Linear Algebra: Example Sheet 2 of 4

1. (Another proof of the row rank column rank equality.) Let A be an $m \times n$ matrix of (column) rank r. Show that r is the least integer for which A factorises as $A = BC$ with $B \in \mathrm{Mat}_{m,r}(\mathbb{F})$ and $C \in \mathrm{Mat}_{r,n}(\mathbb{F})$. Using the fact that $(BC)^T = C^T B^T$, deduce that the (column) rank of A^T equals r.

2. Write down the three types of elementary matrices and find their inverses. Show that an $n \times n$ matrix A is invertible if and only if it can be written as a product of elementary matrices. Use this method to find the inverse of
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & -1 \end{pmatrix}.$$

3. Let A and B be $n \times n$ matrices over a field F. Show that the $2n \times 2n$ matrix
$$C = \begin{pmatrix} I & B \\ -A & 0 \end{pmatrix} \quad \text{can be transformed into} \quad D = \begin{pmatrix} I & B \\ 0 & AB \end{pmatrix}$$
by elementary row operations (which you should specify). By considering the determinants of C and D, obtain another proof that $\det AB = \det A \det B$.

4. (i) Let V be a non-trivial real vector space of finite dimension. Show that there are no endomorphisms α, β of V with $\alpha\beta - \beta\alpha = \mathrm{id}_V$.
(ii) Let V be the space of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$. Find endomorphisms α, β of V which do satisfy $\alpha\beta - \beta\alpha = \mathrm{id}_V$.

5. Find the eigenvalues and give bases for the eigenspaces of the following complex matrices:
$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$
The second and third matrices commute; find a basis with respect to which they are both diagonal.

6. Let $\lambda \in \mathbb{F}$. Consider the $n \times n$ matrix A with each diagonal entry equal to λ and all other entries 1. How does the rank of A depend on λ? Evaluate $\det A$.

7. Let V be a vector space, let $\pi_1, \pi_2, \ldots, \pi_k$ be endomorphisms of V such that $\mathrm{id}_V = \pi_1 + \cdots + \pi_k$ and $\pi_i \pi_j = 0$ for any $i \neq j$. Show that $V = U_1 \oplus \cdots \oplus U_k$, where $U_j = \mathrm{Im}(\pi_j)$.
Let α be an endomorphism on the vector space V, satisfying the equation $\alpha^3 = \alpha$. Prove directly that $V = V_0 \oplus V_1 \oplus V_{-1}$, where V_λ is the λ-eigenspace of α.

8. Let α be an endomorphism of a finite dimensional complex vector space. Show that if λ is an eigenvalue for α then λ^2 is an eigenvalue for α^2. Show further that every eigenvalue of α^2 arises in this way. Are the eigenspaces $\mathrm{Ker}(\alpha - \lambda\iota)$ and $\mathrm{Ker}(\alpha^2 - \lambda^2\iota)$ necessarily the same?

9. (Another proof of the Diagonalisability Theorem.) Let V be a vector space of finite dimension. Show that if α_1 and α_2 are endomorphisms of V, then the nullity $n(\alpha_1\alpha_2)$ satisfies $n(\alpha_1\alpha_2) \leq n(\alpha_1) + n(\alpha_2)$. Deduce that if α is an endomorphism of V such that $p(\alpha) = 0$ for some polynomial $p(t)$ which is a product of distinct linear factors, then α is diagonalisable.

10. Let A be a square complex matrix of finite order - that is, $A^m = I$ for some $m > 0$. Show that A can be diagonalised.

11. Let C be an $n \times n$ matrix over $\mathbb{C}$, and write $C = A + iB$, where A and B are real $n \times n$ matrices. By considering $\det(A + \lambda B)$ as a function of λ, show that if C is invertible then there exists a real number λ such that $A + \lambda B$ is invertible. Deduce that if two $n \times n$ real matrices P and Q are similar when regarded as matrices over $\mathbb{C}$, then they are similar as matrices over $\mathbb{R}$.

12. Let A, B be $n \times n$ matrices, where $n \geq 2$. Show that, if A and B are non-singular, then

$$(i)\ \mathrm{adj}\,(AB) = \mathrm{adj}\,(B)\mathrm{adj}\,(A)\,, \quad (ii)\ \det(\mathrm{adj}\,A) = (\det A)^{n-1}\,, \quad (iii)\ \mathrm{adj}\,(\mathrm{adj}\,A) = (\det A)^{n-2}A\,.$$

What happens if A is singular? *[Hint: Consider $A + \lambda I$ for $\lambda \in \mathbb{F}$.]*

Show that the rank of the adjugate matrix is $r(\mathrm{adj}\,A) = \begin{cases} n & \text{if } r(A) = n \\ 1 & \text{if } r(A) = n-1 \\ 0 & \text{if } r(A) \leq n-2. \end{cases}$

13. Let $f(x) = a_0 + a_1 x + \ldots + a_n x^n$, with $a_i \in \mathbb{C}$, and let C be the *circulant* matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_n \\ a_n & a_0 & a_1 & \ldots & a_{n-1} \\ a_{n-1} & a_n & a_0 & \ldots & a_{n-2} \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Show that the determinant of C is $\det C = \prod_{j=0}^{n} f(\zeta^j)$, where $\zeta = \exp(2\pi i/(n+1))$.

14. Let V denote the space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ and let α be the differentiation endomorphism $f \mapsto f'$.
(i) Show that every real number λ is an eigenvalue of α. Show also that $\ker(\alpha - \lambda\iota)$ has dimension 1.
(ii) Show that $\alpha - \lambda\iota$ is surjective for every real number λ.

15. Let $\alpha : V \to V$ be an endomorphism of a real finite dimensional vector space V with $\mathrm{tr}(\alpha) = 0$.
(i) Show that, if $\alpha \neq 0$, there is a vector $\mathbf{v}$ with $\mathbf{v}, \alpha(\mathbf{v})$ linearly independent. Deduce that there is a basis for V relative to which α is represented by a matrix A with all of its diagonal entries equal to 0.
(ii) Show that there are endomorphisms β, γ of V with $\alpha = \beta\gamma - \gamma\beta$.