GROUPS

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Purpose of notes. Please note that these are not notes of the lectures but notes made by the lecturer in preparation for the lectures. This means they may not exactly correspond to what was said and/or written during the lectures.

Lecture 1

1. Examples of groups

Groups are fundamentally about symmetry. More precisely they are an algebraic tool designed to abstract the notion of symmetry. Symmetry arises all over mathematics; which is to say that groups arise all over mathematics. Roughly speaking a symmetry is a transformation of an object that preserves certain properties of the object.

As I understand it, the purpose of this course is two-fold. First to introduce groups so that those who follow the course will be familiar with them and be better equipped to study symmetry in any mathematical context that they encounter. Second as an introduction to abstraction in mathematics, and to proving things about abstract mathematical objects.

It is perfectly possible to study groups in a purely abstract manner without geometric motivation. But this seems to both miss the point of why groups are interesting and make getting used to reasoning about abstract objects more difficult. So we will try to keep remembering that groups are about symmetry. So what do we mean by that?

1.1. **A motivating example.**

Question. What are the distance preserving functions from the integers to the integers? That is what are the members of the set

$$\mathrm{Isom}(\mathbb{Z}) := \{f\colon \mathbb{Z} \to \mathbb{Z} \text{ such that } |f(n) - f(m)| = |n - m| \text{ for all } n, m \in \mathbb{Z}\}?$$

These functions might reasonably be called the symmetries of the integers; they describe all the ways of 'rearranging' the integers that preserve the distance between any pair.

Let's begin to answer our question by giving some examples of such functions. Suppose that $a \in \mathbb{Z}$ is an integer. We can define the function 'translation by a' by

$$t_a\colon n \mapsto n + a \text{ for } n \in \mathbb{Z}.$$

For any choice of $m, n \in \mathbb{Z}$

$$|t_a(n) - t_a(m)| = |(n + a) - (m + a)| = |n - m|.$$

Thus t_a is an element of $\mathrm{Isom}(\mathbb{Z})$. We might observe that if a and b are both integers then

$$(t_a \circ t_b)(n) = t_a(b + n) = a + b + n = t_{a+b}(n)$$

for every integer n, that is that $t_{a+b} = t_a \circ t_b$[1]. Moreover t_0 is the identity or 'do nothing' function $\mathrm{id}\colon \mathbb{Z} \to \mathbb{Z}$ that maps every integer n to itself. Thus for every $a \in \mathbb{Z}$, t_{-a} is the inverse of t_a, that is $t_a \circ t_{-a} = \mathrm{id} = t_{-a} \circ t_a$.

[1]This is because two functions $f, g\colon X \to Y$ are the same function, i.e. $f = g$, precisely if they have the same effect on every element of the set they are defined on, i.e. $f(x) = g(x)$ for all $x \in X$.

Suppose now that $f \in \mathrm{Isom}(\mathbb{Z})$ is a symmetry of the integers. Consider the function $g := t_{-f(0)} \circ f$. Then for $n, m \in \mathbb{Z}$,

$$|g(n) - g(m)| = |(f(n) - f(0)) - (f(m) - f(0))| = |f(n) - f(m)| = |n - m|,$$

so $g \in \mathrm{Isom}(\mathbb{Z})$ is also a symmetry of the integers.

Moreover $g(0) = t_{-f(0)}(f(0)) = f(0) - f(0) = 0$, i.e. g fixes the integer 0. What does this tell us about g? For example, what does it tell us about $g(1)$? Since g is a symmetry and $g(0) = 0$ it must be the case that

$$|g(1)| = |g(1) - 0| = |g(1) - g(0)| = |1 - 0| = 1.$$

That is $g(1) = \pm 1$.

If $g(1) = 1$, what else can we say? For any $n \in \mathbb{Z}$,

$$|g(n)| = |g(n) - g(0)| = |n - 0| = |n|$$

i.e. $g(n) = \pm n$. But also,

$$|g(n) - 1| = |g(n) - g(1)| = |n - 1|$$

i.e. $g(n) = 1 \pm (n-1)$. These two conditions together force $g(n) = n$ and so $g = \mathrm{id}$. Now in this case

$$t_{f(0)} = t_{f(0)} \circ \mathrm{id} = t_{f(0)} \circ (t_{-f(0)} \circ f) = (t_{f(0)} \circ t_{-f(0)}) \circ f = \mathrm{id} \circ f = f.$$

Thus f is translation by $f(0)$ in this case.

What about the case when $g(1) = -1$? In this case we still must have $g(n) = \pm n$ for every integer n but now also

$$|g(n) + 1| = |g(n) - g(1)| = |n - 1|$$

i.e. $g(n) = -1 \pm (n - 1)$. These two conditions together force $g(n) = -n$ and so g is the 'reflection about 0'-function

$$s \colon n \mapsto -n \text{ for all } n \in \mathbb{Z}.$$

Now we've seen that $s = g = t_{-f(0)} \circ f$ in this case. It follows that $f = t_{f(0)} \circ s$.

We've now proven that every element of $\mathrm{Isom}(\mathbb{Z})$ is either a translation t_a or of the form $t_a \circ s$ (with $a \in \mathbb{Z}$ in either case). That is all symmtries of $\mathbb{Z}$ are of the form $n \mapsto n + a$ or of the form $n \mapsto a - n$.

It is worth reflecting at this point on some key facts we've used in the argument above which is sometimes known as a 'nailing to the wall argument'.

(1) We've used that the composition of two symmetries of the integers is itself a symmetry of the integers. In fact, we've only used this for some special cases but it is true in general since if $f, g \in \mathrm{Isom}(\mathbb{Z})$ and $n, m \in \mathbb{Z}$ then
$$|f(g(n)) - f(g(m))| = |g(n) - g(m)| = |n - m|.$$
We might note that for $a, n \in \mathbb{Z}$,
$$s(t_a(n)) = s(n + a) = -a - n = t_{-a}(s(n))$$
and so $s \circ t_a = t_{-a} \circ s$. Thus order of composition matters.
(2) We've used that there is a 'do nothing' symmetry of the integers id and that for any other symmetry f, $f \circ \mathrm{id} = f = \mathrm{id} \circ f$.
(3) We've used that symmetries are 'undo-able', that is that given any symmetry f there is a symmetry g such that $g \circ f = \mathrm{id} = f \circ g$ (in fact we've only used this for $f = t_a$ and $f = s$ and only that there is a g such that $g \circ f = \mathrm{id}$ but again it is true as stated. (Why?).

(4) We've used that composition of symmetries is associative, that is that for symmetries f, g and h, $(f \circ g) \circ h = f \circ (g \circ h)$.

We'll see that these properties say precisely that $\mathrm{Isom}(\mathbb{Z})$ is a group.

1.2. **Some initial definitions.** First we need to make some definitions.

Definition. Suppose that S is a set. A *binary operation on* S is a function

$$\circ \colon S \times S \to S; (x, y) \mapsto x \circ y.$$

This defintion means that a binary operation is something that takes an ordered pair of elements of S and uses them to produce an element of S. If $x \circ y = y \circ x$ then we say that x and y *commute* (with respect to $\circ$). We say $\circ$ is commutative if every pair of elements of S commute.

Examples.

(1) Composition of functions is a non-commutative binary operation on $\mathrm{Isom}(\mathbb{Z})$.
(2) Addition, multiplication, and subtraction are all binary operations on $\mathbb{Z}$. Note that addition and multiplication are both commutative operations on $\mathbb{Z}$ but distinct integers never commute with respect to subtraction.
(3) Addition and multiplication are also binary operations on $\mathbb{N} := \{1, 2, 3, \ldots\}$. Subtraction is not a binary operation on $\mathbb{N}$ since $2 - 3 \notin \mathbb{N}$.
(4) Exponentiation: $(a, b) \mapsto b^a$ is a binary operation on $\mathbb{N}$.
(5) If X is any set and $S = \{f : X \to X\}$ is the set of all functions from X to itself then composition of functions is a binary operation on S.

Definition. A binary operation $\circ$ on a set S is *associative* if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$.

This means that when $\circ$ is associative there is a well-defined element $x \circ y \circ z \in S$ i.e. it doesn't matter which of the two $\circ$ we use first. It will be instructive to convince yourself that if $\circ$ is an associative binary operation on S and $w, x, y, z \in S$ then

$$w \circ (x \circ y \circ z) = (w \circ x) \circ (y \circ z) = (w \circ x \circ y) \circ z.$$

Having done this you should also convince yourself that there is nothing special about four and the obvious generalisation holds for any (finite) number of elements of S whenever $\circ$ is associative. This means that whenever $\circ$ is an associative binary operation we may (and will!) omit brackets, writing for example $w \circ x \circ y \circ z$ without ambiguity. If it is clear what operation we have in mind we will often omit it too, writing $wxyz$, for example.

Examples.

(1) Addition and multiplication are associative when viewed as binary operations on $\mathbb{Z}$ or $\mathbb{N}$. Subtraction is not associative on $\mathbb{Z}$ since $((0-1)-2) = -3$ but $0 - (1 - 2) = 1 \neq -3$.
(2) Exponentiation $(a, b) \mapsto b^a$ is not associative on $\mathbb{N}$ since $2^{3^2} = 2^9$ but $(2^3)^2 = 2^6 \neq 2^9$.
(3) Composition is always an associative operation on the set of functions from X to X since if f, g and h are three such functions and $x \in X$ then

$$((f \circ g) \circ h)(x) = f(g(h(x))) = (f \circ (g \circ h))(x).$$

Definition. A binary operation $\circ$ on a set S *has an identity* if there is some element $e \in S$ such that for all $x \in S$, $e \circ x = x = x \circ e$.

Examples.

(1) 0 is an identity for addition on $\mathbb{Z}$ but addition has no identity on $\mathbb{N}$. 1 is an identity for multiplication on both these sets. Subtraction on $\mathbb{Z}$ does not have an identity since if $e - x = x$ for all $x \in \mathbb{Z}$ then $e = 2x$ for all $x \in \mathbb{Z}$ and this is absurd. Note however that $x - 0 = x$ for all $x \in \mathbb{Z}$. We sometimes say that 0 is a right identity for subtraction to d9escribe this.
(2) $(a, b) \mapsto b^a$ does not have an identity but 1 is a left identity in the obvious sense.
(3) If X is any set then the identity function $\mathrm{id}\colon X \to X; s \mapsto s$ is an identity for composition of functions from X to X.

Lemma. *If a binary operation $\circ$ on a set S has an identity then it is unique.*

Proof. Suppose that e and e' are both identities on S. Then $e \circ e' = e' = e' \circ e$ since e is an identity. But also $e' \circ e = e = e \circ e'$ since e' is an identity. Thus $e = e'$ as required. □

Lecture 2

Definition. If a binary operation $\circ$ on a set S has an identity e then we say that it *has inverses* if for every $x \in S$ there is some $y \in S$ such that $x \circ y = e = y \circ x$.

Examples.

(1) $+$ on $\mathbb{Z}$ has inverses since for every $n \in \mathbb{Z}$, $n + (-n) = 0 = (-n) + n$. Multiplication does not have inverses on $\mathbb{N}$ or $\mathbb{Z}$ since there is no integer (and therefore no natural number) n such that $2n = 1$.
(2) Multiplication defines an associative binary operation on the rationals $\mathbb{Q}$ with an identity (1) but it still does not have inverses. Although for every non-zero rational q, $1/q$ is also rational and $q \cdot 1/q = 1 = 1/q \cdot q$, 0 is also rational and there is no rational r such that $r \cdot 0 = 1$. However multiplication does have inverses on the set $\mathbb{Q}\backslash\{0\}$.
(3) In general composition on the set of functions $X \to X$ does not have inverses. For example the function $f\colon \mathbb{Z} \to \mathbb{Z}; n \mapsto 0$ has no inverse since if $g\colon \mathbb{Z} \to \mathbb{Z}$ were an inverse then we'd have $f(g(n)) = n$ for all $n \in \mathbb{Z}$ but in fact however g is defined $f(g(1)) = 0$. This idea can be adapted to show that whenever $|X| > 1$ there is a function $f\colon X \to X$ that has no inverse.

Definition. A set G equipped with a binary operation $\circ$ is a *group* if

(i) the operation $\circ$ is associative;
(ii) the operation $\circ$ has an identity;
(iii) the operation $\circ$ has inverses.

Examples.

(1) $\mathrm{Isom}(\mathbb{Z})$ is a group (under composition).
(2) $(\mathbb{Z}, +)$ is a group since $+$ is associative and has an identity and inverses.
(3) $(\mathbb{N}, +)$ is not a group since it does not have an identity.

(4) $(\mathbb{Z}, -)$ is not a group since $-$ is not associative.[2]
(5) $(\mathbb{Z}, \cdot)$ is not a group since it does not have inverses but $(\mathbb{Q}\backslash\{0\}, \cdot)$ is a group.
(6) If X is a set with more than one element then the set of functions $X \to X$ is not a group under composition of functions since not all such functions have inverses.

We will sometimes say that G is a group without specifying the operation $\circ$. This is laziness and the operation will always be implicit and either clear what it is (in concrete settings) or unimportant what it is (in abstract settings). We'll nearly always call the identity of a group e (or e_G if we want to be clear which group it is the identity for) if we don't know it by some other name.

Definition. We say that a group G is *abelian* if any pair of elements of G commute.

Definition. We say that a group G is *finite* if it has finitely many elements as a set. We call the number of elements of a finite group G the *order* of G written $|G|$.

Example. For every integer $n \geqslant 1$ we can define a group that is the set $\mathbb{Z}_n := \{0, 1, \dots, n-1\}$ equipped with the operation $+_n$ where $x +_n y$ is the remainder after dividing $x + y$ by n[3]. It is straightforward to see that $\mathbb{Z}_n$ is an abelian group of order n.

Lemma. *Suppose that G is a group.*

(i) inverses are unique i.e. if $g \in G$ there is precisely one element g^{-1} in G such that $g^{-1}g = e = gg^{-1}$;
(ii) for all $g \in G$, $(g^{-1})^{-1} = g$;
(iii) for all $g, h \in G$, $(gh)^{-1} = h^{-1}g^{-1}$ (the shoes and socks lemma).

Proof. (i) By assumption every element $g \in G$ has at least one inverse. We must show that there is at most one. Suppose that $h, k \in G$ such that $gh = e = hg$ and $gk = e = kg$. Then

$$h = eh = kgh = ke = k$$

i.e. h and k are the same element of G.

(ii) Given $g \in G$, g satisfies the equations for the inverse of g^{-1} i.e.

$$gg^{-1} = e = gg^{-1}$$

so by (i), $(g^{-1})^{-1} = g$.

(iii) Given $g, h \in G$,

$$(gh)(h^{-1}g^{-1}) = ghh^{-1}g^{-1} = geg^{-1} = gg^{-1} = e$$

and similarly $(h^{-1}g^{-1})(gh) = e$. □

Notation. For each element g in a group G and natural number n we define g^n recursively by $g^1 := g$ and $g^n := gg^{n-1}$ for $n > 1$. We'll also write $g^0 := e$ and $g^n := (g^{-1})^{-n}$ for integers $n < 0$. It follows that $g^a g^b = g^{a+b}$ for all $a, b \in \mathbb{Z}$.

Definition. If G is a group then we say that $g \in G$ has *finite order* if there is a natural number n such that $g^n = e$. If g has finite order, we call the smallest natural number n such that $g^n = e$ the *order* of g and write $o(g) = n$.

[2] recall that it also does not have an identity but to see that it is not a group it suffices to see that any one of the three properties fails.

[3] $\mathbb{Z}_{12}$ is familiar from everyday life. When is it used?

1.3. **Further geometric examples.**

1.3.1. *Symmetry groups of regular polygons.* Suppose we want to consider the set D_{2n} of all symmetries of a regular polygon P with n vertices (for $n \geqslant 3$) living in the complex plane $\mathbb{C}$. By symmetry of P we will mean a distance preserving transformation of the plane that maps P to itself. We might as well assume that the centre of P is at the origin 0 and that one of the vertices is the point $1 = 1+0i$[4].

Proposition. *D_{2n} is a group of order $2n$ under composition.*

Lecture 3

Proof. First we observe that if $g, h \in D_{2n}$ then for $z, w \in \mathbb{C}$,

$$|g(h(z) - g(h(w))| = |h(z) - h(w)| = |z - w|,$$

and if $p \in P$ then $g(h(p)) \in P$ since $h(p) \in P$. Thus composition of functions defines a binary operation on D_{2n}. Since composition of functions is always associative this operation on D_{2n} is associative. Moreover the identity function id is obviously a symmetry of P. Thus to see that D_{2n} is a group it remains to show that every element of D_{2n} is invertible. We could do this directly but instead we'll use a nailing to the wall argument as we did for $\mathrm{Isom}(\mathbb{Z})$.

Let $r\colon \mathbb{C} \to \mathbb{C}$; $z \mapsto e^{2\pi i/n} z$ denote rotation anticlockwise about 0 by $2\pi/n$. Then r does preserve distances in $\mathbb{C}$ as

$$|r(z) - r(w)| = |e^{2\pi i/n}||z - w| = |z - w|$$

and $r(P) = P$ i.e. $r \in D_{2n}$. Moreover $r^m \in D_{2n}$ for all $m \in \mathbb{N}$ and $r^n = \mathrm{id}$; i.e. $r^{-1} = r^{n-1}$.

Similarly let $s\colon \mathbb{C} \to \mathbb{C}$; $z \mapsto \bar{z}$ denote complex conjugation; i.e. reflection in the real axis. Once again s preserves distances in $\mathbb{C}$ and $s(P) = P$. Moreover $s^2 = \mathrm{id}$; i.e. $s^{-1} = s$.

We will try to write down all the elements of D_{2n} using r and s.

Consider the set $V := \{e^{2\pi ik/n} \mid k = 0, 1, \ldots n-1\} \subset \mathbb{C}$ of nth roots of unity in the complex plane, the set of vertices of our regular n-gon. Any element of D_{2n} will preserve V; that is to say if $g \in D_{2n}$ then $g(V) = V$[5]. So if we pick any $g \in D_{2n}$ then $g(1) = e^{2\pi ik/n}$ for some $k = 0, 1, \ldots, n-1$. Thus

$$r^{n-k} g(1) = e^{2\pi i(n-k)} e^{2\pi ik/n} = 1.$$

Now $e^{2\pi i/n}$ and $e^{-2\pi i/n}$ are the only two elements of V of distance $|1 - e^{2\pi i/n}|$ from 1. Thus $r^{n-k} g(e^{2\pi i/n}) = e^{\pm 2\pi i/n}$.

If $r^{n-k} g(e^{2\pi i/n}) = e^{2\pi i/n}$ then $r^{n-k} g$ is a distance preserving transformation of $\mathbb{C}$ fixing $0, 1$ and $e^{2\pi i/n}$. Thus $r^{n-k} g = \mathrm{id}$ and $g = r^k$.

If $r^{n-k} g(e^{2\pi i/n}) = e^{-2\pi i/n}$ then $s r^{n-k} g$ is a distance preserving transformation of $\mathbb{C}$ fixing $0, 1$ and $e^{2\pi i/n}$. Thus $s r^{n-k} g = \mathrm{id}$ and $g = r^k s$.

So we have computed that

$$D_{2n} = \{r^k, r^k s | k = 0, \ldots, n-1\}.$$

[4] we'll be able to make precise why this assumption is reasonable later but it should at least seem reasonable already.

[5] Recall that $g(V) := \{g(v) \mid v \in V\}$.

Thus it has $2n$ elements as required. We can compute for $z \in \mathbb{C}$,

$$sr^k(z) = s(e^{2\pi ik/n}z) = e^{-2\pi ik/n}\bar{z} = r^{-k}s(z).$$

Thus $r^k s r^k s = r^k r^{-k} ss = e$ and $r^k s$ is self-inverse. Geometrically $r^k s$ denote reflection in the line through 0 and the point $e^{\pi ik/2n}$. To show this we can compute

$$\begin{aligned} r^k s(0) &= 0, \\ r^k s(1) &= e^{2\pi ik/n} \text{ and} \\ r^k s(e^{\pi ik/n}) &= e^{\pi ik/n}. \end{aligned}$$

More generally if we wish to multiply elements in this standard form,

$$\begin{aligned} r^k \cdot r^l &= r^{k+_n l}, \\ r^k \cdot r^l s &= r^{k+_n l}s, \\ r^k s \cdot r^l &= r^{k+_n(-l)}s \text{ and} \\ r^k s \cdot r^l s &= r^{k+_n(-l)}. \end{aligned}$$

In particular $sr = r^{n-1}s = r^{-1}s$. It would be instructive to reflect on the geometric meaning of these equations. □

1.3.2. *Symmetry groups of regular solids.* Suppose that X is a regular solid in $\mathbb{R}^3$. We can consider $\mathrm{Sym}(X)$, the group of distance preserving transformations ρ of $\mathbb{R}^3$ such that $\rho(X) = X$. These form a group. We will consider the cases X a tetrahedron and X a cube later in the course.

1.3.3. *The Symmetric group.* We might hope that given any set X the set of *invertible* functions from X to X forms a group under composition; that is the set of functions $f\colon X \to X$ such that there is some $g\colon X \to X$ such that $f \circ g = \mathrm{id} = g \circ f$. This is true but not immediate: we need to check that composition of functions is a binary operation on this set; that is that the composition of two invertible functions is invertible. Some people would say that we need to check that the binary operation is *closed* but 'closure' is built into our definition of binary operation.

Lemma. *Suppose that $f_1, f_2\colon X \to X$ are invertible. Then $f_1 \circ f_2\colon X \to X$ is invertible.*

Proof. There are functions $g_1, g_2\colon X \to X$ such that $f_1 \circ g_1 = \mathrm{id} = g_1 \circ f_1$ and $f_2 \circ g_2 = \mathrm{id} = g_2 \circ f_2$. Then since $\circ$ is associative

$$(f_1 \circ f_2) \circ (g_2 \circ g_1) = f_1 \circ \mathrm{id} \circ g_1 = \mathrm{id}$$

and

$$(g_2 \circ g_1) \circ (f_1 \circ f_2) = g_2 \circ \mathrm{id} \circ f_2 = \mathrm{id}\,.$$

□

It follows that for every set X, the set $S(X) = \{f\colon X \to X \mid f \text{ is invertible}\}$ is a group under the composition of functions. It is called the *symmetric group* on X[6]. We call elements of the symmetric group *permutations of* X. If $X = \{1, \ldots, n\}$ we write S_n instead on $S(X)$. We will return to the groups S_n later in the course.

[6]The name comes from the fact that it can be viewed as the set of symmetries of the set X. This is quite a subtle idea but you might like to think further about it when you come to revise the course

1.4. **Subgroups and homomorphisms.** Sometimes when considering the symmetries of an object we want to restrict ourselves to considering symmetries that preserve certain additional properties of the object. In fact we've already seen this, the sets of distance preserving transformations of $\mathbb{C}$ and of $\mathbb{R}^3$ are both groups of symmetries under composition. The groups D_{2n} and Sym(X) for X a regular solid are defined to consist of those symmetries that preserve a certain subset of the whole space. Similary, instead of considering D_{2n}, the group of all symmetries of a regular n-gon we might want to restrict only to those symmetries that preserve orientation, that is the rotations. This idea leads us to the notion of subgroup.

Definition. If $(G, \circ)$ is a group then a subset $H \subset G$ is a *subgroup* if $\circ$ restricts to a binary operation on H[7] and $(H, \circ)$ is a group. We write $H \leqslant G$ to denote that H is a subgroup of G.

Examples.

(1) $\mathrm{Isom}(\mathbb{Z}) \leqslant S(\mathbb{Z})$.
(2) $D_{2n} \leqslant \mathrm{Isom}(\mathbb{C}) \leqslant S(\mathbb{C})$.
(3) $\mathrm{Isom}^+(\mathbb{Z}) := \{f \colon \mathbb{Z} \to \mathbb{Z} \mid f(n) - f(m) = n - m \text{ for all } n, m \in \mathbb{Z}\} \leqslant \mathrm{Isom}(\mathbb{Z})$.
(4) $\mathbb{Z}$ is a subgroup of $(\mathbb{Q}, +)$.
(5) If $H \subset D_{2n}$ consists of all rotations of the n-gon then H is a subgroup.
(6) For any $n \in \mathbb{Z}$, $n\mathbb{Z} := \{an \in \mathbb{Z} \mid a \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$.
(7) For every group G, $\{e\} \leqslant G$ (the *trivial subgroup*) and $G \leqslant G$ (we call a subgroup H of G with $H \neq G$ a *proper subgroup*).

Lecture 4

Lemma (Subgroup criteria)**.** *A subset H of a group G is a subgroup if and only if the following conditions hold*

(i) for every pair of elements $h_1, h_2 \in H$, $h_1 h_2 \in H$;
(ii) the identity $e \in H$;
(iii) for every $h \in H$, $h^{-1} \in H$.

Proof. We must show that if conditions (i)-(iii) hold then H is a group. Condition (i) tells us that the binary operation on G restricts to one on H. That this operation is associative follows immediately from the associativity of its extension to G. Condition (ii) tells us that it has an identity[8] and condition (iii) tells us H has inverses[9].

We must also show that if H is a subgroup of G then conditions (i)-(iii) hold. That condition (i) holds is immediate from the definition of subgroup. If e_H is the identity in H then $e_H e = e_H = e e_H$ since e is the identity in G. But also $e_H e_H = e_H$ since e_H is the identity in H. Thus $e_H e = e_H e_H$. But e_H has an inverse in G and we see that $e_H^{-1} e_H e = e_H^{-1} e_H e_H$ ie $e = e_H \in H$ i.e. condition (ii) holds. That condition (iii) holds follows from uniqueness of inverses in G since for $h \in H$ the inverse of h in H will also be an inverse of h in G. □

Remark. Our subgroup criteria contain no mention of associativity since as noted in the proof it is immediate from the associativity of the operation on G.

[7] precisely $h_1 \circ h_2 \in H$ for all $h_1, h_2 \in H$
[8] the same identity as G since if $h \in H$ then $h \in G$ so $eh = h = he$
[9] the inverse h^{-1} of $h \in H \subset G$ in G is also an inverse in H since $hh^{-1} = e = h^{-1}h$.

There is an even shorter set of criteria for a subset to be a subgroup.

Corollary. *A subset H of G is a subgroup precisely if it is non-empty and $h_1^{-1}h_2 \in H$ for all $h_1, h_2 \in H$.*

Proof. Suppose that conditions (i)-(iii) of the last lemma hold. Condition (ii) tells us that H is non-empty and conditions (iii) and (i) combined give that $h_1^{-1}h_2 \in H$ for all $h_1, h_2 \in H$.

For the converse, since H is non-empty, there is some $h \in H$. By assumption $e = h^{-1}h \in H$ so condition (ii) holds. Now for $h \in H$, $h^{-1}e = h^{-1} \in H$ by assumption i.e. condition (iii) holds. Finally, for $h_1, h_2 \in H$, $h_1h_2 = (h_1^{-1})^{-1}h_2 \in H$. □

Example. The set $H = \{f \in \mathrm{Isom}(\mathbb{Z}) | f(0) = 0\}$ is a subgroup of $\mathrm{Isom}(\mathbb{Z})$. We can see this using the corollary. Certainly $\mathrm{id}(0) = 0$ so $H \neq \emptyset$. Moreover if $h_1, h_2 \in H$ then

$$h_1^{-1}h_2(0) = h_1^{-1}(0) = h_1^{-1}h_1(0) = \mathrm{id}(0) = 0.$$

Note that this argument isn't much simpler than verifying conditions (i)-(iii) of the lemma in practice.

We will also be interested in maps between groups. However we won't typically be interested an arbitary functions between two groups but only those that respect the structure of the two groups. More precisely we make the following definition.

Definition. If $(G, \circ)$ and $(H, *)$ are two groups then $\theta\colon H \to G$ is a *group homomorphism* (or just *homomorphism*) precisely if $\theta(h_1 * h_2) = \theta(h_1) \circ \theta(h_2)$ for all $h_1, h_2 \in H$.

Definition. A group homomorphism $\theta\colon H \to G$ is an *isomorphism* if θ is invertible as a function; ie if there is a function $\theta^{-1}\colon G \to H$ such that $\theta \circ \theta^{-1} = \mathrm{id}_G$ and $\theta^{-1} \circ \theta = \mathrm{id}_H$.

Examples.

(1) If $H \leqslant G$ then the inclusion map $\iota\colon H \to G; h \mapsto h$ is a group homomorphism. It is not an isomorphism unless $H = G$.
(2) The function $\theta\colon \mathbb{Z} \to \mathbb{Z}_n$ such that $\theta(a)$ is the remainder after dividing a by n is always a homomorphism from $(\mathbb{Z}, +)$ to $(\mathbb{Z}_n, +_n)$ but never an isomorphism.
(3) If G is any group and $g \in G$ is any element then $\theta\colon \mathbb{Z} \to G$; $n \mapsto g^n$ is a homomorphism from $(\mathbb{Z}, +)$ to G. Indeed every homomorphism from $(\mathbb{Z}, +)$ to G arises in this way.
(4) $\theta\colon \mathbb{Z} \to \mathrm{Isom}^+(\mathbb{Z})$; $n \mapsto t_n$[10] is an isomorphism.
(5) The exponential function defines an isomorphism

$$\exp\colon (\mathbb{R}, +) \to (\{r \in \mathbb{R} \mid r > 0\}, \cdot); a \mapsto e^a.$$

The inverse map is given by $\log = \log_e$.

If you are alert you will be asking why we don't require homomorphisms $\theta\colon H \to G$ to satisfy $\theta(e_H) = e_G$ and $\theta(h^{-1}) = \theta(h)^{-1}$ for all $h \in H$. The following lemma shows that this is because these properties follow from our definition.

Lemma. *Suppose that $\theta\colon H \to G$ is a group homomorphism.*

(i) $\theta(e_H) = e_G$.
(ii) For all $h \in H$, $\theta(h^{-1}) = \theta(h)^{-1}$.

[10] recall t_n denotes translation by n

Proof. (i) Since θ is a homomorphism $\theta(e_H) = \theta(e_H e_H) = \theta(e_H)\theta(e_H)$. Thus $e_G = \theta(e_H)^{-1}\theta(e_H) = \theta(e_H)^{-1}\theta(e_H)\theta(e_H) = \theta(e_H)$ as required.

(ii) Pick $h \in H$. Then $\theta(h)\theta(h^{-1}) = \theta(hh^{-1}) = \theta(e_H) = e_G$ (by (i)). Similarly $\theta(h^{-1})\theta(h) = \theta(h^{-1}h) = \theta(e_H) = e_G$. Thus $\theta(h^{-1}) = \theta(h)^{-1}$ as required. □

Definition. If $\theta\colon H \to G$ is a group homomorphism then the *kernel of* θ is defined by

$$\ker\theta := \{h \in H \mid \theta(h) = e_G\}$$

and the *image of* θ is defined by

$$\operatorname{Im}\theta := \theta(H).$$

Proposition. *If $\theta\colon H \to G$ is a homomorphism then $\ker\theta$ is a subgroup of H and $\operatorname{Im}\theta$ is a subgroup of G.*

Proof. We've seen $\theta(e_H) = e_G$ so $e_H \in \ker\theta$ and $e_G \in \operatorname{Im}\theta$.

Suppose that $k_1, k_2 \in \ker\theta$. Then $\theta(k_1k_2) = \theta(k_1)\theta(k_2) = e_Ge_G = e_G$, i.e. $k_1k_2 \in \ker\theta$. Similarly $\theta(k_1^{-1}) = \theta(k_1)^{-1} = e_G^{-1} = e_G$, i.e. $k_1^{-1} \in \ker\theta$. So $\ker\theta \leqslant H$.

Suppose now that $\theta(h_1), \theta(h_2) \in \operatorname{Im}\theta$. Then $\theta(h_1)\theta(h_2) = \theta(h_1h_2)$ so $\theta(h_1)\theta(h_2) \in \operatorname{Im}\theta$. Similarly $\theta(h_1)^{-1} = \theta(h_1^{-1})$, so $\theta(h_1)^{-1} \in \operatorname{Im}\theta$. So $\operatorname{Im}\theta \leqslant G$. □

Lecture 5

Theorem (Special case of the isomorphism theorem)**.** *A group homomorphism $\theta\colon H \to G$ is an isomorphism if and only if $\ker\theta = \{e_H\}$ and $\operatorname{Im}\theta = G$. In this case, $\theta^{-1}\colon G \to H$ is a group homomorphism (and so also an isomorphism).*

Proof. Suppose that θ has an inverse. Certainly $e_H \in \ker\theta$ and so $\theta^{-1}(e_G) = \theta^{-1}\theta(e_H) = e_H$. Thus if $k \in \ker\theta$ then $k = \theta^{-1}\theta(k) = \theta^{-1}(e_G) = e_H$ and so $\ker\theta = \{e_H\}$. Moreover for each $g \in G$, $\theta^{-1}(g) \in H$ and $g = \theta(\theta^{-1}(g))$. Thus $G = \operatorname{Im}\theta$.

Conversely, suppose that $\ker\theta = e_H$ and $\operatorname{Im}\theta = G$. By the latter property for each $g \in G$ we may choose an element $h_g \in H$ such that $\theta(h_g) = g$. Now if $s\colon G \to H$ is the function $g \mapsto h_g$ then $\theta s(g) = \theta(h_g) = g$ for all $g \in G$. It follows that for all $h \in H$,

$$\theta(s(\theta(h))h^{-1}) = \theta(s(\theta(h)))\theta(h^{-1}) = \theta(h)\theta(h^{-1}) = \theta(hh^{-1}) = \theta(e_H) = e_G,$$

i.e. $s(\theta(h))h^{-1} \in \ker\theta = \{e_H\}$. It follows that $s\theta(h) = h$ for all $h \in H$ and so s is an inverse of θ.

Finally, suppose that θ is an isomorphism and $g_1, g_2 \in G$. Then

$$\theta(\theta^{-1}(g_1g_2)) = g_1g_2$$

and

$$\theta(\theta^{-1}(g_1)\theta^{-1}(g_2)) = \theta(\theta^{-1}(g_1))\theta(\theta^{-1}(g_2)) = g_1g_2$$

since θ is a homomorphism. Thus

$$\theta^{-1}(g_1g_2) = \theta^{-1}(\theta(\theta^{-1}(g_1g_2))) = \theta^{-1}(\theta(\theta^{-1}(g_1)\theta(\theta^{-1}(g_2)))) = \theta^{-1}(g_1)\theta^{-1}(g_2)$$

as required. □

Lemma. *The composite of two group homomorphisms is a group homomorphism. In particular the composite of two isomorphisms is an isomorphism.*

Proof. Suppose $\theta_1 \colon H \to G$ and $\theta_2 \colon K \to H$ are homomorpisms and $k_1, k_2 \in K$ then $\theta_1(\theta_2(k_1k_2)) = \theta_1(\theta_2(k_1)\theta_2(k_2)) = \theta_1\theta_2(k_1)\theta_1\theta_2(k_2)$ ie $\theta_1\theta_2$ is a homomorphism. Finally, if θ_1 and θ_2 are invertible then $\theta_2^{-1}\theta_1^{-1}$ is an inverse of $\theta_1\theta_2$. □

Definition. We say that a group G is *cyclic* if there is a homomorphism $f \colon \mathbb{Z} \to G$ such that $\operatorname{Im} f = G$. Given such a homomorphism f we call $f(1)$ a *generator* of G.

Note that G is cyclic with generator g if and only if every element of G is of the form g^i with $i \in \mathbb{Z}$. More generally we say that a subset S of G *generates* G if every element of G is a product of elements of S and their inverses — that is if G the unique smallest subgroup of G containing S[11].

Examples.

(1) The identity map $\mathrm{id} \colon \mathbb{Z} \to \mathbb{Z}$; $n \mapsto n$ and the 'reflection about 0' map $s \colon \mathbb{Z} \to \mathbb{Z}$; $n \mapsto -n$ are both homomorphisms with image $\mathbb{Z}$. Thus $\mathbb{Z}$ is cyclic and both 1 and -1 are generators. No other element generates $\mathbb{Z}$.
(2) $\mathbb{Z}_n$ is cyclic. In Numbers and Sets it is proven that an element of $\{0, 1, \dots, n-1\}$ generates $\mathbb{Z}_n$ if and only if it is coprime to n[12]. The 'if' part is a conseqeunce from Euclid's algorithm; the only if part is elementary.

Lemma. *Suppose that G is a group containing an element g with $g^n = e$. There is a unique group homomorphism $f \colon \mathbb{Z}_n \to G$ such that $f(1) = g$. In particular every group of order n with an element of order n is isomorphic to $\mathbb{Z}_n$.*

Proof. Suppose that $f \colon \mathbb{Z}_n \to G$ is a homomorphism such that $f(1) = g$. Then for $a = 0, 1, \dots n-1$, $f(a +_n 1) = f(a)f(1) = f(a)g$. Thus we can see inductively that $f(a) = g^a$ for all $a \in \mathbb{Z}_n$. Thus if f exists then it is unique.

We now see how to construct f. We define $f(a) = g^a$ for all $a \in \mathbb{Z}_n$ and we must prove that this defines a homomorphism. That is we must show that $g^{a+_n b} = f(a +_n b)$ and $g^{a+b} = f(a)f(b)$ are equal for all $a, b \in Z_n$. Since $a + b - (a +_n b) = kn$ for some integer k and $g^n = e$, we see that

$$g^{a+b}g^{-(a+_n b)} = (g^n)^k = e^k = e.$$

Thus $g^{a+b} = g^{a+_n b}$ as claimed.

Suppose now that G has order n and $g \in G$ has order n. By the previous part there is a homomorphism $f \colon \mathbb{Z}_n \to G$ such that $f(a) = g^a$ for each $a \in Z_n$. Suppose that $f(a) = f(b)$ for $a, b \in Z_n$. Then $f(b -_n a) = g^{a-_n b} = e$ thus $a = b$ else g would have order strictly smaller than n. It follows that $\ker f = \{e\}$ and $|\operatorname{Im} f|$ has n elements and so must be the whole of G. Thus f is an isomorphism. □

Notation. We'll write C_n for any group that is cyclic of order n. We've verified that any two such groups are isomorphic.

[11] The curious will be reflecting on why G should have a unique smallest subgroup containing S. Their reflections will do them good

[12] Recall that non-negative integers a, b are coprime if and only if their only common factor is 1.

Recall that we showed that $D_{2n} = \{r^i, r^i s \mid i = 0, 1, \ldots, n-1\}$ where r denotes a rotation by $2\pi/n$ and s denotes a reflection. And that

$$\begin{aligned} r^k \cdot r^l &= r^{k+_n l}, \\ r^k \cdot r^l s &= r^{k+_n l} s, \\ r^k s \cdot r^l &= r^{k+_n (-l)} s \text{ and} \\ r^k s \cdot r^l s &= r^{k+_n (-l)}. \end{aligned}$$

Lemma. *Let $n > 2$ and suppose that G is a group containing elements g, h such that $g^n = e$, $h^2 = e$ and $hgh^{-1} = g^{-1}$. There is a unique group homomorphism $f\colon D_{2n} \to G$ such that $f(r) = g$ and $f(s) = h$. Moreover if $o(g) = n$ and $|G| = 2n$ then f is an isomorphism.*

Proof. This is similar to the last proof. Any homomorphsim $f\colon D_{2n} \to G$ such that $f(r) = g$ and $f(s) = h$ must satisfy $f(r^i) = g^i$ and $f(r^i s) = g^i h$. So we must show that this does define a homomorphism i.e. show that $f(xy) = f(x)f(y)$ when $x, y \in \{r^i, r^i s\} = D_{2n}$. We'll do the most difficult case and leave the rest as an exercise. We can compute

$$f(r^k s) f(r^l s) = g^k h g^l h = g^k (hgh^{-1})^l = g^k g^{-l} = g^{k+_n (-l)} = f(r^k s r^l s)$$

as required.

For the last part suppose that $o(g) = n$ and $|G| = 2n$. If $r^i \in \ker f$ with $i \in \{0, 1, \ldots n-1\}$ then $g^i = e$ so as $o(g) = n$, $i = 0$. If $r^i s \in \ker f$ then $g^i h = e$ so $h = g^i$. Then $g^{-1} = hgh^{-1} = g^i g g^{-i} = g$ and so $g^2 = e$ contradicting $o(g) = n$. Thus $\ker f = \{e\}$. This means that f can't take the same value twice: if $f(x) = f(y)$ then $f(xy^{-1}) = e$ so $x = y$. That $\operatorname{Im} f = G$ now follows from the pigeonhole principle. □

Lecture 6

1.5. **The Möbius Group.** Informally, a Möbius transformation is a function $f\colon \mathbb{C} \to \mathbb{C}$ of the form

$$f\colon z \mapsto \frac{az + b}{cz + d}$$

with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. The reason for the condition $ad - bc \neq 0$ is that for such a function if $z, w \in \mathbb{C}$ then

$$f(z) - f(w) = \frac{az + b}{cz + d} - \frac{aw + b}{cw + d} = (ad - bc) \frac{(z - w)}{(cz + d)(cw + d)}$$

so f would be constant if $ad - bc$ were 0.[13]

Unfortunately the function is not well defined if $z = -d/c$ since we may not divide by zero in the complex numbers. This makes composition of Möbius transformations problematic since the image of one Möbius transformation may not coincide with the domain of defintion of another. We will fix this by adjoining an additional point to $\mathbb{C}$ called ∞.[14]

Notation. Let $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$ which we call the *extended complex plane.*

[13] This is bad because we're really interested in invertible functions

[14] And pronounced 'infinity'.

Definition. Given $(a,b,c,d) \in \mathbb{C}^4$ such that $ad-bc \neq 0$ we can define a function $f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ as follows:

if $c \neq 0$ then

$$f(z) := \begin{cases} \frac{az+b}{cz+d} & \text{if} \quad z \in \mathbb{C}\backslash\{-d/c\}; \\ \infty & \text{if} \quad z = -d/c; \\ a/c & \text{if} \quad z = \infty; \end{cases}$$

if $c = 0$ then

$$f(z) := \begin{cases} \frac{az+b}{cz+d} & \text{if} \quad z \in \mathbb{C} \\ \infty & \text{if} \quad z = \infty. \end{cases}$$

We call all functions from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$ that arise in this way *Möbius transformations* and let

$$\mathcal{M} := \{f\colon \mathbb{C}_\infty \to \mathbb{C}_\infty \mid f \text{ is a Möbius transformation}\}.$$

We'll see another way to interpret this definition later in the course involving projective geometry. But for now we'll work with it as it stands and also take for granted a result that will will prove later.[15]

Theorem. *The set $\mathcal{M}$ defines a subgroup of $S(\mathbb{C}_\infty)$.*

Lemma. *Suppose that $f \in \mathcal{M}$ such that $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Then $f = \mathrm{id}$.*

Proof. If $f = \frac{az+b}{cz+d}$ then $f(\infty) = \infty$ forces $c = 0$ and $f(0) = 0$ forces $b = 0$ and then $f(1) = 1$ forces $a/d = 1$ and so $f\colon z \mapsto \frac{az+0}{0z+a} = z$ for all $z \in \mathbb{C}$ as required. □

Theorem (Strict triple transitivity of Möbius transformations)**.** *If (z_1,z_2,z_3) and (w_1,w_2,w_3) are two sets of three distinct points then there is a unique $f \in \mathcal{M}$ such that $f(z_i) = w_i$ for $i = 1,2$ and 3.*

Proof. First we consider the case $w_1 = 0$, $w_2 = 1$ and $w_3 = \infty$.

If none of z_1, z_2, z_3 are ∞ then $f(z) = \frac{(z_2-z_3)(z-z_1)}{(z_2-z_1)(z-z_3)}$ does the job.

If some $z_i = \infty$ then choose $z_4 \in \mathbb{C}_\infty\backslash\{z_1,z_2,z_3\}$ and let $s(z) = \frac{1}{z-z_4}$ then $(s(z_1), s(z_2), s(z_3))$ does not contain ∞ so by the above we may find $g \in \mathcal{M}$ such that $g(s(z_1)) = 0$, $g(s(z_2)) = 1$ and $g(s(z_3)) = \infty$. Then $f = gs$ does the job.

In the general case we can find $g, h \in \mathcal{M}$ such that $g(z_1) = 0$, $g(z_2) = 1$, $g(z_3) = \infty$ and $h(w_1) = 0$, $h(w_2) = 1$ and $h(w_3) = \infty$. Then $f = h^{-1}g(z_i) = w_i$ for $i = 1,2,3$ and we've shown existence in general. Moreover if k is another such map then hkg^{-1} fixes $0, 1$ and ∞ so by the lemma is the identity. Thus $k = h^{-1}g = f$ and we've shown uniqueness. □

Definition. Given distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ the *cross-ratio* of z_1, z_2, z_3, z_4 written $[z_1,z_2,z_3,z_4] := f(z_4)$ where f is the unique Möbius transformation such that $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$.[16]

Lemma. *If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ then $[z_1,z_2,z_3,z_4] = \frac{(z_4-z_1)(z_2-z_3)}{(z_2-z_1)(z_4-z_3)}$.*[17]

[15] There is a straight-forward if slightly fiddly way to prove it directly that demands care with the point ∞. We will give a slightly more sophisticated but less fiddly proof.

[16] There are 6 essentially different definitions of cross-ratio depending on how we order 0, 1 and ∞ in this definition. It doesn't really matter which we choose as long as we are consistent.

[17] We could've defined the cross-ratio by this formula but we'd need to be more careful when some $z_i = \infty$.

Proof. We've already computed that $f(z) = \frac{(z_2-z_3)(z-z_1)}{(z_2-z_1)(z-z_3)}$ is the unique Möbius transformation such that $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$ and so

$$[z_1, z_2, z_3, z_4] = f(z_4)$$

is given by the required formula. □

Theorem (Invariance of Cross-Ratio)**.** *For all $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ and $g \in \mathcal{M}$, $[g(z_1), g(z_2), g(z_3), g(z_4)] = [z_1, z_2, z_3, z_4]$.*

Proof. Suppose that $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$. Then $(fg^{-1})(g(z_1)) = 0$, $(fg^{-1})(g(z_2)) = 1$ and $(fg^{-1})(g(z_3)) = \infty$. Thus

$$[g(z_1), g(z_2), g(z_3), g(z_4)] = fg^{-1}(g(z_4)) = f(z_4) = [z_1, z_2, z_3, z_4]$$

as required. □

Proposition. *Every element of $\mathcal{M}$ is a composite of Möbius transformations of the following forms.*

(a) $D_a \colon z \mapsto az = \frac{az+0}{0z+1}$ *with* $a \in \mathbb{C}\backslash\{0\}$ *(rotation/dilations);*
(b) $T_b \colon z \mapsto z + b = \frac{1z+b}{0z+1}$ *with* $b \in \mathbb{C}$ *(translations);*
(c) $S \colon z \mapsto 1/z = \frac{0z+1}{1z+0}$ *(inversion).*

Lecture 7

Proof. We'll give two proofs. First a 'nailing to the wall' type argument.

Suppose $f \in \mathcal{M}$. If $f(\infty) \in \mathbb{C}$ then $ST_{-f(\infty)}f(\infty) = S(0) = \infty$. So, by replacing f by $ST_{-f(\infty)}f$ if necessary, without loss of generality $f(\infty) = \infty$[18].

Now since $f(\infty) = \infty$, $f(0) \neq \infty$. Thus $T_{-f(0)}$ exists, $T_{-f(0)}f(0) = 0$ and $T_{-f(0)}(\infty) = \infty$ so we may assume f fixes both 0 and ∞.

Now since $f(\infty) = \infty$ and $f(0) = 0$, $f(1) \in \mathbb{C}\backslash\{0\}$. Thus $D_{1/f(1)}$ exists and $D_{1/f(1)}(1) = 1$, $D_{1/f(1)}(0) = 0$ and $D_{1/f(1)}(\infty) = \infty$. Thus $f = D_{f(1)}$ and we're done.

Alternatively if $c \neq 0$ then $\frac{az+b}{cz+d} = \frac{a}{c} + \frac{(bc-ad)}{c(cz+d)}$ so $f = T_{a/c}D_{(bc-ad)/c}ST_dD_c$. If $c = 0$ then $f = T_{b/d}D_{a/d}$. □

Definition. A *circle* in $\mathbb{C}_\infty$ is a subset that is either of the form $\{z \in \mathbb{C} \mid |z-a| = r\}$ for some $a \in \mathbb{C}$ and $r > 0$[19] or of the form $\{z \in \mathbb{C} \mid a\mathrm{Re}(z) + b\mathrm{Im}(z) = c\} \cup \{\infty\}$ for some $a, b, c \in \mathbb{R}$ with $(a, b) \neq (0, 0)$[20]

It follows that any three distinct points in $\mathbb{C}_\infty$ determine a unique circle in $\mathbb{C}_\infty$.

Lemma. *The general equation of a circle in $\mathbb{C}_\infty$ is*

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0$$

with $A, C \in \mathbb{R}$, $B \in \mathbb{C}$ and $AC < |B|^2$.[21]

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[18] Since if we can prove that $ST_{-f(\infty)}f$ is a product of the special maps then f can be obtained by postcomposing this product with $T_{f(\infty)}S$

[19] i.e. a usual circle in $\mathbb{C}$

[20] i.e. a line in $\mathbb{C}$ together with ∞

[21] where ∞ is understood to be a solution of this equation precisely if $A = 0$

Proof. First consider the case $A \neq 0$. Then

$$Az\bar{z} + B\bar{z} + \bar{B}z + C = 0 \iff |z + B/A|^2 = |B|^2/A^2 - C/A.$$

This latter defines a usual circle if and only if $|B|^2/A^2 > C/A$ i.e. $|B|^2 > AC$.

Next consider the case $A = 0$. Then

$$B\bar{z} + \bar{B}z + C = 0 \iff 2\mathrm{Re}(B)\mathrm{Re}(z) + 2\mathrm{Im}(B)\mathrm{Im}(z) = -C.$$

The latter defines a line in $\mathbb{C}$ if and only if $B \neq 0$ ie $|B|^2 > AC = 0$. □

Theorem (Preservation of circles)**.** *If $f \in \mathcal{M}$ and C is a circle in $\mathbb{C}_\infty$ then $f(C)$ is a circle in $\mathbb{C}_\infty$.*

Proof. If f is a rotation/dilation or translation the result is easy. Thus in light of the last proposition we may assume that $f\colon z \mapsto 1/z$.

But z is a solution to $Az\bar{z} + B\bar{z} + \bar{B}z + C$ if and only if $w = 1/z$ is a solution to $Cw\bar{w} + Bw + \bar{B}\bar{w} + A$. □

Corollary. *Four distinct points z_1, z_2, z_3 and z_4 in $\mathbb{C}_\infty$ lie on a circle if and only if $[z_1, z_2, z_3, z_4] \in \mathbb{R}$.*

Proof. Using triple transitivity we can find $f \in \mathcal{M}$ such that $f(z_1) = 0$, $f(z_2) = 1$ and $f(z_3) = \infty$. By preservation of circles z_1, z_2, z_3 and z_4 lie on a circle if and only if $f(z_1), f(z_2), f(z_3)$ and $f(z_4)$ lie on a circle i.e. if and only if $f(z_4)$ is real. But $[z_1, z_2, z_3, z_4] = f(z_4)$. □

Remark. It is possible to prove the corollary directly and then use a similar argument to deduce that Möbius transformations preserve of circles from it.

Definition. Given two elements x, y of a group G we say y is *conjugate* to x if there is some $g \in G$ such that $y = gxg^{-1}$.

Note that if $y = gxg^{-1}$ then $x = g^{-1}y(g^{-1})^{-1}$ so the notion of being conjugate is symmetric in x and y. Morever if also $z = hyh^{-1}$ then $z = (gh)x(gh)^{-1}$ so if z is conjugate to y and y is conjugate to x then z is conjugate to x.

Proposition. *Every Möbius transformation f except the identity has precisely one or two fixed points. If f has precisely one fixed point it is conjugate to the translation $z \mapsto z + 1$. If f has precisely two fixed points it is conjugate to a map of the form $z \mapsto az$ with $a \in \mathbb{C}\backslash\{0\}$.*

Proof. Let f be the map extending $z \mapsto \frac{az+b}{cz+d}$ and assume that $f \neq \mathrm{id}$.

Suppose first that $c = 0$. In this case ∞ is a fixed point. Moreover for $z \in \mathbb{C}$, f fixes z if and only if $dz = az + b$. The possible sets of solutions to this equation are $\{b/(d-a)\}$ (when $d \neq a$), $\mathbb{C}$ (when $d = a$ and $b = 0$), or $\emptyset$ (when $d = a$ and $b \neq 0$). Thus when $c = 0$ and f has either one or two fixed points.

Next suppose that $c \neq 0$ then f does not fix ∞. For $z \in \mathbb{C}$, $z = f(z)$ if and only if $cz^2 + dz - az - b = 0$. This is a quadratic equation and the coefficient of z^2 is non-zero so it has one or two solutions (two unless there is a repeated root). Thus again f has 1 or 2 fixed points in this case.

Now suppose that f has precisely one fixed point w and pick $x \in \mathbb{C}$ that is not fixed by f. Then $f(x) \notin \{x, w\}$.[22] Let g be the unique Möbius transformation such that $g(w) = \infty$, $g(x) = 0$ and $g(f(x)) = 1$. Now for $z \in \mathbb{C}_\infty$, $gfg^{-1}(z) = z$ if and

[22] If $f(x) = w$ then $x = f^{-1}(w) = w$

only if $fg^{-1}(z) = g^{-1}(z)$, i.e. if and only if $g^{-1}(z)$ is fixed by f. Since w is the only fixed point of f, $\infty = g(w)$ is the only fixed point of gfg^{-1}. Thus by the discussion above gfg^{-1} extends $z \mapsto \frac{az+b}{0z+a} = z + b/a$ for some $a, b \in \mathbb{C}\backslash\{0\}$. Since

$$gfg^{-1}(0) = gf(x) = 1$$

we see that $b/a = 1$ as required.

Suppose instead that f has two distinct fixed points z_1 and z_2. Let g be any Möbius transformation such that $g(z_1) = 0$ and $g(z_2) = \infty$. Then

$$gfg^{-1}(0) = gf(z_1) = g(z_1) = 0$$

and

$$gfg^{-1}(\infty) = gf(z_2) = g(z_2) = \infty.$$

Thus $gfg^{-1}(z) = az$ for some $a \in \mathbb{C}_\infty$. □

Remark. Suppose that $f \in \mathcal{M}$. If $gfg^{-1} \colon z \mapsto z + 1$ then for each $n \geqslant 0$,

$$gf^n g^{-1} = (gfg^{-1})^n \colon z \mapsto z + n$$

so $f^n(z) = g^{-1}(g(z)+n)$ for $z \in \mathbb{C}_\infty$ not fixed by f. Similarly if $gfg^{-1} \colon z \mapsto az$ then for $n \geqslant 0$, $f^n(z) = g^{-1}(a^n g(z))$ for z not fixed by f. Thus we can use conjugation to compute iterates of $f \in \mathcal{M}$ in a simple manner.

Lecture 8

2. Lagrange's Theorem

2.1. **Cosets.**

Definition. Suppose that $(G, \circ)$ is a group and H is a subgroup. A *left coset* of H in G is a set of the form $g \circ H := \{g \circ h \mid h \in H\}$ for some $g \in G$. Similarly a *right coset* of H in G is a set of the form $H \circ g := \{hg \mid h \in H\}$ for some $g \in G$. We write G/H to denote the set of left cosets of H in G and $H\backslash G$ to denote the set of right cosets of H in G[23].

As usual we will often suppress the $\circ$ and write gH or Hg.

Examples.

(1) Suppose $n \in \mathbb{Z}$, so that $n\mathbb{Z} := \{an \mid a \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$. Then $0 + n\mathbb{Z} = n\mathbb{Z} = n + n\mathbb{Z}$. $1 + n\mathbb{Z} = \{1 + an \mid a \in \mathbb{Z}\} = (1 - n) + n\mathbb{Z}$. More generally, $b + n\mathbb{Z}$ is the set of integers x such that $x - b$ is a multiple of n.[24]
(2) Suppose that $G = D_6 = \{e, r, r^2, s, rs, r^2s\}$ and $H = \{e, s\}$ then

$$\begin{aligned} eH &= \{e, s\} &&= sH \\ rH &= \{r, rs\} &&= rsH \\ r^2H &= \{r^2, r^2s\} &&= r^2sH. \end{aligned}$$

[23]This latter is a little unfortunate in the \ is normally used to denote set-theoretic difference but this should not cause confusion. Why?

[24]Note that because the operation on $\mathbb{Z}$ is addition we don't suppress it when we name cosets, i.e. we write $a + n\mathbb{Z}$ rather than $an\mathbb{Z}$ because the latter would create confusion.

However,

$$\begin{aligned} He &= \{e, s\} &&= Hs \\ Hr &= \{r, r^2 s\} &&= Hr^2 s \\ Hr^2 &= \{r^2, rs\} &&= Hrs. \end{aligned}$$

Thus left cosets and right cosets need not agree when the group is not abelian. However if $K = \{e, r, r^2\} \leqslant D_6$ then $K = eK = rK = r^2K = Ke = Kr = Kr^2$ and $\{s, rs, r^2 s\} = sK = rsK = r^2 sK = Ks = Krs = Kr^2 s$. So in this case the left and right cosets are the same.

(3) Suppose that $\mathcal{M}$ is the Möbius group and $H = \{f \in \mathcal{M} \mid f(0) = 0\}$. Then, for $g \in \mathcal{M}$,

$$\begin{aligned} gH &= \{f \in \mathcal{M} \mid f(0) = g(0)\} \text{ whereas} \\ Hg &= \{f \in \mathcal{M} \mid f^{-1}(0) = g^{-1}(0)\}. \end{aligned}$$

We'll return to this idea later in the course.

2.2. **Lagrange's Theorem.**

Theorem (Lagrange's Theorem)**.** *Suppose that G is a group and H is a subgroup of G then the left cosets of H in G partition G. In particular if G is finite then $|H|$ divides $|G|$.*

Proof. To show that the left cosets of H in G partition G we must show (i) that every element of G lives inside some left coset; and (ii) if two left cosets have a common element then they are equal.

For (i) notice that for all $g \in G$, $g \in gH$ since $e \in H$ and $ge = g$.

For (ii) suppose that $g \in g_1H \cap g_2H$. Then we may find $h_1, h_2 \in H$ such that $g = g_1h_1 = g_2h_2$. Then for all $h \in H$, $g_1h = gh_1^{-1}h = g_2h_2h_1^{-1}h$. Since $h_2h_1^{-1}h \in H$, we see that $g_1H \subset g_2H$. By symmetry we may conclude that $g_2H \subset g_1H$ and so $g_1H = g_2H$ as required.

Now for the last part we observe that when G is finite, H must also be finite since it is a subset of G. Moreover, $|G| = \sum_{gH \in G/H} |gH|$. Now for each left coset gH there is an invertible function $l_g \colon H \to gH$; $h \mapsto gh$ — the function l_g^{-1} is given by $gh \mapsto g^{-1}gh = h$. Thus $|gH| = |H|$ for every left coset gH and $|G| = |G/H| \cdot |H|$. □

Remark. By a very similar argument the right cosets of H in G also partition G.

Corollary. *If G is a finite group, then every element of G has order dividing $|G|$.*

Proof. Suppose $g \in G$ and let $f \colon \mathbb{Z} \to G$ be the homomorphism defined by $f(n) = g^n$. We claim that the order of g is precisely $|\operatorname{Im} f|$ which divides $|G|$ by Lagrange. To prove the claim we first observe that $\operatorname{Im} f = \{e, g, g^2, \ldots, g^{o(g)-1}\}$ since if $n \in \mathbb{Z}$ there are integers q, r with $0 \leqslant r < o(g)$ and $n = qo(g) + r$ and then $g^n = (g^{o(g)})^q g^r = g^r$. Moreover the elements $e, g, \ldots, g^{o(g)-1}$ are distinct since if $g^i = g^j$ with $0 \leqslant i < j < o(g)$ then $g^{j-i} = e$ contradicting the definition of $o(g)$. □

Proposition. *Suppose that p is prime. Then every group of order p is isomorphic to C_p.*

Proof. Let G be a group of order p and $g \in G \backslash \{e\}$. Since $|G| = p$ and $o(g)$ divides $|G|$ and is not 1 we must have $o(g) = p$. Thus g generates G by counting and an earlier lemma tells us that G is isomorphic to C_p. □

2.3. **Groups of order at most 8.** In this section we will classify all groups of order at most 8 under the perspective that two groups are the same precisely if they are isomorphic. We've already seen that every group of order $2, 3, 5$ or 7 is isomorphic to the cyclic group of the same order. It is evident that the trivial group is the only group of order 1 up to isomorphism.

Before we begin this we'll need the following construction that enables us to build new groups from old ones.

Example. Suppose that G and H are groups. We can define a binary operation on $G \times H$ via $(g_1, h_1)(g_2, h_2) = (g_1 g_2, h_1 h_2)$ for $g_1, g_2 \in G$ and $h_1, h_2 \in H$. We claim that this makes $G \times H$ into a group.

Proof of claim. Since

$$((g_1, h_1)(g_2, h_2))(g_3, h_3) = (g_1 g_2 g_3, h_1 h_2 h_3) = (g_1, h_1)((g_2, h_2)(g_3, h_3))$$

for all $g_1, g_2, g_3 \in G$ and $h_1, h_2, h_3 \in H$, the operation on $G \times H$ is associative.

Since $(e_G, e_H)(g, h) = (g, h) = (g, h)(e_G, e_H)$, (e_G, e_H) is an identity for the operation on $G \times H$.

Finally since $(g^{-1}, h^{-1})(g, h) = (e_G, e_H) = (g, h)(g^{-1}, h^{-1})$ the operation on $G \times H$ has inverses. □

Exercise. Show that if G_1, G_2 and G_3 are groups then $G_1 \times G_2$ is isomorphic to $G_2 \times G_1$ and $(G_1 \times G_2) \times G_3$ is isomorphic to $G_1 \times (G_2 \times G_3)$.

Lecture 9

Theorem (Direct Product Theorem)**.** *Suppose that $H_1, H_2 \leqslant G$ such that*

(i) $H_1 \cap H_2 = \{e\}$;
(ii) if $h_1 \in H_1$ and $h_2 \in H_2$ then h_1 and h_2 commute;
(iii) for all $g \in G$ there are $h_1 \in H_1$ and $h_2 \in H_2$ such that $g = h_1 h_2$.

Then there is an isomorphism $H_1 \times H_2 \to G$.

Proof. Let $f \colon H_1 \times H_2 \to G$ be given by $f(h_1, h_2) = h_1 h_2$. Now if $h_1, k_1 \in H_1$ and $h_2, k_2 \in H_2$ then

$$f((h_1, k_1)(h_2, k_2)) = f((h_1 h_2, k_1 k_2)) = h_1 h_2 k_1 k_2$$

and

$$f((h_1, k_1))f((h_2, k_2)) = h_1 k_1 h_2 k_2.$$

Since k_1 and h_2 commute (by (ii)), we see that f is a homomorphism. Now if $(h_1, h_2) \in \ker f$ then $h_1 h_2 = e$ so $h_1 = h_2^{-1} \in H_1 \cap H_2 = \{e\}$ (by (i)). Thus $\ker f = \{e\}$. Finally $\operatorname{Im} f = \{h_1 h_2 \mid h_1 \in H_1, h_2 \in H_2\} = G$ (by (iii)) and so f is the required isomorphism. □

We also need the following result that also appeared on the first example sheet.

Lemma. *If G is a group such that every non-identity element has order two*[25] *then G is abelian.*

[25] Of course in any group the identity has order 1

Proof. Suppose that $x, y \in G$. We must show that $xy = yx$.

Note that for all $g \in G$, $g^2 = e$. So by uniqueness of inverses every element of G is self-inverse, i.e. $g = g^{-1}$. In particular $(xy)^{-1} = xy$. By the shoes and socks lemma it follows that $y^{-1}x^{-1} = xy$. Since $x^{-1} = x$ and $y^{-1} = y$, $yx = xy$ as required. □

We also recall that every a group of order n with an element of order n is isomorphism to C_n and that every group of order $2n$ that has an element g of order n and an element h of order 2 such that $hg = g^{-1}h$ is isomorphic to D_{2n}.

Proposition. *Every group of order* 4 *is isomorphic to precisely one of* C_4 *and* $C_2 \times C_2$.

Proof. First we'll show that C_4 and $C_2 \times C_2$ are not isomorphic. Suppose $(g, h) \in C_2 \times C_2$. Then $(g, h)^2 = (e, e)$. So every element of $C_2 \times C_2$ has order 2 but C_4 has an element of order 4 thus they cannot be isomorphic.[26]

Now suppose that G is any group of order 4. If G contains an element of order 4 then it is isomorphic to C_4 so suppose not. By Lagrange every non-identity element of G has order 2. Let g, h be two distinct such elements. Let $H_1 = \{e, g\} \simeq C_2$ and $H_2 = \{e, h\} \simeq C_2$. It suffices to show that the three conditions of the direct product theorem hold. Condition (i) is immediate since $g \neq h$. Condition (ii) follows from the last lemma since it tells us that G must be abelian. Finally, since $g \notin H_2$, $gH_2 \neq H_2$ and G is partitioned by H_2 and gH_2 by Lagrange. Thus condition (iii) holds. □

Proposition. *Every group of order* 6 *is isomorphic to precisely one of* C_6 *or* D_6.

Proof. We can easily see that C_6 and D_6 are not isomorphic since C_6 is abelian and D_6 is not[27].

Now suppose that G is any group of order 6. If G contains an element of order 6 then it is isomorphic to C_6 so suppose not. By Lagrange every non-identity element of G has order 2 or 3. If there were no element of order 3 then any two non-identity elements would generate a subgroup of order 4 contradicting Lagrange. Thus G contains an element g of order 3. Let $K = \{e, g, g^2\}$ be the subgroup of G generated by g which is isomorphic to C_3. By Lagrange for any $h \notin K$, G is partitioned by K and hK i.e. $G = \{e, g, g^2, h, hg, hg^2\}$. Now consider h^2: $h^2 \notin hK$ else $h \in K$[28]. Thus $h^2 \in K$. If h^2 is g or g^2 then h has order 6 contradicting our assumption that G has no elements of order 6. Thus $h^2 = e$.

By the right coset version of Lagrange, G is also partitioned by K and Kh thus $Kh = hK$ and $\{h, gh, g^2h\} = \{h, hg, hg^2\}$. So $gh \in \{hg, hg^2\}$. If $gh = hg$ then $(gh)^2 = g^2$ and $(gh)^3 = h$. Thus gh does not have order 1, 2 or 3, a contradiction[29]. Thus $gh = hg^{-1}$ And we can deduce that $G \simeq D_6$. □

Example. The following set of matrices form an non-abelian group Q_8 of order 8

$$\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \pm \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}.$$

[26] If $f\colon \mathbb{Z}_4 \to C_2 \times C_2$ were an isomorphism then $f(2) = f(1)^2 = e$ giving f a non-trivial kernel.

[27] If $f\colon D_6 \to C_6$ were an isomorphism then for all $g_1, g_2 \in D_6$, $f(g_1g_2) = f(g_1)f(g_2) = f(g_2)f(g_1) = f(g_2g_1)$ so $g_2g_1 = g_1g_2$.

[28] if $h^2 = hk$ then $h = k$

[29] had we not made the assumption about no elements of order 6 then in this case we'd get that gk has order 6 so $G \simeq C_6$

It is common to write

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then $Q_8 = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. Then we can compute that $\mathbf{1}$ is an identity, $-\mathbf{1}$ has order 2 commutes with everything and multiplies as you'd expect given the notation. Morever $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ all have order 4 (all of them square to $-\mathbf{1}$), $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{ki} = \mathbf{j} = \mathbf{ik}$ and $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$.

Exercise. Verify that Q_8 is a group.

Lecture 10

Proposition. *Every group of order* 8 *is isomorphic to precisely one of* C_8, $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_8 *or* Q_8.

Proof. Let G be a group of order 8.

If G has an element of order 8 then $G \simeq C_8$.

If every non-identity element of G has order 2 then G is abelian. Moreover any two non-identity elements g_1, g_2 generate a subgroup H_1 of G of order 4 necessarily isomorphic to $C_2 \times C_2$.[30] If $g_3 \in G \backslash H_1$ then $H_2 = \{e, g_3\}$ is a subgroup of order 2. One can easily verify that $G \simeq H_1 \times H_2 \simeq C_2 \times C_2 \times C_2$ in a similar fashion to the argument above for $C_2 \times C_2$.[31]

So we're reduced to classifying groups G of order 8 with no element of order 8 and at least one element g of order 4. In this setting the set $K = \{e, g, g^2, g^3\}$ is a subgroup of G isomorphic to C_4.

Let $h \in G \backslash K$. Then $G = K \cup hK$ by Lagrange so $G = \{e, g, g^2, g^3, h, hg, hg^2, hg^3\}$. Moreover $h^2 \in K$ else $h^2 \in hK$ whence $h \in K$.

If $h^2 = g$ or $h^2 = g^{-1}$ then h has order 8 contradicting our assumption that G has no such elements. So there are two cases remaining: case A where $h^2 = e$ and case B where $h^2 = g^2$.

Suppose that we're in case A and $h^2 = e$. Then consider the product gh. By Lagrange again, $gh \in hK = \{h, hg, hg^2, hg^3\}$. If $gh = hg^i$ then $h^{-1}gh = g^i$ and

$$g = h^{-2}gh^2 = h^{-1}(h^{-1}gh)h = h^{-1}g^i h = (h^{-1}gh)^i = (g^i)^i = g^{i^2}.$$[32]

Thus $i = 1$ or 3. If $i = 1$ then G is abelian and we can use the direct product theorem to show that $G \simeq K \times \{e, h\} \simeq C_4 \times C_2$. If instead $i = 3$ then G has an element g of order 4 and an element h of order 2 such that $gh = hg^{-1}$ so as G has order 8 we know that $G \simeq D_8$.

Finally suppose we're in case B and $h^2 = g^2$. Again consider the product $gh \in \{h, gh, g^2h, g^3h\} = Kh = hK = \{h, hg, hg^2, hg^3\}$. Since $g^2h = h^3 = hg^2$, $gh = hg$ or $gh = hg^3$. If $gh = hg$ then

$$(gh)^2 = g^2h^2 = g^4 = e$$

so $\{e, gh\} \leqslant G$ and $G \simeq K \times \{e, gh\} \simeq C_4 \times C_2$ by the direct product theorem.

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[30] the elements are e, g_1, g_2 and g_1g_2 it is easy to check that these are distinct and form a subgroup.

[31] See also Example Sheet 1 Q14

[32] since $h^2 = e$

If $gh = hg^3$ then do some renaming. Write $\mathbf{1} := e$, $-\mathbf{1} := h^2 = g^2$, $\mathbf{i} := g$, $\mathbf{j} := h$, $\mathbf{k} := gh$. Then

$$G = \{\pm\mathbf{1}, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$$

and we can verify that multiplication is as for Q_8.

Exercise. Complete the proof by showing that no two of the listed groups are isomorphic. □

2.4. **The Quaternions.**

Definition. The *quaternions* are the set of matrices

$$\mathbb{H} := \{a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\} \subset \mathrm{Mat}_2(\mathbf{C})$$

where as before

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The sum or product of two elements of $\mathbb{H}$ lives in $\mathbb{H}$ and $+$ and $\cdot$ obey the same associativity and distributivity laws as $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ with identities 0 and $\mathbf{1}$ respectively. Although the multiplication in $\mathbb{H}$ is not commutative (since $\mathbf{ij} = -\mathbf{ji}$), $(\mathbf{H}, +)$ is an abelian group and $(\mathbf{H}\backslash\{0\}, \cdot)$ is a (non-abelian) group.[33] To see the latter we can define 'quaternionic conjugation' by

$$(a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})^* = a\mathbf{1} - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}$$

and then verifying that if $x = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ then

$$xx^* = x^*x = (a^2 + b^2 + c^2 + d^2)\mathbf{1}$$

so $x^{-1} = \frac{1}{a^2+b^2+c^2+d^2}x^*$ for $x \neq 0$.

2.5. **Fermat–Euler theorem.** We can define a multiplication operation on the set $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ by setting $a \cdot_n b$ to be the remainder after dividing ab by n.

Definition. Let $U_n := \{a \in \mathbb{Z}_n \mid \exists b \in \mathbb{Z}_n \text{ s.t. } a \cdot_n b = 1\}$ be the set of invertible elements of $\mathbb{Z}_n$ with respect to $\cdot_n$.

It is a result from Numbers and Sets that follows from Euclid's algorithm that $|U_n| = \varphi(n)$ where $\varphi(n)$ denotes the number of elements of $\mathbb{Z}_n$ coprime to n. Indeed $U_n = \{a \in \mathbb{Z}_n \mid (a, n) = 1\}$.

Lemma. *$(U_n, \cdot_n)$ is an abelian group.*

Proof. $\cdot_n$ defines an associative and commutative operation on $\mathbb{Z}_n$ with an identity 1.[34] If $a_1, a_2 \in U_n$ then there are $b_1, b_2 \in U_n$ such that $a_i \cdot_n b_i = 1$. Then

$$(a_1 \cdot_n a_2) \cdot_n (b_1 \cdot_n b_2) = (a_1 \cdot_n b_1) \cdot_n (a_2 \cdot_n b_2) = 1 \cdot_n 1 = 1$$

so $\cdot_n$ restricts to an associative binary operation on U_n. $1 \in U_n$ is an identity and if $a \in U_n$ then there is $b \in \mathbb{Z}_n$ with $a \cdot_n b = b \cdot_n a = 1$. Then $b \in U_n$ and $b = a^{-1}$. So U_n has inverses. □

Theorem (Fermat–Euler Theorem)**.** *If $(a, n) = 1$ then $a^{\varphi(n)} \equiv 1 \mod n$.*

[33] We say that $\mathbb{H}$ is a division algebra. $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ are the only division algebras that are finite dimensional as vector spaces over $\mathbb{R}$.

[34] for $a, b, c \in \mathbb{Z}_n$, $a \cdot_n (b \cdot_n c) \equiv abc \equiv (a \cdot_n b) \cdot_n c \mod n$

Proof. Since $(U_n, \cdot_n)$ is a group of order $\varphi(n)$ it is consequence of Lagrange's theorem that every element of $(U_n, \cdot_n)$ has order dividing $\varphi(n)$. The result follows immediately. □

Lecture 11

3. Group Actions

We started the course by saying that groups are fundamentally about symmetry but the connection has been opaque for the last three lectures. In this chapter we will discuss how to recover the notion of symmetry from the group axioms.

3.1. **Definitions and examples.**

Definition. An *action* of a group G on a set X is a function

$$\cdot\colon G \times X \to X; (g, x) \mapsto g \cdot x$$

such that for all $x \in X$

(i) $e \cdot x = x$;
(ii) $g \cdot (h \cdot x) = (gh) \cdot x$ for all $g, h \in G$.

Examples.

(1) Isom($\mathbb{Z}$) acts on $\mathbb{Z}$ via $f \cdot n = f(n)$.
(2) The Möbius group $\mathcal{M}$ acts on the extended complex plane $\mathbb{C}_\infty$ via $f \cdot z = f(z)$.
(3) Generalising both the examples above, if $H \leqslant S(X)$ then H acts on X via $h \cdot x = h(x)$. We call this the *natural action* of H on X.
(4) $\mathcal{M}$ also acts on the set of circles in $\mathbb{C}_\infty$. We proved in §1.5 that if $f \in \mathcal{M}$ and $C \subset \mathbb{C}_\infty$ is a circle then $f(C) \subset \mathbb{C}_\infty$ is also a circle so $(f, C) \mapsto f(C)$ is a function. Moreover for all circles C the conditions $\mathrm{id}(C) = C$ and $f(g(C)) = (fg)(C)$ for $f, g \in \mathcal{M}$ are both clear.
(5) D_{2n} acts on the set of points a regular n-gon. D_{2n} also acts on the set of vertices of a regular n-gon and on the set of edges of a regular n-gon.
(6) If X is a regular solid then Sym(X) acts on the set of points (and on the sets of vertices/edges/faces) of X.
(7) If $H \leqslant G$ then G acts on G/H, the set of left cosets of G in H via $g \cdot kH = gkH$ for $g, k \in G$. To see this we need to check that if $kH = k'H$ then $gkH = gk'H$. But if $kH = k'H$ then $k' = kh$ for some $h \in H$ so $gk' = gkh \in gk'H \cap gkH$ and $gkH = gk'H$ by Lagrange. Given this we see that for all $k \in G$, $ekH = kH$ and $g_1(g_2kH) = (g_1g_2)kH$ for all $g_1, g_2 \in G$.
(8) For any group G and set X we can define the *trivial action* via $g \cdot x = x$ for all $g \in G$ and $x \in X$.

Theorem. *For every group G and set X there is a $1-1$ correspondence*

$$\{\textit{actions of } G \textit{ on } X\} \longleftrightarrow \{\theta\colon G \to S(X) \mid \theta \textit{ is a homomorphism}\}$$

such that an action $\cdot\colon G \times X \to X$ corresponds to the homomorphism $\theta\colon G \to S(X)$ given by $\theta(g)(x) = g \cdot x$.

Proof. First we must show that if $\cdot\colon G \times X \to X$ is a action then the formula $\theta(g)(x) = g \cdot x$ does define a homomorphism $G \to S(X)$.

For each $g \in G$, define $\theta(g)\colon X \to X$ via $\theta(g)(x) = g \cdot x$. So that

$$\theta\colon G \to \{f\colon X \to X\}.$$

Then for $g, h \in G$ we can compute

$$\theta(gh)(x) = (gh) \cdot x = g \cdot (h \cdot x) = \theta(g) \circ \theta(h)(x)$$

ie $\theta(gh) = \theta(g)\theta(h)$. So to show that θ defines a homomorphism $G \to S(X)$ it suffices to show that $\theta(g) \in S(X)$ for each $g \in G$; i.e. that each $\theta(g)$ is invertible. But

$$\theta(g)\theta(g^{-1}) = \theta(e) = \theta(g^{-1})\theta(g)$$

so to show this it suffices to show that $\theta(e)$ is id_X. But $\theta(e)(x) = e \cdot x = x = \mathrm{id}_X(x)$. So we're done.

To finish we must show that every homomorphism $\theta\colon G \to S(X)$ arises like this in precisely one way. So suppose that θ is any such homomorphism, θ corresponds to an action $\cdot\colon G \times X \to X$ precisely if $g \cdot x = \theta(g)(x)$ defines an action. So it remains to check that $e \cdot x = x$ and $g \cdot (h \cdot x) = (gh) \cdot x$ for all $x \in X$ and $g, h \in G$. But

$$e \cdot x = \theta(e)(x) = \mathrm{id}_X(x) = x$$[35]

and

$$g \cdot (h \cdot x) = \theta(g)(\theta(h)(x)) = \theta(gh)(x) = (gh) \cdot x$$

as required. □

Definition. We say that an action of G on X is *faithful* if the kernel of the corresponding homomorphism $G \to S(X)$ is the trivial group.

3.2. **Orbits and Stabilisers.**

Definition. Suppose a group G acts on a set X and that $x \in X$. The *orbit* of x under the action is given by

$$\mathrm{Orb}_G(x) := \{g \cdot x \mid g \in G\} \subset X.$$

The *stabiliser* of x under the action is given by

$$\mathrm{Stab}_G(x) := \{g \in G \mid g \cdot x = x\} \subset G.$$

Thus an action is faithful precisely if $\bigcap_{x \in X} \mathrm{Stab}_G(x) = \{e\}$.

Examples.

(1) Under the natural action of $\mathrm{Isom}(\mathbb{Z})$ on $\mathbb{Z}$, for all $n \in \mathbb{Z}$

$$\mathrm{Orb}_{\mathrm{Isom}(\mathbb{Z})}(n) = \mathbb{Z}$$

and

$$\mathrm{Stab}_{\mathrm{Isom}(\mathbb{Z})} = \{\mathrm{id}, m \mapsto 2n - m\}$$

(2) Under the natural action of $\mathcal{M}$ on $\mathbb{C}_\infty$, for all $z \in \mathbb{C}_\infty$

$$\mathrm{Orb}_{\mathcal{M}}(z) = \mathbb{C}_\infty$$

and

$$\mathrm{Stab}_{\mathcal{M}}(\infty) = \left\{ z \mapsto \frac{az + b}{0c + d} \mid ad \neq 0 \right\}.$$

(3) Under the action of D_{2n} on the set of points of a regular n-gon the orbit of a vertex of the n-gon is the set of all vertices of the n-gon and the stabliser of a vertex consists of the identity and reflection in the line through the centre of the n-gon and the vertex.[36]

[35] Since θ is a homomorphism it must send the identity of G to the identity of $S(X)$.

[36] It would be instructive to think about what the orbits and stabilisers of other points of n-gon are under the action of D_{2n}.

(4) For the left coset action of G on G/H defined earlier

$$\mathrm{Orb}_G(eH) = G/H$$

and

$$\mathrm{Stab}_G(eH) = \{g \in G \mid gH = eH\} = H.$$

More generally

$$\mathrm{Stab}_G(kH) = \{g \in G \mid gkH = kH\} = \{g \in G \mid k^{-1}gkH = H\} = kHk^{-1}.$$

(5) For the trivial action of G on X and any $x \in X$,

$$\mathrm{Orb}_G(x) = \{x\} \text{ and } \mathrm{Stab}_G(x) = G.$$

Lemma. *Suppose that G is a group acting on a set X.*

(i) Each stabiliser $\mathrm{Stab}_G(x)$ is a subgroup of G.

(ii) The orbits $\mathrm{Orb}_G(x)$ partition X. In particular if X is finite and the distinct orbits are $\mathcal{O}_1, \ldots, \mathcal{O}_m$ then

$$|X| = \sum_{i=1}^{m} |\mathcal{O}_i|$$

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Proof. (i) Let $x \in X$. Since $e \cdot x = x$, $e \in \mathrm{Stab}_G(x)$. Suppose that $g, h \in \mathrm{Stab}_G(x)$. Then

$$(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$$

so $gh \in \mathrm{Stab}_G(x)$ and

$$x = e \cdot x = (g^{-1}g) \cdot x = g^{-1} \cdot (g \cdot x) = g^{-1} \cdot x$$

so $g^{-1} \in \mathrm{Stab}_G(x)$. Thus $\mathrm{Stab}_G(x) \leqslant G$ as required.[37]

(ii) Since for any $x \in X$, $x = e \cdot x$ so $x \in \mathrm{Orb}_G(x)$, it suffices to prove that for any $x, y \in X$ either $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$ or $\mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y) = \emptyset$. Suppose that $z \in \mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y)$. Then there are some elements $g, h \in G$ such that $g \cdot x = z = h \cdot y$. Thus $x = (g^{-1}h) \cdot y$. Now if $w \in \mathrm{Orb}_G(x)$ then there is some $f \in G$ such that $w = f \cdot x$. Whence we can compute $w = (fg^{-1}h) \cdot y$ and so $\mathrm{Orb}_G(x) \subset \mathrm{Orb}_G(y)$. By symmetry $\mathrm{Orb}_G(y) \subset \mathrm{Orb}_G(x)$. Now we can see that $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(y)$ as required.[38] □

Definition. We say that an action of G on X is *transitive* if there is only one orbit i.e. if $X = \mathrm{Orb}_G(x)$ for any $x \in X$.

Theorem (Orbit-Stabiliser Theorem)**.** *Suppose a group G acts on a set X and $x \in X$. There is a (natural) invertible function*

$$G/\mathrm{Stab}_G(x) \to \mathrm{Orb}_G(x).$$

In particular if G is finite

$$|G| = |\mathrm{Orb}_G(x)| \cdot |\mathrm{Stab}_G(x)|.$$

[37] We used the same argument for a special case of this in Lecture 4 when we showed that $\mathrm{Stab}_{\mathrm{Isom}(\mathbb{Z})}(0) \leqslant \mathrm{Isom}(\mathbb{Z})$.

[38] You should spend some time thinking about the relationship between this proof and the proof of Lagrange's Theorem. Can you find an action of H on G making Lagrange a special case of this result?

Proof. For $y \in \mathrm{Orb}_G(x)$, $\{g \in G \mid g \cdot x = y\}$ is a left coset of $\mathrm{Stab}_G(x)$ in G since it is non-empty and if $g \cdot x = y$ then for $h \in G$

$$\begin{aligned} h \cdot x = y &\iff g^{-1} \cdot (h \cdot x) = x \\ &\iff g^{-1}h \in \mathrm{Stab}_G(x) \\ &\iff h \in g\,\mathrm{Stab}_G(x). \end{aligned}$$

Thus there is a function $G/\mathrm{Stab}_G(x) \to \mathrm{Orb}_G(x)$ given by $h\,\mathrm{Stab}_G(x) \mapsto h \cdot x$ with inverse $\mathrm{Orb}_G(x) \to G/\mathrm{Stab}_G(x)$ given by $y \mapsto \{g \in G \mid g \cdot x = y\}$.

Now if G is finite then $|\mathrm{Orb}_G(x)| = |G/\mathrm{Stab}_G(x)| = |G|/|\mathrm{Stab}_G(x)|$ by Lagrange.[39] □

Examples.

(1) For the natural action of $\mathrm{Isom}(\mathbb{Z})$ on $\mathbb{Z}$ the set of left cosets of $\mathrm{Stab}_{\mathrm{Isom}(\mathbb{Z})}(0) = \{e, n \mapsto -n\}$ in $\mathrm{Isom}(\mathbb{Z})$ is in bijection with $\mathbb{Z}$. We secretly used this fact when we computed $\mathrm{Isom}(\mathbb{Z})$ in the first lecture.

(2) For the usual action of D_{2n} on the vertices of the n-gon and v such a vertex we see that $|D_{2n}| = |\mathrm{Stab}_{D_{2n}}(v)||\mathrm{Orb}_{D_{2n}}(v)| = 2n$. Again we secretly used this when we computed $|D_{2n}| = 2n$.

(3) The symmetric group S_n acts on $X = \{1, 2, \dots, n\}$ via the natural action $f \cdot x = f(x)$. Then $\mathrm{Orb}_{S_n}(n) = X$ since for each $i \in X$ the function $f_i \colon X \to X$; $f_i(i) = n$, $f_i(n) = i$, $f_i(x) = x$ for $x \notin \{i, n\}$ is an element of S_n. Thus $|S_n| = n\,\mathrm{Stab}_{S_n}(n)$. But $\mathrm{Stab}_{S_n}(n)$ is isomorphic to S_{n-1} by restricting $f \in S_n$ that fixes n to a permutation of $\{1, \dots, n-1\}$. Thus $|S_n| = n|S_{n-1}|$. Since $|S_1| = 1$[40] we deduce that $|S_n| = n!$.

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Fact. If $f \colon \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry that fixes 4 non-coplanar points then f is the identity.

(4) Let X be a regular tetrahedron. Then $\mathrm{Sym}(X)$ acts transitively on the set of 4 vertices of X and the stabiliser of a vertex $v \in X$ consists of three rotations and three reflections. Thus $|\mathrm{Sym}(X)| = 6 \cdot 4 = 24$.

This calculation enables us to prove that $\mathrm{Sym}(X) \simeq S_4$: if we label the vertices by the numbers $1, 2, 3, 4$ then the action of $\mathrm{Sym}(X)$ on the vertices defines a homomorphism $\theta \colon \mathrm{Sym}(X) \to S_4$. Since any isometry of $\mathbb{R}^3$ fixing all four vertices is the identity we can conclude that $\ker\theta = \{\mathrm{id}\}$. By counting we can deduce $\mathrm{Im}\,\theta = S_4$.

(5) Let X be a cube. Then $\mathrm{Sym}(X)$ acts transitively on the set of 6 faces of X and the stabiliser $H := \mathrm{Stab}_{\mathrm{Sym}(X)}(F)$ of a face F acts transitively on the set of 4 vertices contained in it[41]. If v is one of these vertices and w is the diagonally opposite vertex in F then

$$\mathrm{Stab}_H(v) = \{e, \text{reflection in plane containing } v, w \text{ and the centre of } X\}$$[42].

[39] You might like to think about whether you can do this last part without recourse to Lagrange.

[40] Or if you prefer $|S_0| = 1$

[41] This can be seen by considering rotations about an axis through the centre of F and the centre of its opposite face.

[42] Since if an isometry of $\mathbb{R}^3$ fixes all vertices of F and the centre of X then it is the identity.

Thus

$$\mathrm{Sym}(X) = 6|H| = 6 \cdot |\operatorname{Orb}_H(v)||\operatorname{Stab}_H(v)| = 6 \cdot 4 \cdot 2 = 48.$$

3.3. **Conjugacy classes.**

Definition. If G is a group then the *conjugation action* of G on itself is given by $\cdot\colon G \times G \to G$; $g \cdot x = gxg^{-1}$.

Note that the conjugation action is indeed an action since for $g, h, x \in G$,

$$e \cdot x = exe^{-1} = x$$

and

$$g \cdot (h \cdot x) = g(hxh^{-1})g^{-1} = (gh)x(gh)^{-1} = (gh) \cdot x.$$

Definition. The orbits of G on itself under the conjugation action are called the *conjugacy classes* of G: the orbit of $x \in G$ will be denoted $\mathrm{ccl}(x)$; i.e.

$$\mathrm{ccl}(x) = \{gxg^{-1} \mid g \in G\}.^{43}$$

The stabliser of $x \in G$ under this action is called the *centraliser* of x and will be denoted $C_G(x)$.

Examples.

(1) Suppose $G = \mathrm{Isom}(\mathbb{Z}) = \{t_a\colon n \mapsto a + n, s_a\colon n \mapsto a - n \mid a \in \mathbb{Z}\}$. Let $H := \{t_a \mid a \in \mathbb{Z}\}$ denote the subgroup of translations We know that for any $a, b \in \mathbb{Z}$,

$$t_b t_a t_b^{-1} = t_a$$

and for $n \in \mathbb{Z}$,

$$s_b t_a s_b^{-1}(n) = s_b t_a(b - n) = s_b(a + b - n) = n - a$$

ie

$$s_b t_a s_b^{-1} = t_{-a}$$

so for $a \neq 0$

$$C_G(t_a) = H \text{ and } \mathrm{ccl}(t_a) = \{t_a, t_{-a}\}^{44}.$$

Similarly for $n \in \mathbb{Z}$

$$t_b s_a t_b^{-1}(n) = t_b s_a(n - b) = t_b(a - (n - b)) = (2b + a - n) = s_{2b+a}(n)$$

and

$$s_b s_a s_b^{-1}(n) = s_b s_a(b - n) = s_b(a - (b - n)) = b - (a + n - b) = 2b - a - n = s_{2b-a}(n)$$

so as $a \equiv -a \mod 2$

$$C_G(s_a) = \{t_0, s_a\} \text{ and } \mathrm{ccl}(s_a) = \{s_{a+2b} \mid b \in \mathbb{Z}\}.$$

That is there are two conjugacy classes of reflections $\mathrm{ccl}(s_0)$ and $\mathrm{ccl}(s_1)$.[45]

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[43] Since conjugacy classes are orbits of an action they partition G; that is every element of G lies in precisely one conjugacy class.

[44] Of course $C_G(t_0) = G$ and $\mathrm{ccl}(t_0) = \{t_0\}$.

[45] What is the geometric meaning of this?

(2) We saw in section 1.5 that in the Möbius group $\mathcal{M}$ the conjugacy class of $z \mapsto z+1$ consists of all Möbius transformations with precisely one fixed point i.e.

$$\mathrm{ccl}(z \mapsto z+1) = \{f \in \mathcal{M} \mid f \text{ has precisely one fixed point}\}$$

and that every Möbius transformation with precisely two fixed points is in the same conjugacy class as a Möbius transformation of the form $z \mapsto az$. We will return later to the question of when $\mathrm{ccl}(z \mapsto az) = \mathrm{ccl}(z \mapsto bz)$ and what the centralisers of these elements are.[46] Of course $\mathrm{ccl}(\mathrm{id}) = \{\mathrm{id}\}$ and $C_{\mathcal{M}}(\mathrm{id}) = \mathcal{M}$.

Definition. The kernel of the homomorphism $G \to S(G)$ given by the conjugation action of G on itself is called the *centre* of G and written $Z(G)$.

Lemma. *Suppose that G is a group.*

(a) For $x \in G$, $C_G(x) = \{g \in G \mid xg = gx\}$.
(b) $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\} = \bigcap_{x \in G} C_G(x)$.
(c) $Z(G) = \{g \in G \mid |\mathrm{ccl}(g)| = 1\}$.

Proof. (a) For $g, x \in G$, $g \in C_G(x)$ if and only if $gxg^{-1} = x$ that is if and only if $gx = xg$.

(b) For $g \in G$, $g \in Z(G)$ if and only if $gxg^{-1} = x$ for all $x \in G$ i.e. if and only if $gx = xg$ for all $x \in G$. The other equality follows immediately from (a).

(c) This follows easily from (b): $g \in Z(G)$ if and only if $gx = xg$ for all $x \in G$ i.e. if and only if $xgx^{-1} = g$ for all $x \in G$. □

3.4. **Cayley's Theorem.** Cayley's Theorem will tell us that every group is isomorphic to a subgroup of a symmetric group.

Definition. If G is a group then the *left regular action* of G on itself is given by the function $\cdot : G \times G \to G$; $g \cdot x = gx$.

Example. The left regular action of $\mathbb{Z}$ on itself is by translations. i.e. the corresponding homomorphism $\mathbb{Z} \to S(\mathbb{Z})$ is given by $n \mapsto t_n$.[47]

Lemma. *The left regular action of G on G is an action that is both transitive and faithful.*

Proof. First we should check that the left regular action is indeed an action: for all $g, h, x \in G$, $e \cdot x = ex = x$ and $g \cdot (h \cdot x) = ghx = (gh) \cdot x$.

Next we observe that $\mathrm{Orb}_G(e) = G$ and $\mathrm{Stab}_G(e) = \{e\}$ since for all $g \in G$, $g \cdot e = g$. Thus the left regular action is transitive and faithful. □

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Theorem (Cayley's Theorem)**.** *If G is a group then G is isomorphic to a subgroup of $S(G)$.*

[46] Spoiler: $\mathrm{ccl}(z \mapsto az) = \mathrm{ccl}(z \mapsto bz)$ if and only if $b \in \{a, 1/a\}$, $C_G(z \mapsto z+1) = \{\text{translations in } \mathcal{M}\}$ and, for $a \neq 1$, $C_G(z \mapsto az) = \{\text{dilations/rotations in } \mathcal{M}\}$. Can you prove these facts now? Hint: if $g(z \mapsto az)g^{-1} = z \mapsto bz$ for $g \in \mathcal{M}$ what can you say about $g(0)$ and $g(\infty)$?

[47] recall t_n denotes translation by n

Proof. The left-regular action defines a homomorphism $\theta\colon G \to S(G)$. Since $\operatorname{Im}\theta$ is a subgroup of G we may view this as a homomorphism $G \to \operatorname{Im}\theta$ whose image is still $\operatorname{Im}\theta$. Since the action is faithful, $\ker\theta = \{e\}$ and we see that G is isomorphic to $\operatorname{Im}\theta$. □

It perhaps should be said that this theorem is simultaneously deep and almost useless. Deep because it tells us that anything satisfying our abstract definition of a group can be viewed as symmetries of something. Almost useless because knowing this doesn't really help prove things about groups.

3.5. **Cauchy's Theorem.**

Theorem (Cauchy's Theorem)**.** *Supppose that p is a prime and G is a finite group whose order is a multiple of p. Then G contains an element of order p.*

Proof. Consider the action of $\mathbb{Z}_p$ on G^p via

$$i \cdot (g_0, g_1, \dots, g_{p-1}) = (g_i, g_{1+_p i}, \dots, g_{(p-1)+_p i})$$

i.e. by cyclic permutation. This is an action since $0 \cdot x = x$ for all $x \in G^p$ and

$$\begin{aligned}(i +_p j) \cdot (g_0, g_1, \dots, g_{p-1}) &= (g_{i+_p j}, g_{1+_p i+_p j}, \dots, g_{(p-1)+_p i+_p j}) \\ &= i \cdot (j \cdot (g_0, g_1, \dots, g_{p-1}))\end{aligned}$$

for all $i, j \in \mathbb{Z}_p$ and $g_0, \dots, g_{p-1} \in G$.

Let $X = \{(g_0, \dots, g_{p-1}) \in G^p \mid g_0 g_1 \cdots g_{p-1} = e\} \subset G^p$. Then $|X| = |G|^{p-1}$ since however we choose $g_0, \dots, g_{p-2} \in G$ there is precisely one choice of $g_{p-1} \in G$ such $g_0 g_1 \cdots g_{p-1} = e$[48]. Moreover the 'constant tuple' $(g, g, \dots, g)$ is in X if and only $g^p = e$ — and if $g \neq e$ this is equivalent to $o(g) = p$.

Now the cyclic permutation action of $\mathbb{Z}_p$ on G^p restricts to an action on X since if $g_0 g_1 \cdots g_{p-1} = e$ and $i \in \mathbb{Z}_p$ then

$$g_i g_{1+_p i} \cdots g_{(p-1)+_p i} = (g_0 \cdots g_{i-1})^{-1}(g_0 \cdots g_{p-1})(g_0 \cdots g_{i-1}) = e.$$

Since $\mathbb{Z}_p$ has prime order the orbit-stabiliser theorem tells us that every orbit $\operatorname{Orb}_{\mathbb{Z}_p}(x)$ has order 1 or p for $x \in X$. Since the orbits partition X and p is a factor of $|X|$ it follows that the number of orbits of size 1 in X is a multiple of p.

Since the constant tuple $(e, e, \dots, e)$ is an orbit in X of size 1 there must be at least $p - 1$ other such orbits. If $(g_0, \dots, g_{p-1})$ is one such orbit then, since $i \cdot (g_0, \dots, g_{p-1}) = (g_0, \dots, g_{p-1})$ for all $i \in \mathbb{Z}_p$, we can conclude that $g_i = g_0$ for all $i \in \mathbb{Z}_p$ and so $g_0^p = e$. □

4. Quotient Groups

4.1. **Normal subgroups.**

Suppose that G is a group. Let $\mathscr{P}(G)$ denote the set of subsets of G, i.e. the power set of G. There is a natural binary operation on $\mathscr{P}(G)$ given by

$$AB := \{ab \mid a \in A, b \in B\}.$$

Examples.

(1) If $A \in \mathscr{P}(G)$ then $A\emptyset = \emptyset = \emptyset A$. If A is non-empty then $AG = G = GA$.
(2) If $H \leqslant G$ then the binary operation on $\mathscr{P}(G)$ restricts to a binary operation on $\mathscr{P}(H)$.
(3) If $H \leqslant G$ then the sets $\{g\}H$ are precisely the left cosets gH of H in G.

[48] That is $g_{p-1} = (g_0 g_1 \cdots g_{p-2})^{-1}$.

Lemma. *This operation on $\mathscr{P}(G)$ is associative and has an identity but does not have inverses.*

Proof. Suppose that $A, B, C \in \mathscr{P}(G)$ then

$$\begin{aligned}(AB)C &= \{(ab)c \mid a \in A, b \in B, c \in C\}\\ &= \{a(bc) \mid a \in A, b \in B, c \in C\}\\ &= A(BC)\end{aligned}$$

If A is any subset of G then $\{e\}A = A$ so $\{e\}$ is an identity. Also always $A\emptyset = \emptyset$, so $\emptyset$ has no inverse. [49] □

We'll be particularly interested in the product of two cosets under this operation — in particular if $H \leqslant G$ we'd like to use it to put a group structure on the set of left cosets G/H of H in G. If G is abelian then this is straightforward:

$$g_1Hg_2H = \{g_1h_1g_2h_2 \mid h_1, h_2 \in H\} = \{g_1g_2h_1h_2 \mid h_1, h_2 \in H\} = g_1g_2H$$

and one can easily[50] show that this does define a group structure on G/H. However in general things are not so straightforward.

Example. Consider $G = D_6 = \{e, r, r^2, s, rs, r^2s\}$ where r denotes a non-trivial rotation in the group and s a reflection.

If H is the subgroup of rotations $\{e, r, r^2\}$ then the cosets of H in G are H and sH. We can compute

$$\begin{aligned}HH &= H\\ HsH &= sH\\ sHH &= sH \text{ and}\\ sHsH &= H.\end{aligned}$$

So G/H with this operation is isomorphic to C_2.

However if K is the subgroup $\{e, s\}$ of G then

$$rKr^2K = \{r, rs\}\{r^2, r^2s\} = \{e, r^2s, s, r^2\}$$

which is not a left coset of K in G.

Proposition. *Suppose $H \leqslant G$. The product of two left cosets of H in G is always a left coset of H in G if and only if $gHg^{-1} = H$[51] for all $g \in G$. In this case $g_1Hg_2H = g_1g_2H$ for all $g_1, g_2 \in G$.*

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Proof. For $g_1, g_2 \in G$ we compute

$$g_1Hg_2H = \{g_1h_1g_2h_2 \mid h_1, h_2 \in H\} = \{g_1g_2g_2^{-1}h_1g_2h_2 \mid h_1, h_2 \in H\}.$$

Thus if $g_2^{-1}Hg_2 = H$ then $g_1Hg_2H = g_1g_2H$ as claimed.

Moreover in general $g_1g_2H \subset g_1Hg_2H$ since we may take $h_1 = e$ above. Thus for g_1Hg_2H to be a left coset we must have $g_1Hg_2H = g_1g_2H$ as claimed. For this

[49] We note in passing that we didn't use that G has inverses so a version of this result is still true for G any set with an associative binary operation with an identity such as $(\mathbb{Z}, \cdot)$ or $(\mathbb{N}_0, +)$.

[50] and we will later

[51] Here gHg^{-1} means $\{g\}H\{g^{-1}\}$

we need $g_2^{-1}h_1g_2 \in H$ for all $h_1 \in H$ since $(g_1g_2)g_2^{-1}h_1g_2e \in g_1Hg_2H$. So if the product of any two left cosets is a left coset then $g^{-1}Hg \subset H$ for all $g \in G$. Then

$$H = g(g^{-1}Hg)g^{-1} \subset gHg^{-1} \subset H$$

so we have equalities throughout. □

Remark. Notice that along the way we proved that whenever $gHg^{-1} \subset H$ for all $g \in G$, in fact $gHg^{-1} = H$ for all $g \in G$.

Definition. We say that a subgroup H of a group G is *normal* if $gHg^{-1} = H$ for all $g \in G$.

Warning. To show that a subset of G is a normal subgroup we must show that it is a subgroup as well as that it satisfies the above conditon.

Examples.

(1) If G is abelian then every subgroup is normal.
(2) The group $\mathrm{Isom}^+(\mathbb{Z})$ is normal in $\mathrm{Isom}(\mathbb{Z})$ but the subgroup $\{\mathrm{id}_{\mathbb{Z}}, s\colon n \mapsto -n\}$ is not normal is $\mathrm{Isom}(\mathbb{Z})$.
(3) The subgroup of rotations in D_{2n} is normal in D_{2n} but no subgroup generated by a reflection is normal in D_{2n}.
(4) $\mathrm{Stab}_{\mathcal{M}}(\infty)$ is not a normal subgroup of $\mathcal{M}$.

Lemma. *A subgroup H of a group G is normal if and only if every left coset is a right coset.*[52]

Proof. Suppose that $gHg^{-1} = H$. Then $gH = gHg^{-1}g = Hg$ so the left coset gH is a right coset.

Conversely suppose that every left coset is a right coset and $g \in G$. Then $gH = Hk$ for some $k \in G$. So $g = hk$ for some $h \in H$. Thus

$$gH = Hk = Hkg^{-1}g = Hh^{-1}g = Hg$$

and $gHg^{-1} = H$ as required. □

Proposition. *If H is a normal subgroup of G then the restriction of the binary operation on $\mathscr{P}(G)$ makes G/H into a group such that $g_1Hg_2H = g_1g_2H$.*

Definition. We call G/H the *quotient group* of G by H.

Proof of Proposition. We've already seen that the binary operation on $\mathscr{P}(G)$ restricts to an associative binary operation on G/H when H is normal and moreover that $g_1Hg_2H = g_1g_2H$.

Suppose that $gH \in G/H$. Then $eHgH = gH = gHeH$ so $eH = H$ is an identity in G/H. Finally $gHg^{-1}H = eH = g^{-1}HgH$ so G/H has inverses. □

[52] We'll often just say coset in this case.

4.2. **The isomorphism theorem.**

Theorem (The (first) isomorphism theorem)**.** *Suppose that $f\colon G \to H$ is a group homomorphism. Then* $\ker f$ *is a normal subgroup of* G, $\operatorname{Im} f$ *is a subgroup of* H *and* f *induces an isomorphism*

$$\overline{f}\colon G/\ker f \xrightarrow{\simeq} \operatorname{Im} f$$

given by $\overline{f}(g\ker f) = f(g)$.

Proof. We've already seen that $\ker f \leqslant G$ and $\operatorname{Im} f \leqslant H$. So first we must show that $g(\ker f)g^{-1} = \ker f$ for all $g \in G$.

Suppose that $k \in \ker f$ and $g \in G$. Then

$$f(gkg^{-1}) = f(g)f(k)f(g^{-1}) = f(g)ef(g)^{-1} = e$$

since $f(k) = e$. Thus $g(\ker f)g^{-1} \subset \ker f$. As remarked above this suffices to see that $\ker f$ is normal.

Now if $g\ker f = g'\ker f$ then $g^{-1}g' \in \ker f$ and so $f(g)^{-1}f(g') = f(g^{-1}g') = e$. Thus $f(g) = f(g')$ and $\overline{f}(g\ker f) = f(g)$ is a well-defined function whose image is $\operatorname{Im} f$. Moreover

$$\overline{f}(g_1\ker f g_2 \ker f) = \overline{f}(g_1g_2\ker f) = f(g_1g_2) = f(g_1)f(g_2) = \overline{f}(g_1\ker f)\overline{f}(g_2\ker f)$$

so $\overline{f}$ is a homomorphism from G/H to $\operatorname{Im} f$ with $\operatorname{Im}\overline{f} = \operatorname{Im} f$. Finally if $\overline{f}(g\ker f) = e$ then $f(g) = e$ i.e. $g \in \ker f$. So $\ker\overline{f} = \{\ker f\} = \{e_{G/\ker f}\}$ and the result follows from the special case of the first isomorphism theorem proven in section 1.4. □

Remark. Often a good way to prove that a subset of a group G is a normal subgroup is to show that it is the kernel of some homomorphism from G to another group.

Lecture 16

Example. The homomorphism $\mathbb{Z} \to \mathbb{Z}_n$ that sends a to the remainder after dividing a by n has kernel $n\mathbb{Z}$ and image $\mathbb{Z}_n$. Thus it induces an isomorphism $\mathbb{Z}/n\mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}_n$.[53]

Example. Let $\theta\colon (\mathbb{R}, +) \to (\mathbb{C}\backslash\{0\}, \cdot)$ be given by $\theta(r) = e^{2\pi ir}$. Then $\theta(r+s) = e^{2\pi i(r+s)} = \theta(r)\theta(s)$ so θ is a homomorphism. Moreover

$$\operatorname{Im}\theta = S^1 := \{z \in \mathbb{C} \mid |z| = 1\},$$

the unit circle in $\mathbb{C}$ and

$$\ker\theta = \mathbb{Z}$$

thus we can deduce that $\mathbb{R}/\mathbb{Z} \simeq S^1$.

Example. Let $\theta\colon D_{2n} \to \{\pm 1\}$ such that

$$\theta(g) := \begin{cases} +1 & \text{if} \quad g \text{ is a rotation} \\ -1 & \text{if} \quad g \text{ is a reflection.} \end{cases}$$

Then we can verify that θ is a homomorphism since the product of two reflections or two rotations is a rotation and the product of a rotation and a reflection in either order is a reflection. Moreover $\operatorname{Im}\theta = \{\pm 1\}$ and $\ker\theta$ is the subgroup of all rotations of the regular n-gon. Thus $D_{2n}/\{\text{rotations in } D_{2n}\} \simeq C_2$.

[53]The notation $\mathbb{Z}_n$ as we have defined it is rarely used and instead $\mathbb{Z}/n\mathbb{Z}$ is used to describe essentially the same thing.

Example (Group-theoretic understanding of qth powers mod p). Let p and q be distinct primes and $G = (\mathbb{Z}_p \backslash \{0\}, \cdot_p)$. Define

$$\theta \colon G \to G; x \mapsto x^q.$$

Then for $x, y \in G$, $\theta(xy) = (xy)^q = x^q y^q = \theta(x)\theta(y)$ i.e. θ is a homomorphism. Then

$$\ker\theta = \{x \in G \mid x^q = 1\} = \{x \in G \mid o(x) = 1 \text{ or } q\}.$$

We now divide into two cases.

First suppose that q is not a factor of $p-1$. Since $|G| = p-1$, G has no elements of order q by Lagrange. Thus $\ker\theta = \{1\}$. It follows that θ induces an isomorphism $G \simeq \operatorname{Im}\theta$. By counting we can conclude that $\operatorname{Im}\theta = G$. In particular we see that every element of $\mathbb{Z}_p$ is a qth power when p is not 1 mod q.

Next suppose that q is a factor of $p-1$. In this case G does have an element of order q by Cauchy's Theorem. Thus $|\ker\theta| \geqslant q$.[54] Since $G/\ker\theta \simeq \operatorname{Im}\theta$ and $|G/\ker\theta| = |G|/|\ker\theta| \leqslant \frac{p-1}{q}$ we see that $\mathbb{Z}_p$ has at most $\frac{p-1}{q}+1$ qth-powers when p is 1 mod q.[55]

Example. If G acts on a set X and $K = \{g \in G \mid g(x) = x \text{ for all } x \in X\} = \bigcap_{x\in X} \operatorname{Stab}_G(x)$ then the homomorphism $G \to S(X)$ given by the action induces an isomorphism from G/K to a subgroup of $S(X)$. Thus the action of G on X induces a faithful action of G/K on X.[56]

Example. Suppose that X is a regular tetrahedron in $\mathbb{R}^3$. X has six edges and each edge has four neighbours.[57] Thus we can partition the set of edges into three pairs with each pair consisting of non-adjacent edges. Let P denote the set of such pairs. Then the action of $\operatorname{Sym}(X)$ on X induces an action on P since if $f \in \operatorname{Sym}(X)$ and v and w are non-adjacent edges of X then $f(v)$ and $f(w)$ are also non-adjacent edges of X. Thus by the last example there is a homomorphism $\theta \colon \operatorname{Sym}(X) \to S(P)$. It is easy to verify by hand that $\operatorname{Im}\theta = S(P)$. Then the isomorphism theorem we can deduce that $\operatorname{Sym}(X)/\ker\theta \simeq S(P)$. We showed earlier than $\operatorname{Sym}(X) \simeq S_4$ and it is straightforward to see that $S(P) \simeq S_3$.[58] Thus we can deduce that S_4 has a normal subgroup K such that $S_4/K \simeq S_3$.[59]

Lecture 17

5. Matrix groups

Suppose that throughout this section $\mathbb{F}$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

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[54] Since an element of order q generates a subgroup of order q contained in the kernel. In fact it is not too hard to prove that $\ker\theta$ has precisely q elements.

[55] In fact precisely this many.

[56] This means that to understand all actions of a group G it is equivalent to understand all faithful actions of all quotients of G.

[57] There are two edges sharing each vertex of a given edge.

[58] Or $S(P) \simeq D_6$ if you prefer

[59] Can you say which elements of S_4 live in K? There must be four of them by Lagrange. If you find this too hard at this stage then try again when you revise the course having studied the groups S_n in more detail.

5.1. **The general and special linear groups.** Let $M_n(\mathbb{F})$ denote the set of $n \times n$ matrices with entries in $\mathbb{F}$.

Here are some facts proven in Vectors and Matrices.

Facts.

(1) Every element A of $M_n(\mathbb{F})$ defines a linear map $\underline{A}\colon \mathbb{F}^n \to \mathbb{F}^n$ via $\underline{A}\colon v \mapsto Av$.[60] Moreover every linear map $\mathbb{F}^n \to \mathbb{F}^n$ arises in this way and A can be recovered from $\underline{A}$ since the ith column of A is $\underline{A}(e_i)$ where e_i denotes the element of $\mathbb{F}^n$ with ith entry 1 and all other entries 0.
(2) $\underline{AB}$ corresponds to the composite $\underline{A} \circ \underline{B}$. Thus associativity of multiplication of (square) matrices follows from associativity of composition of functions $\mathbb{F}^n \to \mathbb{F}^n$.
(3) The matrix I_n with 1s down the main diagonal and 0s elsewhere is an identity for matrix multiplication on $M_n(\mathbb{F})$. Moreover $\underline{I_n} = \mathrm{id}_{\mathbb{F}^n}$.
(4) There is a function $\det\colon M_n(\mathbb{F}) \to \mathbb{F}$ such that A has an inverse in $M_n(\mathbb{F})$ if and only if $\det A \neq 0$. Moreover $\det(AB) = \det(A)\det(B)$ for any $A, B \in M_n(\mathbb{F})$ and $\det I_n = 1$.

Definition. The *general linear group* $GL_n(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid \det A \neq 0\}$ is the group of invertible $n \times n$ matrices with entries in $\mathbb{F}$.

Proposition. *$GL_n(\mathbb{F})$ is a group under matrix multiplication.*

Proof. Since for $A, B \in M_n(\mathbb{F})$, $\det AB = \det A \det B$, if $A, B \in GL_n(\mathbb{F})$ then $\det A \neq 0$ and $\det B \neq 0$ so $\det AB \neq 0$. Thus $AB \in GL_n(\mathbb{F})$ and matrix multiplication defines an associative binary operation on $GL_n(\mathbb{F})$.

Since $\det I_n = 1$, $I_n \in GL_n(\mathbb{F})$ so $GL_n(\mathbb{F})$ has an identity.

Finally if $A \in GL_n(\mathbb{F})$, since $\det A \neq 0$, there is a matrix B such that $AB = I_n = BA$. Then $\det A \det B = \det AB = \det I_n = 1$. So $\det B \neq 0$ and $B \in GL_n(\mathbb{F})$. □

Remark. There is a natural action of $GL_n(\mathbb{F})$ on $\mathbb{F}^n$ via $(A, v) \mapsto Av$. One can show that the homomorphism $GL_n(\mathbb{F}) \to S(\mathbb{F}^n)$ coming from this action induces an isomorphism $GL_n(\mathbb{F})$ with the subgroup of $S(\mathbb{F}^n)$ consisting of all invertible linear maps $\mathbb{F}^n \to \mathbb{F}^n$.

Lemma. *The function* $\det\colon GL_n(\mathbb{F}) \to (\mathbb{F}\backslash\{0\}, \cdot)$ *is a group homomorphism with image* $\mathbb{F}\backslash\{0\}$.

Proof. That it is a homomorphism follows immediately from fact 4 above: for $A, B \in GL_n(\mathbb{F})$, $\det AB = \det A \det B$. To see its image we compute

$$\det \begin{pmatrix} \lambda & 0 & \ddots & 0 \\ 0 & 1 & \ddots & \ddots \\ \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & 0 & 1 \end{pmatrix} = \lambda.$$

□

Definition. The *special linear group* $SL_n(\mathbb{F})$ is the kernel of $\det\colon GL_n(\mathbb{F}) \to \mathbb{F}\backslash\{0\}$ i.e.

$$SL_n(\mathbb{F}) := \{A \in M_n(\mathbb{F}) \mid \det A = 1\}.$$

[60] Recall that $\underline{A}$ is linear means that $\underline{A}(\lambda v + \mu w) = \lambda \underline{A}(v) + \mu \underline{A}(w)$ for all $\lambda, \mu \in \mathbb{F}$ and $v, w \in \mathbb{F}^n$.

Remarks.

(1) The action of $GL_n(\mathbb{F})$ on $\mathbb{F}^n$ induces an action of $SL_n(\mathbb{F})$ on $\mathbb{F}^n$ by restriction and the resulting homomorphism $SL_n(\mathbb{F}) \to S(\mathbb{F}^n)$ induces an isomorphism of $SL_n(\mathbb{F})$ with the subgroup of $S(\mathbb{F}^n)$ consisting of volume preserving linear maps $\mathbb{F}^n \to \mathbb{F}^n$.

(2) $SL_n(\mathbb{F})$ a normal subgroup of $GL_n(\mathbb{F}$ and $GL_n(\mathbb{F})/SL_n(\mathbb{F}) \simeq \mathbb{F}\backslash\{0\}$.

Examples.

$$\begin{aligned} GL_2(\mathbb{F}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \right\} \text{ and} \\ SL_2(\mathbb{F}) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - dc = 1 \right\} \end{aligned}$$

5.2. **Möbius maps as projective linear transformations.**

Notation. Given $v \in \mathbb{C}^2\backslash\{0\}$ let $[v]$ denote the (unique) line $\{\lambda v \mid \lambda \in \mathbb{C}\}$ through 0 and v in $\mathbb{C}^2$. The set of all such lines is called the *complex projective line* typically written $\mathbb{P}^1(\mathbb{C})$.

The following lemma gives a parameterisation of the elements of $\mathbb{P}^1(\mathbb{C})$.

Lemma. *Every element of* $\mathbb{P}^1(\mathbb{C})$ *is either of the form* $\left[\begin{pmatrix} z \\ 1 \end{pmatrix}\right]$ *with* $z \in \mathbb{C}$ *or* $\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$. *Moreover these lines are all distinct.*

Proof. Suppose $v \in \mathbb{C}^2\backslash\{0\}$. Then $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ for some $v_1, v_2 \in \mathbb{C}$ not both 0.

If $v_2 \neq 0$ then $[v] = \left[\begin{pmatrix} \frac{v_1}{v_2} \\ 1 \end{pmatrix}\right]$ since $\lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v_2\lambda \begin{pmatrix} \frac{v_1}{v_2} \\ 1 \end{pmatrix}$.

If $v_2 = 0$ then $[v] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$ since $\lambda v = v_1\lambda \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Finally if $\lambda \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} w \\ 1 \end{pmatrix}$ then $\lambda = 1$ and $z = w$. And $\lambda \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is evidently impossible. □

It follows that we may identify $\mathbb{C}_\infty$ and $\mathbb{P}^1(\mathbb{C})$ via $z \mapsto \left[\begin{pmatrix} z \\ 1 \end{pmatrix}\right]$ for $z \in \mathbb{C}$ and $\infty \mapsto \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$.

Proposition. $GL_2(\mathbb{C})$ *acts on* $\mathbb{P}^1(\mathbb{C})$ *via* $(A, [v]) \mapsto [Av]$ *for* $v \in \mathbb{C}^2\backslash\{0\}$.

Proof. First we must show that $(A, [v]) \mapsto [Av]$ is a well-defined function

$$GL_2(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$$

i.e. that if $A \in GL_2(\mathbb{C})$ and $v \in \mathbb{C}^2\backslash\{0\}$ then $Av \neq 0$, and if $[v] = [w]$ then $[Av] = [Aw]$. Now, if $Av = 0$ then $v = A^{-1}Av = A^{-1}(0) = 0$ and if $[v] = [w]$ there is some non-zero $\mu \in \mathbb{C}$ such that $v = \mu w$. Then for $\lambda \in \mathbb{C}$, $A(\lambda w) = \lambda A(\mu v) = (\lambda\mu)Av$ so $[Aw] \subset [Av]$. By symmetry we must have $[Av] = [Aw]$.

It remains to observe that $[I_2 v] = [v]$ and that $[A(Bv)] = [(AB)v]$ for all $[v] \in \mathbb{P}^1(\mathbb{C})$ and $A, B \in GL_2(\mathbb{C})$. □

We note that under this action of $GL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left[\begin{pmatrix} z \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} az+b \\ cz+d \end{pmatrix}\right] = \left[\begin{pmatrix} \frac{az+b}{cz+d} \\ 1 \end{pmatrix}\right] \text{ when } z \neq -d/c,$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left[\begin{pmatrix} -d/c \\ 1 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right],$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} a \\ c \end{pmatrix}\right] = \left[\begin{pmatrix} \frac{a}{c} \\ 1 \end{pmatrix}\right] \text{ when } c \neq 0, \text{ and}$$

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right].$$

Thus, under the identification of $\mathbb{C}_\infty$ with $\mathbb{P}^1(\mathbb{C})$, the homomorphism

$$\theta\colon GL_2(\mathbb{C}) \to S(\mathbb{C}_\infty)$$

corresponding to this action sends the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the Möbius map represented by $z \mapsto \frac{az+b}{cz+d}$, and so $\operatorname{Im}\theta = \mathcal{M}$. Thus $\mathcal{M}$ is a subgroup of $S(\mathbb{C}_\infty)$.

Moreover $\ker\theta$ consists of invertible matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fixing every line through the origin in $\mathbb{C}^2$.

Now

$$\operatorname{Stab}_{GL_2(\mathbb{C})}\left(\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]\right) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) \mid c = 0\right\},$$

$$\operatorname{Stab}_{GL_2(\mathbb{C})}\left(\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]\right) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C} \mid b = 0\right\} \text{ and}$$

$$\operatorname{Stab}_{GL_2(\mathbb{C})}\left(\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]\right) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) \mid a+b = c+d\right\}.$$

Since a Möbius transformation that fixes three distinct points is the identity, $\ker\theta$ is the intersection of these three sets i.e.

$$\ker(GL_2(\mathbb{C}) \to \mathcal{M}) = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \neq 0\right\}$$

is the group of non-zero scalar matrices.[61]

Thus $PGL_2(\mathbb{C}) := GL_2(\mathbb{C})/\{\lambda I \mid \lambda \in \mathbb{C} \neq 0\} \simeq \mathcal{M}$. It is not hard to see that a similar argument shows that $PSL_2(\mathbb{C}) := SL_2(\mathbb{C})/\{\pm I\} \simeq \mathcal{M}$.

We can summarize this discussion with the following theorem.

Theorem. *The action of $GL_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$ induces an isomorphism from $PGL_2(\mathbb{C})$ to $\mathcal{M}$. In particular $\mathcal{M}$ is a subgroup of $S(\mathbb{C}_\infty)$.*

[61] We can see this another way: the kernel of θ is certainly contains in the intersection of these three stabilisers so it would suffice to check that any scalar matrix is in the kernel ie $[\lambda I_2 v] = [v]$ for all non-zero λ in $\mathbb{C}$. Indeed this is how we showed that a Möbius map that fixes 0, 1 and ∞ is the identity in §1.5.

Lecture 18

5.3. **Change of basis.** Recall that if $\underline{A}$ is a linear map $\mathbb{F}^n \to \mathbb{F}^n$ corresponding to the matrix A and $e_1, \ldots, e_n$ is the standard basis for $\mathbb{F}^n$ then $\underline{A}(e_i) = \sum_{j=1}^n A_{ji} e_j$.

If $f_1, \ldots, f_n$ is another basis for $\mathbb{F}^n$ then there is an invertible linear map $\underline{P}$ such that $\underline{P}(e_i) = f_i$ for $i = 1, \ldots, n$. i.e. $\underline{P}$ corresponds to the matrix P whose columns $f_1, \ldots, f_n$ and $f_i = \sum_{j=1}^n P_{ji} e_j$ for $i = 1, \ldots, n$. It follows that for $j = 1, \ldots, n$,

$$\sum_{k=1}^n P^{-1}_{kj} f_k = \sum_{k=1}^n P^{-1}_{kj} \sum_{l=1}^n P_{lk} e_l = \sum_{l=1}^n (PP^{-1})_{lj} e_l = e_j.$$

Then

$$\begin{aligned} \underline{A}(f_i) &= \underline{AP}(e_i) \\ &= \sum_{j=1}^n (AP)_{ji} e_j \\ &= \sum_{j=1}^n (AP)_{ji} (\sum_{k=1}^n P^{-1}_{kj} f_k) \\ &= \sum_{k=1}^n (P^{-1}AP)_{ki} f_k \end{aligned}$$

Thus $P^{-1}AP$ represents $\underline{A}$ with respect to the basis $f_1, \ldots, f_n$.

Proposition. *$GL_n(\mathbb{F})$ acts on $M_n(\mathbb{F})$ by conjugation.*

Proof. Consider

$$\cdot : GL_n(\mathbb{F}) \times M_n(\mathbb{F}) \to M_n(\mathbb{F}); (P, X) \mapsto PXP^{-1}.$$

Then for $X \in \mathrm{Mat}_n(\mathbb{F})$,

$$I_n \cdot X = I_n X I_n^{-1} = X$$

and

$$P(QXQ^{-1})P^{-1} = (PQ)X(PQ)^{-1}$$

for all $P, Q \in GL_n(\mathbb{F})$. □

It is now straightforward to see that two distinct matrices in $M_n(\mathbb{F})$ represent the same linear map with respect to different bases if and only if they are in the same $GL_n(\mathbb{F})$-orbit under this conjugation action.

Example (See Vectors and Matrices). If $\underline{A} : \mathbb{C}^2 \to \mathbb{C}^2$ is a linear map then precisely one of the following three things is true:

(i) there is a basis for $\mathbb{C}^2$ such that $\underline{A}$ is represented by a matrix of the form

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

with $\lambda, \mu \in \mathbb{C}$ distinct — in this case $\{\lambda, \mu\}$ is determined by $\underline{A}$[62] but they may appear in either order in the matrix;

[62] λ and μ are its eigenvalues and the basis vectors are the corresponding eigenvectors

(ii) there is a basis for $\mathbb{C}^2$ such that $\underline{A}$ is represented by a matrix of the form
$$\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$$
with $\lambda \in \mathbb{C}$ — in this case λ is determined by $\underline{A}$ indeed $\underline{A} = \lambda \mathrm{id}_{\mathbb{C}^2}$ and $\underline{A}$ is represented by this matrix with respect to every basis;

(iii) there is a basis for $\mathbb{C}^2$ such that $\underline{A}$ is represented by a matrix of the form
$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$$
again λ is determined by $\underline{A}$.[63]

We may interpret this group-theoretically: every $GL_2(\mathbb{C})$-orbit in $M_2(\mathbb{C})$ with respect to the conjugation action is one of the following:
$$\mathcal{O}_{\lambda,\mu} := \mathrm{Orb}_{GL_2(\mathbb{C})}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) \text{ with } \lambda, \mu \in \mathbb{C} \text{ distinct},$$
$$\mathcal{O}^{(1)}_{\lambda} := \mathrm{Orb}_{GL_2(\mathbb{C})}\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}\right) \text{ with } \lambda \in \mathbb{C} \text{ and}$$
$$\mathcal{O}^{(2)}_{\lambda} := \mathrm{Orb}_{GL_2(\mathbb{C})}(\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) \text{ with } \lambda \in \mathbb{C}.$$
These are all disjoint except that $\mathcal{O}_{\lambda,\mu} = \mathcal{O}_{\mu,\lambda}$. More explicitly,
$$\mathcal{O}_{\lambda,\mu} = \{A \in M_2(\mathbb{C}) \mid \det(tI_2 - A) = (t-\lambda)(t-\mu) \text{ for all } t \in \mathbb{C}\},$$
$$\mathcal{O}^{(1)}_{\lambda} = \{\lambda I_2\} \text{ and}$$
$$\mathcal{O}^{(2)}_{\lambda} = \{A \in M_2(\mathbb{C}) \mid \det(tI - A) = (t-\lambda)^2 \text{ for all } t \in \mathbb{C}, A \neq \lambda I_2\}.$$
We can also compute
$$\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a\lambda & b\lambda\\ c\mu & d\mu\end{pmatrix}$$
and
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} = \begin{pmatrix}a\lambda & b\mu\\ c\lambda & d\mu\end{pmatrix}$$
so that for $\lambda \neq \mu$,
$$\mathrm{Stab}_{GL_2(\mathbb{C})}\left(\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}\right) = \left\{\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \mid ad \neq 0\right\}$$
and $\mathrm{Stab}_{GL_2(\mathbb{C})}(\lambda I_2) = GL_2(\mathbb{C})$.

Similarly
$$\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a\lambda + c & b\lambda + d\\ c\lambda & d\lambda\end{pmatrix}$$
and
$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix} = \begin{pmatrix}a\lambda & a + b\lambda\\ c\lambda & c + d\lambda\end{pmatrix}$$
so
$$\mathrm{Stab}_{GL_2(\mathbb{C})}\left(\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}\right) = \left\{\begin{pmatrix}a & b\\ 0 & a\end{pmatrix} \mid a \neq 0\right\}.$$
All other stabilisers are conjugate to these ones. We can easily read off the conjugacy classes and centralisers in $GL_2(\mathbb{C})$ by restricting to the case $\lambda, \mu \neq 0$.

[63] it is the unique eigenvalue of $\underline{A}$ and $\underline{A} \neq \lambda \mathrm{id}_{\mathbb{C}}$.

Exercise. Deduce that in $\mathcal{M} \simeq PGL_2(\mathbb{C})$, $\mathrm{ccl}(z \mapsto az) = \mathrm{ccl}(z \mapsto z \mapsto bz)$ if and only if $b \in \{a, 1/a\}$ thus provide a description of all the conjugacy classes in $\mathcal{M}$ and compute centralisers of suitable representatives of each class.

Lecture 19

5.4. **The orthogonal and special orthogonal groups.** Recall that any (square) matrix A has a transpose A^T with $A^T_{ij} = A_{ji}$ and $\det A^T = \det A$. Moreover if A, B are square matrices of the same size then $(AB)^T = B^T A^T$.

Definition. The *orthogonal group* $O(n) := \{A \in M_n(\mathbb{R}) \mid A^T A = I_n = AA^T\} \subset GL_n(\mathbb{R})$ is the group of orthogonal $n \times n$ matrices.

Lemma. *$O(n)$ is a subgroup of $GL_n(\mathbb{R})$.*

Proof. $I_n \in O(n)$ and if $A, B \in O(n)$ then $B^T = B^{-1}$ so

$$(AB^{-1})^T AB^{-1} = (B^{-1})^T A^T AB^{-1} = (B^{-1})^T IB^T = (BB^{-1})^T = I_n^T = I_n$$

and similarly $(AB^{-1})(AB^{-1})^T = I_n$. □

Recall that $\mathbb{R}^n$ comes with an inner product $v \cdot w = \sum_{i=1}^n v_i w_i$ which defines a length function on $\mathbb{R}^n$ via $|v| = (v \cdot v)^{1/2}$. We also recall the definition of Kronecker's delta

$$\delta_{ij} := \begin{cases} 1 & \text{if} \quad i = j; \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

A basis $f_1, \ldots, f_n$ of $\mathbb{R}^n$ is said to be *orthonormal* if $f_i \cdot f_j = \delta_{ij}$.[64]

Lemma.

(a) If $\{f_1, \ldots, f_n\} \subset \mathbb{R}^n$ such that $f_i \cdot f_j = \delta_{ij}$ for all $1 \leqslant i, j \leqslant n$, then $\{f_1, \ldots, f_n\}$ is an orthonormal basis for $\mathbb{R}^n$.

(b) If $v, w \in \mathbb{R}^n$ then $v \cdot w = \frac{1}{4}(|v + w|^2 - |v - w|^2)$.

Proof. (a) Suppose $\sum_{i=1}^n \lambda_i f_i = 0$. Then for $j = 1, \ldots, n$,

$$0 = \left(\sum_{i=1}^n \lambda_i f_i\right) \cdot f_j = \sum_{i=1}^n \lambda_i (f_i \cdot f_j) = \lambda_j.$$

Thus $\{f_1, \ldots, f_n\}$ is linearly independent. Since it has n elements it must span $\mathbb{R}^n$.[65]

(b) We compute

$$(v + w) \cdot (v + w) = v \cdot v + v \cdot w + w \cdot v + w \cdot w$$

and

$$(v - w) \cdot (v - w) = v \cdot v - v \cdot w - w \cdot v - w \cdot w$$

subtracting and using symmetry of the inner product we see that

$$|v + w|^2 - |v - w|^2 = 4v \cdot w$$

as required. □

[64] There is a little apparent notational ambiguity here since we use subscripts to index the basis vectors as well as to index the coordinates of a vector. Each f_i is in $\mathbb{R}^n$ so can be written as $\sum_{k=1}^n (f_i)_k e_k$ and $f_i \cdot f_j = \sum_{k=1}^n (f_i)_k (f_j)_k$.

[65] In fact if $v \in \mathbb{R}^n$, $v = \sum_{i=1}^n (v \cdot f_i) f_i$.

Proposition. *Suppose that $A \in M_n(\mathbb{R})$. The following are equivalent:*

(i) $A \in O(n)$;
(ii) $Av \cdot Aw = v \cdot w$ *for all* $v, w \in \mathbb{R}^n$;
(iii) the columns of A form an orthonormal basis;
(iv) $|Av| = |v|$ *for all* $v \in \mathbb{R}^n$.

Proof. Suppose that $A \in O_n$ and $v, w \in \mathbb{R}^n$. Then

$$\begin{aligned} Av \cdot Aw &= \sum_{i=1}^{n} (Av)_i (Aw)_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij} v_j \right) \left(\sum_{k=1}^{n} A_{ik} w_k \right) \\ &= \sum_{j,k=1}^{n} v_j (A^T A)_{jk} w_k \\ &= \sum_{j} v_j w_j = v \cdot w. \end{aligned}$$

Thus (i) $\implies$ (ii).

Suppose now that $Av \cdot Aw = v \cdot w$ for all $v, w \in \mathbb{R}^n$. Then in particular $Ae_i \cdot Ae_j = e_i \cdot e_j = \delta_{ij}$ for each $1 \leqslant i, j \leqslant n$ and $Ae_1, \dots, Ae_n$[66] is an orthonormal basis for $\mathbb{R}^n$ by part (a) of the last lemma. Thus (ii) $\implies$ (iii).

Next, if $Ae_1, \dots, Ae_n$ form an orthonormal basis then for $1 \leqslant i, j \leqslant n$

$$\begin{aligned} \delta_{ij} &= Ae_i \cdot Ae_j \\ &= \sum_{k=1}^{n} A_{ki} A_{kj} = (A^T A)_{ij} \end{aligned}$$

and $A^T A = I_n$. Thus (iii) $\implies$ (i).

(ii) $\implies$ (iv) is immediate from the definitions. To see that (iv) $\implies$ (ii) we use (b) of the lemma:

$$Av \cdot Aw = \frac{1}{4}(|A(v+w)|^2 - |A(v-w)|^2$$

and

$$v \cdot w = \frac{1}{4}(|v+w|^2 - |v-w|^2).$$

If (iv) holds then RHSs of these equations to be equal for all v, w and thus the LHSs are equal for all v, w and (ii) also holds. □

Thus $O(n)$ is isomorphic to the subgroup of $S(\mathbb{F}^n)$ consisting of linear maps that preserve the scalar product or equivalently to the subgroup of $S(\mathbb{F}^n)$ consisting of linear maps that preserve length.

The conjugation action $GL_n(\mathbb{R})$ on $M_n(\mathbb{R})$ restricts to an action of $O(n)$ on $M_n(\mathbb{R})$. The equivalence of (i) and (iii) in the propostion shows that two distinct matrices in $M_n(\mathbb{R})$ are in the same $O(n)$-orbit if and only if they represent the same linear map with respect to two different orthonormal bases (see the last lecture).

Proposition. $\det\colon O(n) \to (\mathbb{R}\backslash\{0\}, \cdot)$ *has image* $\{\pm 1\}$.

[66] i.e. the set of columns of A

Proof. If $A \in O(n)$ then $\det A = \det A^T$ so $1 = \det I_n = \det AA^T = (\det A)^2$ and $\det A = \pm 1$. Since

$$\begin{pmatrix} -1 & 0 & & \\ 0 & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{pmatrix} \in O(n)$$

has determinant -1 both 1 and -1 are in the image. □

Definition. The *special orthogonal group*

$$SO(n) := O(n) \cap SL_n(\mathbb{R}) = \ker(\det\colon O(n) \to \{\pm 1\}).$$

$SO(n)$ is isomorphic to the subgroup of $S(\mathbb{R}^n)$ consisting of linear maps that preserve the scalar product and orientation.[67] It is a normal subgroup of $O(n)$ and $O(n)/SO(n) \simeq C_2$.

There are complex versions of the orthogonal group and the special orthogonal group called the unitary group and the special unitary group. We won't have time to discuss them but they do appear on Example Sheet 4.

Lecture 20

5.5. Reflections.

Definition. Suppose that $n \in \mathbb{R}^m$ has length 1 then the *reflection in the plane normal to* n is the function $R_n\colon \mathbb{R}^m \to \mathbb{R}^m$ given by

$$R_n(x) = x - 2(x \cdot n)n.$$

Note that if $y \cdot n = 0$ then $R_n(y) = y$, and $R_n(n) = n - 2n = -n$.

Lemma. *Suppose $n \in \mathbb{R}^m$ has length 1 then*

(a) $R_n \in O(m)$;
(b) $\mathbb{R}^m$ has a basis with respect to which R_n is represented by a diagonal matrix D such that $D_{11} = -1$, $D_{ii} = 1$ for $2 \leqslant i \leqslant m$;
(c) $(R_n)^2 = \mathrm{id}_{\mathbb{R}^m}$ and;
(d) $\det R_n = -1$.

Proof. (a) First we show that R_n is linear: if $x, y \in \mathbb{R}^m$ and $\lambda, \mu \in \mathbb{R}$ then

$$\begin{aligned} R_n(\lambda x + \mu y) &= (\lambda x + \mu y) - 2((\lambda x + \mu y) \cdot n)n \\ &= \lambda(x - (2x \cdot n)n) + \mu(y - (2y \cdot n)n) \\ &= \lambda R_n(x) + \mu R_n(y). \end{aligned}$$

Now we show that R_n preserves the inner product: if $x, y \in \mathbb{R}^m$ then

$$\begin{aligned} R_n(x) \cdot R_n(y) &= (x - 2(x \cdot n)n) \cdot (y - 2(y \cdot n)n) \\ &= x \cdot y - 2(x \cdot n)(n \cdot y) - 2(y \cdot n)(x \cdot n) + 4(x \cdot n)(y \cdot n)(n \cdot n) \\ &= x \cdot y. \end{aligned}$$

(b) Notice that $U := \{y \in \mathbb{R}^m \mid y \cdot n = 0\} = \ker(\mathbb{R}^m \to \mathbb{R}; y \mapsto y \cdot n)$ is a subspace of $\mathbb{R}^m$ of dimension $m - 1$ by the rank-nullity theorem. Moreover $n \notin U$.

[67] I have not defined an orientation of $\mathbb{R}^n$. One way would be as an $SO(n)$-orbit of orthonormal bases for $\mathbb{R}^n$ which would make this completely tautological. There are more sophisticated ways that make it less so. With this definition the next sentence gives that there are exactly two orientations of $\mathbb{R}^n$.

So we may find a basis $f_1, \dots, f_m$ for $\mathbb{R}^m$ such that $f_1 = n$ and $f_2, \dots, f_m \in U$. Then $R_n(f_1) = -f_1$ and $R_n(f_i) = f_i$ for $2 \leqslant i \leqslant m$ as required.

(c), (d) If P is the change of basis matrix so that $P^{-1}R_nP = D$ then $P^{-1}(R_n)^2P = (P^{-1}R_nP)^2 = I_m$. Thus $(R_n)^2 = PI_mP^{-1} = I_m$. Moreover

$$-1 = \det D = (\det P)^{-1} \det R_n \det P = \det R_n$$

as required. □

Proposition. *If $x, y \in \mathbb{R}^m$ with $x \neq y$ but $x \cdot x = y \cdot y$ then there is $n \in \mathbb{R}^m$ of unit length such that $R_n(x) = y$. Moreover n may be chosen to be parallel to $x - y$.*

Proof. Let $n = \frac{(x-y)}{((x-y)\cdot(x-y))^{1/2}}$ so that $n \cdot n = 1$.[68] Then $R_n(x) = x - 2(x \cdot n)n$. But

$$2x \cdot (x - y) = 2x \cdot x - 2x \cdot y = (x - y) \cdot (x - y)$$

since $x \cdot x = y \cdot y$. Thus $2(x \cdot n)n = x - y$ as required. □

Theorem. *Every element of $O(3)$ is a product of at most three reflections of the form R_n with $n \in \mathbb{R}^3$ of length 1.* [69]

Proof. Let $A \in O(3)$. For this proof only we'll write R_0 to denote $\mathrm{id}_{\mathbb{R}^3}$.

Consider $A(e_1)$. If $A(e_1) = e_1$ then let $n_1 = 0$. Otherwise use the proposition to find n_1 of unit length such that $R_{n_1}A(e_1) = e_1$. In either case, $R_{n_1}A(e_1) = e_1$.

Next consider $R_{n_1}A(e_2)$. If $R_{n_1}A(e_2) = e_2$ let $n_2 = 0$. Otherwise

$$R_{n_1}A(e_2) \cdot e_1 = e_2 \cdot e_1 = 0$$

since $R_{n_1}A$ is orthogonal and fixes e_1. Thus we may use the proposition to find n_2 of unit length parallel to $R_{n_1}A(e_2) - e_2$ such that $R_{n_2}R_{n_1}A(e_2) = e_2$. Morever $n_2 \cdot e_1 = 0$ since n_2 is parallel to $R_{n_1}A(e_2) - e_2$ and $R_{n_1}A(e_2)$ and e_2 are both orthogonal to e_1. Thus $R_{n_2}R_{n_1}A$ fixes e_1 and e_2 in every case.

Now $R_{n_2}R_{n_1}A(e_3)$ has length 1 and is orthogonal to both e_1 and e_2 since e_3 is orthogonal to both e_1 and e_2, and $R_{n_2}R_{n_1}A$ is orthogonal and fixes e_1 and e_2. Thus $R_{n_2}R_{n_1}A(e_3) = \pm e_3$. If $R_{n_2}R_{n_1}A(e_3) = e_3$ let $n_3 = 0$, otherwise let $n_3 = e_3$. In either case we can compute that $R_{n_3}R_{n_2}R_{n_1}A$ fixes e_1, e_2 and e_3. Thus as it is a linear map it must be the identity. We conclude $A = R_{n_1}R_{n_2}R_{n_3}$ since each R_{n_i} has order 1 or 2 according as $n_i = 0$ or $n_i \neq 0$. In particular A is a product of at most three reflections. □

Proposition. *If $A \in O(2)$ then either*

(i) $A = SO(2)$ and there is some $0 \leqslant \theta < 2\pi$ such that

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$ [70] *or*

(ii) $A \notin SO(2)$ and $A = R_n$ for some $n \in \mathbb{R}^2$ of unit length.

---

[68] $x \neq y$ means $(x - y) \cdot (x - y) > 0$.

[69] There is nothing special about three here. In general every element of $O(m)$ is a product of at most m reflections of the form R_n. The proof is exactly similar to this one.

[70] i.e. A is a rotation

Proof. In either case we know

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = I_2.$$

Thus $a^2 + b^2 = c^2 + d^2 = 1$, $ac + bd = 0$ and $ad - bc = \pm 1$.

Let $a = \cos\theta$, $b = \sin\theta$, $c = \sin\phi$ and $d = \cos\phi$ with $0 \leqslant \theta, \phi < 2\pi$. Then

$$0 = \cos\theta\sin\phi + \cos\phi\sin\theta = \sin(\theta + \phi)$$

and

$$ad - bc = \cos\theta\cos\phi - \sin\theta\sin\phi = \cos(\theta + \phi).$$

In case (i) $ad - bc = 1$, $\theta + \phi$ is a multiple of 2π and $\cos(\phi) = \cos(-\theta) = \cos(\theta)$ and $\sin(\phi) = \sin(-\theta) = -\sin\theta$ as required.

In case (ii) $ad - bc = -1$, and $\theta + \phi$ is π or 3π and $\cos(\phi) = \cos(\pi - \theta) = -\cos(\theta)$. Then

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix} = \begin{pmatrix} \sin -\theta/2 \\ \cos\theta/2 \end{pmatrix} = -\begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$$

and

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix}.$$

Since

$$\begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix} \cdot \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2 \end{pmatrix} = 0$$

we see that $A = R_n$ for $n = \begin{pmatrix} \sin\theta/2 \\ -\cos\theta/2 \end{pmatrix}$.[71] □

Theorem. *If $A \in SO(3)$ then there is some non-zero $v \in \mathbb{R}^3$ such that $Av = v$.*[72]

Proof. By the last theorem A is a product of at most three reflections R_n. Since $\det A = 1$ and each R_n has determinant -1, A must be a product of either 0 or 2 reflections. If 0 then $A = \mathrm{id}_{\mathbb{R}^3}$ and any v will work. If $A = R_{n_1}R_{n_2}$ with n_1, n_2 of unit length. Then consider the linear map

$$\mathbb{R}^3 \to \mathbb{R}^2; x \mapsto \begin{pmatrix} x \cdot n_1 \\ x \cdot n_2 \end{pmatrix}.$$

By the rank-nullity theorem it has non-trivial kernel. i.e. there is $v \in \mathbb{R}^3$ such that $v \cdot n_1 = v \cdot n_2 = 0$. Then $R_{n_1}R_{n_2}(v) = R_{n_1}(v) = v$. □

[71] To handle case (ii) we could instead appeal to the result that any element of $O(2)$ is a product of at most 2 reflections and the product of 0 or 2 reflections is in $SO(2)$.

[72] That is every rotation in $\mathbb{R}^3$ has an axis.

Lecture 21

Corollary. *Every A in $SO(3)$ is conjugate in $SO(3)$ to a matrix of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

Proof. Suppose $A \in SO(3)$ and $v_1 \in \mathbb{R}^3$ non-zero such that $Av_1 = v_1$. By replacing v_1 by $v_1/|v_1|$ we may assume $|v_1| = 1$. Let $U := \ker(\mathbb{R}^3 \to \mathbb{R}; x \mapsto x \cdot v_1)$. By the rank-nullity theorem $\dim U = 2$. We can choose an orthonormal basis v_2, v_3 for U. Then v_1, v_2, v_3 is an orthonormal basis for $\mathbb{R}^3$ and so the matrix P with columns v_1, v_2 and v_3 is in $O(3)$. If $\det P = -1$ then we may swap v_2 and v_3 so that $P \in SO(3)$. We claim that $B := P^{-1}AP$ is of the required form.

Certainly $B \in SO(3)$ and $B(e_1) = P^{-1}Av_1 = P^{-1}(v_1) = e_1$. Moreover

$$e_1 \cdot Be_2 = Be_1 \cdot Be_2 = e_1 \cdot e_2 = 0$$

and similarly $e_1 \cdot Be_3 = 0$. So

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{11} & C_{12} \\ 0 & C_{21} & C_{22} \end{pmatrix}$$

for some $C \in M_2(\mathbb{R})$.

Then $B^TB = I_3$ gives $C^TC = I_2$.[73] Since $\det B = 1$, $\det C = 1$ and $C \in SO(2)$. The result follows from the last proposition about $O(2)$. □

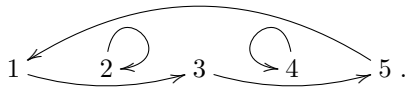
6. Permutations

Recall that a *permutation* of a set X is an element of the group $S(X)$; that is an invertible function $X \to X$. In this chapter we will study permutations of finite sets. More particularly we will study permutations of $[n] := \{1, 2, \dots, n\}$. Since there is a 1-1 correspondence (i.e. invertible function) between any finite set and $[n]$ for some value of n this amounts to the same thing.

6.1. Permutations as products of disjoint cycles.

Definition. We say that a permutation $\sigma\colon [n] \to [n]$ is a *cycle* if the natural action of the cyclic subgroup of S_n generated by σ has precisely one orbit of size greater than one.

Example. If $\sigma\colon [5] \to [5]$ such that $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 5$, $\sigma(4) = 4$ and $\sigma(5) = 1$ then $\sigma \in S_5$. We can draw σ as follows:



We can compute $\sigma^k(2) = 2$ and $\sigma^k(4) = 4$ for all $k \in \mathbb{Z}$. We can also compute $\sigma^2(1) = \sigma(3) = 5$, $\sigma^2(3) = \sigma(5) = 1$ and $\sigma^2(5) = \sigma(1) = 3$. Finally $\sigma^3(1) = \sigma(5) = 1$, $\sigma^3(3) = \sigma(1) = 3$ and $\sigma^3(5) = \sigma(3) = 5$. So $\sigma^3 = \mathrm{id}$, the group generated by σ is $\{\mathrm{id}, \sigma, \sigma^2\}$ and the orbits are $\{1, 3, 5\}$, $\{2\}$ and $\{4\}$. Thus σ is a cycle.

[73] Alternatively we could've shown that B acts on the span of e_2 and e_3 by a length preserving transformation C and so $C \in O(2)$.

Suppose that σ is a cycle of order k. Then σ generates the subgroup

$$\langle\sigma\rangle := \{\mathrm{id}, \sigma, \ldots, \sigma^{k-1}\}.$$

For any $b \in [n]$ in an orbit of size 1

$$\sigma^i(b) = b \text{ for all } i \in \mathbb{Z}$$

and if $a \in [n]$ is in the orbit of size greater than one then for $c = \sigma^j(a) \in \mathrm{Orb}_{\langle\sigma\rangle}(a)$,

$$\sigma^i(c) = \sigma^{i+j}(a) = \sigma^j(\sigma^i(a)).$$

Thus $\sigma^i(c) = c$ whenever $\sigma^i(a) = a$. i.e. $\sigma^i \in \mathrm{Stab}_{\langle\sigma\rangle}(a)$ only if $\sigma^i = \mathrm{id}$. Thus $\mathrm{Stab}_{\langle\sigma\rangle}(a) = \{e\}$ and $|\mathrm{Orb}_{\langle\sigma\rangle}(a)| = k$.

Notation. If σ is a cycle of order k such that the orbit of size greater than one contains the element $a \in [n]$ then we write

$$\sigma = (a\sigma(a)\sigma^2(a)\cdots\sigma^{k-1}(a)).$$

The discussion above shows that the elements $a, \sigma(a), \ldots, \sigma^{k-1}(a)$ are all distinct and exhaust the orbit of a under $\langle\sigma\rangle$. Thus $(a\sigma(a)\cdots\sigma^{k-1}(a))$ uniquely determines σ since $\sigma(b) = b$ for $b \notin \{a, \sigma(a), \ldots, \sigma^{k-1}(a)\}$ and $\sigma(\sigma^i(a)) = \sigma^{i+_k 1}(a)$.

Example. If $\sigma\colon [5] \to [5]$ is as in the example above then $\sigma = (135) = (351) = (513)$.

Definition. We say that $(a_1, \ldots, a_k)$ and $(b_1, \ldots, b_l)$ are *disjoint cycles* if

$$\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_l\} = \emptyset.$$

Lemma.
(a) For $a_1, \ldots, a_m \in [n]$ distinct

$$(a_1a_2\cdots a_m) = (a_2a_3\cdots a_ma_1) = (a_3a_4\cdots a_ma_1a_2) = \cdots$$

i.e. cycles can be cycled.
(b) If σ and τ are disjoint cycles then $\sigma\tau = \tau\sigma$ i.e. disjoint cycles commute.

Proof. (a) comes from choosing different elements of the non-trivial orbit to start the cycle.

(b) Suppose that $k \in [n]$. Since σ and τ are disjoint either σ fixes k or τ fixes k (or possibly both).

Without loss of generality we may assume $\sigma(k) = k$. Then $\tau\sigma(k) = \tau(k)$. Thus to show $\tau\sigma(k) = \sigma\tau(k)$ it suffices to show that σ fixes $\tau(k)$.

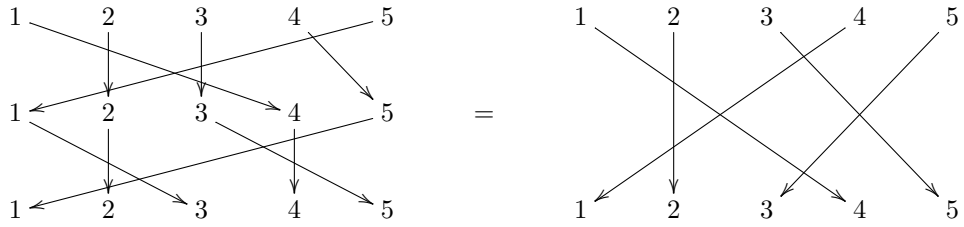
If $\tau(k) = k$ then $\sigma\tau(k) = \sigma(k) = k = \tau(k)$.

If $\tau(k) \neq k$ then $\mathrm{Orb}_{\langle\tau\rangle}(\tau(k)) = \mathrm{Orb}_{\langle\tau\rangle}(k)$ is the non-trivial $\langle\tau\rangle$-orbit so by disjointness of σ and τ any element of it, in particular $\tau(k)$, is fixed by σ as required. □

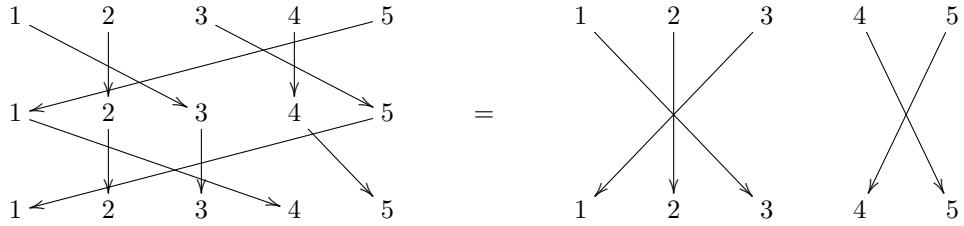
Theorem (Disjoint cycle decomposition)**.** *Every $\pi \in S_n$ may be written as a (possibly empty) product of disjoint cycles. Moreover the representation of π as a product of disjoint cycles is unique up to reordering.*

Example. Consider (135) and (145) in S_5. How is (135)(145) expressed as a product of disjoint cycles? We can chase elements one at a time. (145) sends 1 to 4 and (135) fixes 4. (145) sends 4 to 5 and (135) sends 5 to 1. Thus (14) is one of the cycles in the disjont cycle decompositon of (135)(145). 2 is fixed by both (145) and (135) so we can ignore it. (145) fixes 3 and (135) sends 3 to 5. (145) sends 5 to 1

and (135) sends 1 to 3. So (35) is another cycle in the decomposition. It follows that (135)(145) = (14)(35). Pictorally



whereas



i.e. (145)(135) = (13)(45).

Lecture 22

Proof of disjoint cycle decomposition theorem. Suppose that $\pi \in S_n$. The $\langle\pi\rangle$-orbits partition $[n]$,

$$[n] = \bigcup_{i=1}^{m} \mathcal{O}_i$$

say. By re-ordering the orbits we may assume that $k_i := |\mathcal{O}_i| > 1$ for $i = 1, \dots l$ and $|\mathcal{O}_i| = 1$ for $i = l+1, \dots m$.[74].

For $1 \leqslant i \leqslant l$ pick $a_i \in \mathcal{O}_i$ and define

$$\sigma_i := (a_i \pi(a_i) \cdots \pi^{k_i - 1}(a_i)).$$

Since the $\mathcal{O}_i$ are disjoint, the σ_i are disjoint cycles. We claim that $\pi = \prod_{i=1}^{l} \sigma_i$.[75]

To prove the claim suppose that $a \in [n]$. Since the $\mathcal{O}_i$ partition $[n]$, $a \in \mathcal{O}_{i_a}$ for precisely one $i_a \in [m]$.

If $i_a > l$ then $\pi(a) = a$ and $\sigma_i(a) = a$ for all $i \in [l]$. Thus

$$\pi(a) = a = \left(\prod_{i=1}^{l} \sigma_i\right)(a).$$

If $i_a \leqslant l$ then $\sigma_i(a) = a$ for $i \in [l] \backslash \{i_a\}$ and $\sigma_{i_a}(a) = \pi(a)$ by definition. Thus

$$\left(\prod_{i=1}^{l} \sigma_i\right)(a) = \sigma_{i_a} \prod_{\substack{i=1 \\ i \neq i_a}}^{l} \sigma_i(a) = \sigma_{i_a}(a) = \pi(a)$$

as required.

[74] we allow $l = 0$ or $l = m$ in which cases one of these lists is empty

[75] Since the σ_i are disjoint the order in the product does not matter by part (b) of the last lemma.

Uniqueness follows easily from the construction above — a cycle σ will appear in the product if and only if its non-trivial orbit $\mathcal{O}$ is a non-trivial orbit of $\langle\pi\rangle$ and $\sigma(a) = \pi(a)$ for all $a \in \mathcal{O}$. □

Lemma. *If π is a product of disjoint cycles of order $n_1, n_2, \ldots, n_k$ then*

$$o(\pi) = \operatorname{lcm}(n_1, \ldots, n_k).$$

Proof. Let $\pi = \prod_{i=1}^{k} \sigma_i$ with σ_i pairwise disjoint and $o(\sigma_i) = n_i$. Then $\pi^n = \prod_{i=1}^{k} \sigma_i^n$ since the σ_i commute. Moreover $\pi^n = \mathrm{id}$ if and only if $\sigma_i^n = \mathrm{id}$ for each i.[76] Thus $\pi^n = \mathrm{id}$ if and only if n_i divides n for each $i = 1, \ldots k$ i.e. if and only if n is a multiple of the lcm of the n_i. □

6.2. **Permuations as products of transpositions.**

Definition. We call a cycle of order 2 a *transposition*.

Lemma. *Every $\pi \in S_n$ is a product of transpositions.*

Proof. By the decomposition into disjoint cycles theorem it suffices to show that every cycle is a product of transpositions. One may easily check that

$$(a_1 a_2 \cdots a_k) = (a_1 a_2)(a_2 a_3) \cdots (a_{k-1} a_k).$$

□

Remark. The representation of π as a product of transpositions is not unique. For example

$$(12)(23)(34) = (1234) = (14)(13)(12).$$

Despite the remark it is true that $\pi \in S_n$ cannot be written both as a product of an even number of transpositions and as a product of an odd number of transpositions.

Theorem. *Given $\pi \in S_n$ let $l(\pi)$ be the number of orbits of $\langle\pi\rangle$ in $[n]$. For any $\pi \in S_n$ and any transposition $(ab) \in S_n$,*

$$l(\pi(ab)) = l(\pi) \pm 1.$$

Proof. Suppose first that a and b are in the same orbit in π. i.e. when we write π as a product of disjoint cycles $\pi = \prod_{i=1}^{m} \sigma_i$ one of the $\sigma_j = (a x_2 \cdots x_k b x_{k+2} \cdots x_m)$. Then we can compute

$$\sigma_j(ab) = (a x_{k+2} \cdots x_m)(b x_2 \cdots x_k).$$

Since a and b don't appear in any of the other cycles we see that $l(\pi(ab)) = l(\pi) + 1$ in this case.[77]

Suppose instead that a and b lie in different orbits in π. Then some $\sigma_i = (a x_2 \cdots x_k)$ and some $\sigma_j = (b y_2 \cdots y_l)$.[78] Then $\sigma_i \sigma_j (ab) = (a y_2 \cdots y_l b x_2 \cdots x_k)$. Since a and b don't appear in any of the other cycles we see that $l(\pi(ab)) = l(\pi) - 1$ in this case. [79]

Thus in every case $l(\pi(ab)) = l(\pi) \pm 1$. □

[76] This can be seen by considering what π^n does to the non-trivial orbit of each σ_i.

[77] The orbits are the same except that the orbit containing a and b splits into two.

[78] If a or b is fixed by π this isn't quite true but one can check that the argument can be adapted is a straightforward manner by allowing 'cycles of length 1' in the decomposition.

[79] This time the orbits are the same except that the orbits containing a and b are joined together.

Corollary. *There is a well-defined group homomorphism*

$$\epsilon\colon S_n \to (\{\pm 1\}, \cdot)$$

such that $\epsilon(\pi) = 1$ *if* π *is a product of an even number of transpositions and* $\epsilon(\pi) = -1$ *if* π *is a product of an odd number of transpositions. Moreover, for* $n \geqslant 2$, $\operatorname{Im}\epsilon = \{\pm 1\}$.

Proof. Suppose that $\pi = \prod_{i=1}^{m} \tau_i$ with each τ_i a transposition. By induction on m we see that $l(\pi) \equiv l(\mathrm{id}) + m \mod 2$. Thus if also $\pi = \prod_{i=1}^{m'} \sigma_i$ with each σ_i a transposition then $m \equiv m' \mod 2$ and the function ϵ as defined in the statement makes sense.[80] It is now easy to verify that ϵ is a homomorphism and, since $\epsilon((12)) = -1$, that $\operatorname{Im}\epsilon = \{\pm 1\}$ for $n \geqslant 2$. □

Definition. Given $\pi \in S_n$ we say that π is *even* if $\epsilon(\pi) = 1$ and that π is *odd* if $\epsilon(\pi) = -1$.

Remark. Notice that a cycle of odd order is even and a cycle of even order is odd.[81]

Definition. The *alternating group* on $[n]$, $A_n := \ker(\epsilon\colon S_n \to \{\pm 1\})$ is the normal subgroup of S_n consisting of all even permutations.

Since $|S_n| = n!$ it follows easily from the isomorphism theorem that, for $n \geqslant 2$, $|A_n| = \frac{n!}{2}$.

Lecture 23

6.3. **Conjugacy in S_n and in A_n.** We now try to understand the conjugacy classes in S_n and in A_n. In S_n they have a remarkably simple description.

Lemma. *If* $\sigma \in S_n$ *and* $(a_1 \cdots a_m)$ *is a cycle then*

$$\sigma(a_1 \cdots a_m)\sigma^{-1} = (\sigma(a_1) \cdots \sigma(a_m)).$$

Proof. If $a \in [n]$ then consider

$$\sigma(a_1 \cdots a_m)\sigma^{-1}(\sigma(a)) = \begin{cases} \sigma(a_{i+_m 1}) & \text{if} \quad a = a_i \\ \sigma(a) & \text{if} \quad a \notin \{a_1, \ldots, a_m\}. \end{cases}$$

Since every $b \in [n]$ is uniquely $\sigma(a)$ for some $a \in [n]$ we deduce that

$$\sigma(a_1 \cdots a_m)\sigma^{-1}(b) = (\sigma(a_1) \cdots \sigma(a_m))(b)$$

for all $b \in [n]$ as claimed. □

Theorem (Conjugacy classes in S_n)**.** *Two elements of* S_n *are conjugate if and only if they are the product of the same number of disjoint cycles of each length.*[82]

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[80] This is really the content of the corollary. Without the Theorem, or something like it, this is not at all clear.

[81] This is just another of those frustrating facts of life.

[82] We sometimes say that they have the same *cycle type*.

Proof. The lemma tells us that if $\pi = \prod_{i=1}^{k}(a_{i1}a_{i2}\cdots a_{im(i)})$ is a decomposition of π into a product of disjoint cycles then

$$\begin{aligned}\sigma\pi\sigma^{-1} &= \prod_{i=1}^{m}(\sigma(a_{i1}a_{i2}\cdots a_{im(i)})\sigma^{-1}) \\ &= \prod_{i=1}^{m}(\sigma(a_{i1})\sigma(a_{i2})\cdots\sigma(a_{im(i)}).\end{aligned}$$

Since the right-hand-side is a product of disjoint cycles we see that anything conjugate to π must be a product of the same number of disjoint cycles of the same length.

Suppose now that $\tau = \prod_{i=1}^{k}(b_{i1}b_{i2}\cdots b_{im(i)})$ is a decomposition of τ into a product of disjoint cycles (with the same number of each length as for π). Then let

$$\{a_{01},\ldots,a_{0m(0)}\} = [n]\backslash\{a_{ij} \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant m(i)\}$$[83]

and

$$\{b_{01},\ldots b_{0m(0)}\} = [n]\backslash\{b_{ij} \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant m(i)\}.$$[84]

Then we define $\sigma(a_{ij}) = b_{ij}$ for $0 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m(i)$. Thus $\sigma \in S_n$ and $\tau = \sigma\pi\sigma^{-1}$ by the lemma. □

Example. Conjugacy classes in S_4

representative element	e	(12)	(12)(34)	(123)	(1234)
cycle type	1^4	2.1^2	2^2	3.1	4
number of elements in class	1	6	3	8	6
size of centraliser	24	4	8	3	4
sign	1	-1	1	1	-1

Corollary (Conjugacy classes in A_n)**.** *If $\pi \in A_n$ then its conjugacy class in A_n is equal to its conjugacy class in S_n if and only if $C_{S_n}(\pi)$ contains an odd element. Moreover if $C_{S_n}(\pi) \subset A_n$ then the conjugacy class of π in S_n is a union of two conjugacy classes in A_n of equal size.*

Proof. Under the conjugation action of A_n and S_n respectively $\operatorname{Orb}_{A_n}(\pi) \subset \operatorname{Orb}_{S_n}(\pi)$. It follows that it suffices to show that

$$|\operatorname{Orb}_{A_n}(\pi)| = \begin{cases} |\operatorname{Orb}_{S_n}(\pi)| & \text{if } C_{S_n}(\pi) \not\subset A_n \\ \frac{1}{2}|\operatorname{Orb}_{S_n}(\pi)| & \text{if } C_{S_n}(\pi) \subset A_n. \end{cases}$$

Moreover by the orbit-stabiliser theorem

$$|\operatorname{Orb}_{A_n}(\pi)| = \frac{|A_n|}{|C_{A_n}(\pi)|} = \frac{|S_n|}{2|(C_{A_n}(\pi))|} = |\operatorname{Orb}_{S_n}(\pi)|\frac{|C_{S_n}(\pi)|}{2|C_{S_n}(\pi)\cap A_n|}.$$

Thus if $C_{S_n}(\pi) \subset A_n$ then we see that $|\operatorname{Orb}_{A_n}(\pi)| = \frac{1}{2}|\operatorname{Orb}_{S_n}(\pi)|$.

Otherwise consider $\epsilon|_{C_{S_n}(\pi)}\colon C_{S_n}(\pi) \to \{\pm 1\}$. Since $\epsilon(C_{S_n}(\pi)) \neq \{1\}$ we see that $\operatorname{Im}\epsilon|_{C_{S_n}(\pi)} = \{\pm 1\}$ and

$$C_{S_n}(\pi)/C_{A_n}(\pi) = C_{S_n}(\pi)/\ker\epsilon|_{C_{S_n}(\pi)} \simeq \{\pm 1\}.$$

It follows by Lagrange that $|C_{S_n}(\pi)| = 2|C_{A_n}(\pi)|$ and $\operatorname{Orb}_{A_n}(\pi) = \operatorname{Orb}_{S_n}(\pi)$. □

[83] So $\sum_{i=0}^{n} m(i) = n$ and every element of $[n]$ is equal to precisely one a_{ij}.

[84] So every element of $[n]$ is also equal to precisely one b_{ij}.

Example (Conjugacy classes in A_4).

The even cycle types in S_4 are 1^4, 2^2 and 3.1. Now $(12) \in C_{S_4}(e)$ and $(12) \in C_{S_4}((12)(34))$ so the centralisers of elements of conjugacy classes of cycle type 1^4 and 2^2 contain elements of odd order. Thus these classes are the same in A_4 and S_4.

Since $C_{S_4}((123))$ has order 3 it must be generated by (123) and so it is equal to $C_{A_4}((123))$. Thus the conjugacy class with cycle type 3.1 splits into two parts of equal size.

representative element	e	$(12)(34)$	(123)	(132)
cycle type	1^4	2^2	3.1	3.1
number of elements in class	1	3	4	4
size of centraliser	12	4	3	3

Corollary. *A_4 has no subgroup of index 2.*

Proof. If $H \leqslant A_4$ were index 2 then it would be normal (Example Sheet 3 Q1) and so be a union of conjugacy classes (Example Sheet 3 Q3). But $|A_4|/2 = 12/2 = 6$ and H must contain id. No set of conjugacy classes including $\{e\}$ has a total of 6 elements. □

Lecture 24

6.4. **Simple groups.**

Definition. We say a non-trivial group G is *simple* if G has no normal subgroups except itself and its trivial subgroup.

Since if N is a normal subgroup of G one can view G as 'built out of' N and G/N, one way to try to understand all groups is to first understand all simple groups and then how they can fit together.

Example. If p is prime then C_p is simple since C_p has no non-trivial proper subgroups. These are the only abelian simple groups.

Theorem. *A_5 is simple.*

Proof. The cycle types of even elements of S_5 are 1^5, $2^2.1$, 3.1^2 and 5. These have 1, $5 \times 3 = 15$, $\binom{5}{2} \times 2 = 20$ and $4! = 24$ elements respectively — note that $1 + 15 + 20 + 24 = 60 = |A_5|$ so we have got them all.

The element (12) is in the centralisers of id, (12)(34) and (345) so the classes with cycle type 1^5, $2^2.1$, and 3.1^2 have centralisers containing odd elements and so these classes do not split as conjugacy classes in A_5.

Since there are 4! 5-cycles in S_5, $|C_{S_5}((12345))| = 5!/4! = 5$. But all powers of (12345) commute with (12345) so $C_{S_5}((12345)) = \langle(12345)\rangle \subset A_5$. Thus the class with cycle type 5 in S_5 splits into two classes of order 12 in A_5.

In summary,

representative element	e	$(12)(34)$	(123)	(12345)	$(12345)^2$
cycle type	1^5	$2^2.1$	3.1^2	5	5
number of elements in class	1	15	20	12	12
size of centraliser	60	4	3	5	5

Now if N is a non-trivial proper normal subgroup of A_5 then it is a union of conjugacy classes including the class 1^5. Moreover $|N|$ is a factor of 60 by Lagrange.

If we take the identity and one of the smallest non-trivial classes (of size 12) we already have 13 elements so such an N must have order 15, 20 or 30. It is straightforward to check that no union of conjugacy classes including $\{e\}$ can have these orders. □

The remainder of the course is non-examinable.

In fact we can prove a stronger result.

Theorem. *A_n is simple for all $n \geqslant 5$.*

Remark. A_4 is not simple since it has a normal subgroup of order 4 namely $V := \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. $A_3 \simeq C_3$ is simple, A_2 is trivial so not simple.

Proof. First we show that all 3-cycles are conjugate in A_n (for $n \geqslant 5$). Using our theorem about conjugacy classes in A_n this follows from the fact that $(45) \in C_{A_n}(123)$ is odd.

Next we show that A_n is generated by 3-cycles. We know that every element of A_n is a product of an even number of transpositions. So we must show that every product of two transpositions is also a product of 3-cycles. Suppose a, b, c and d are distinct. Then $(ab)(ab) = \mathrm{id}$, $(ab)(bc) = (abc)$ and $(ab)(cd) = (abc)(bcd)$. Thus A_n is indeed generated by 3-cycles.

Now it suffices to show that every non-trivial normal subgroup N of A_n contains a 3-cycle — if N contains one it must contain all and so it generates A_n and so it is A_n. To this end we pick $\pi \in N \backslash \{e\}$ with the most fixed points. We show that π is a 3-cycle.

If π has two transpositions in its decomposition as a product of disjoint cycles say π swaps a and b and also swaps c and d. Let $\sigma = (cde)$ for some $e \notin \{a, b, c, d\}$. Then $\tau := \pi^{-1}(\sigma\pi\sigma^{-1}) \in N$ since N is a normal subgroup of A_n. Moreover $\tau(a) = a$, $\tau(b) = b$, $\tau(e) = c \neq e$, and if $\pi(i) = i$ for $i \neq e$ then $\tau(i) = i$. Thus $\tau \neq \mathrm{id}$ and τ has more fixed points than π which contradicts our definition of π.

Now we suppose that $\langle\pi\rangle$ has an orbit of size at least 3 and is not a 3-cycle. Suppose $a, \pi(a), \pi^2(a), b, c$ are all distinct and not fixed by π. [85] Let $\sigma = (\pi^2(a)bc)$ and $\tau = (\sigma\pi\sigma^{-1})\pi^{-1} \in N$ since N is a normal subgroup of A_n this time we compute $\tau(\pi(a)) = \pi(a)$, $\tau(\pi^2(a)) = b \neq \pi^2(a)$, and if i is fixed by π then i is also fixed by τ. Thus $\tau \neq \mathrm{id}$ has more fixed points than π. □

A triumph of late 20th century mathematics was the classification of all finite simple groups. Roughly speaking this says that every finite simple group is either

- cyclic of prime order;
- an alternating group;
- a matrix group over a field of finite order (for example $PSL_n(\mathbb{Z}/p\mathbb{Z})$);
- one of 26 so-called sporadic simple groups the largest of which is known as 'the monster' and has approximately 8×10^{53} elements.

One first important step in the proof was a result by Feit and Thompson that there is no non-abelian simple group of odd order that first appeared as a circa 250 page paper in 1963. The first proof of the whole classification theorem was annouced in the early 1980s. It ran to over ten of thousand pages spread across a large number of journal articles by over 100 authors. It turned out not to be quite complete. In 2004 it appeared to experts to be complete.

[85] if there were precisely 4 points that π does not fix then π would be a 4-cycle which would be odd or a product of disjoint transpositions which we have already ruled out.

In this course we have seen a little of how symmetry can be understood using the language of groups. But even when considering only finite groups there is much more to learn.