prove that the inverse of a Jordan block \( J_m(\lambda) \) with \( \lambda \neq 0 \) has Jordan normal form a Jordan block \( J_m(\lambda^{-1}) \). For an arbitrary invertible square matrix \( A \), describe the Jordan normal form of \( A^{-1} \) in terms of that of \( A \).

Prove that any square complex matrix is similar to its transpose.

7. Let \( V \) be a vector space of dimension \( n \) and \( \alpha \) an endomorphism of \( V \) with \( \alpha^n = 0 \) but \( \alpha^{n-1} \neq 0 \). Show that there is a vector \( y \) such that \( \langle y, \alpha(y), \alpha^2(y), \ldots, \alpha^{n-1}(y) \rangle \) is a basis for \( V \).

Show that if \( \beta \) is an endomorphism of \( V \) which commutes with \( \alpha \), then \( \beta = p(\alpha) \) for some polynomial \( p \). [Hint: consider \( \beta(y) \).] What is the form of the matrix for \( \beta \) with respect to the above basis?

8. Let \( A \) be an \( n \times n \) matrix all the entries of which are real. Show that the minimal polynomial of \( A \) over the complex numbers has real coefficients.

9. Let \( V \) be a 4-dimensional vector space over \( \mathbb{R} \), and let \( \{\xi_1, \xi_2, \xi_3, \xi_4\} \) be the basis of \( V^* \) dual to the basis \( \{x_1, x_2, x_3, x_4\} \) for \( V \). Determine, in terms of the \( \xi_i \), the bases dual to each of the following:

(a) \( \{x_2, x_1, x_4, x_3\} \);
(b) \( \{x_1, 2x_2, \frac{1}{2}x_3, x_4\} \);
(c) \( \{x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4\} \);
(d) \( \{x_1 , x_2 - x_1, x_3 - x_2 + x_1, x_4 - x_3 + x_2 - x_1\} \).

10. Let \( P_n \) be the space of real polynomials of degree at most \( n \). For \( x \in \mathbb{R} \) define \( \varepsilon_x \in P_n^* \) by \( \varepsilon_x(p) = p(x) \).

Show that \( \varepsilon_0, \ldots, \varepsilon_n \) form a basis for \( P_n^* \), and identify the basis of \( P_n \) to which it is dual.

11. Let \( \alpha : V \to V \) be an endomorphism of a finite dimensional complex vector space and let \( \alpha^* : V^* \to V^* \) be its dual. Show that a complex number \( \lambda \) is an eigenvalue for \( \alpha \) if and only if it is an eigenvalue for \( \alpha^* \). How are the algebraic and geometric multiplicities of \( \lambda \) for \( \alpha \) and \( \alpha^* \) related? How are the minimal and characteristic polynomials for \( \alpha \) and \( \alpha^* \) related?
12. (a) Show that if \( x \neq y \) are vectors in the finite dimensional vector space \( V \), then there is a linear functional \( \theta \in V^* \) such that \( \theta(x) \neq \theta(y) \).
(b) Suppose that \( V \) is finite dimensional. Let \( A, B \leq V \). Prove that \( A \leq B \) if and only if \( A^o \geq B^o \).
Show that \( A = V \) if and only if \( A^o = \{0\} \).

13. For \( A \in \text{Mat}_{n,m}(\mathbb{F}) \) and \( B \in \text{Mat}_{m,n}(\mathbb{F}) \), let \( \tau_A(B) \) denote \( \text{tr}AB \). Show that, for each fixed \( A \), \( \tau_A: \text{Mat}_{n,m}(\mathbb{F}) \to \mathbb{F} \) is linear. Show moreover that the mapping \( A \mapsto \tau_A \) defines a linear isomorphism \( \text{Mat}_{n,m}(\mathbb{F}) \to \text{Mat}_{m,n}(\mathbb{F})^* \).

14. Show that the dual of the space \( P \) of real polynomials is isomorphic to the space \( \mathbb{R}^N \) of all sequences of real numbers, via the mapping which sends a linear form \( \xi : P \to \mathbb{R} \) to the sequence \( (\xi(1), \xi(t), \xi(t^2), \ldots) \).
In terms of this identification, describe the effect on a sequence \((a_0, a_1, a_2, \ldots)\) of the linear maps dual to each of the following linear maps \( P \to P \):
(a) The map \( D \) defined by \( D(p)(t) = p'(t) \).
(b) The map \( S \) defined by \( S(p)(t) = p(t^2) \).
(c) The map \( E \) defined by \( E(p)(t) = p(t - 1) \).
(d) The composite \( DS \).
(e) The composite \( SD \).
Verify that \((DS)^* = S^*D^* \) and \((SD)^* = D^*S^* \).

The remaining two questions are based on non-examinable material

15. Let \( V \) be a vector space of finite dimension over a field \( F \). Let \( \alpha \) be an endomorphism of \( V \) and let \( U \) be an \( \alpha \)-invariant subspace of \( V \) is a subspace such that \( \alpha(U) \leq U \). Define \( \overline{\alpha} \in \text{End}(V/U) \) by \( \overline{\alpha}(v + U) = \alpha(v) + U \). Check that \( \overline{\alpha} \) is a well-defined endomorphism of \( V/U \).
Consider a basis \((v_1, \ldots, v_n)\) of \( V \) containing a basis \((v_1, \ldots, v_k)\) of \( U \). Show that the matrix of \( \alpha \) with respect to \((v_1, \ldots, v_n)\) is \( \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \), where \( A \) the matrix of the restriction \( \alpha_U: U \to U \) of \( \alpha \) to \( U \) with respect to \((v_1, \ldots, v_k)\), and \( B \) the matrix of \( \overline{\alpha} \) with respect to \((v_{k+1} + U, \ldots, v_n + U) \). Deduce that \( \chi_\alpha = \chi_{\alpha_U} \chi_{\overline{\alpha}} \).

16. (Another proof of the Cayley Hamilton Theorem.) Assume that the Cayley Hamilton Theorem holds for any endomorphism on any vector space over the field \( \mathbb{F} \) of dimension less than \( n \). Let \( V \) be a vector space of dimension \( n \) and let \( \alpha \) be an endomorphism of \( V \). If \( U \) is a proper \( \alpha \)-invariant subspace of \( V \), use the previous question and the induction hypothesis to show that \( \chi_\alpha(\alpha) = 0 \). If no such subspace exists, show that there exists a basis \( \langle v, \alpha(v), \ldots, \alpha^{n-1}(v) \rangle \) of \( V \).
Show that \( \alpha \) has matrix
\[
\begin{pmatrix}
0 & -a_0 \\
1 & \ddots & -a_1 \\
& \ddots & 0 \\
& & 1 & -a_{n-1}
\end{pmatrix}
\]
with respect to this basis, for suitable \( a_i \in \mathbb{F} \). Show that \( \chi_\alpha(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0 \) and that \( \chi_\alpha(\alpha)(v) = 0 \). Deduce that \( \chi_\alpha(\alpha) = 0 \) as an element of \( \text{End}(V) \).