

10. FILTRATIONS, GRADED RINGS AND COMPLETIONS

Definitions. A (\mathbb{Z}) -filtration on a ring A is a set of additive subgroups $(A_i)_{i \in \mathbb{Z}}$ such that

- $A_i \subset A_{i+1}$ for all $i \in \mathbb{Z}$;
- $A_i A_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}$;
- $1 \in A_0$;
- $A = \bigcup_{i \in \mathbb{Z}} A_i$.

We say A with such a filtration is a *filtered ring*.

The filtration is

- *separated* if $\bigcap A_i = 0$;
- *positive* if $A_i = 0$ for all $i < 0$;
- *negative* if $A_i = A$ for all $i \geq 0$.

We will be most interesting in negative filtrations. Notice that always A_0 is a subring of A and A_i is an ideal in A_0 for $i < 0$.

Examples.

- (1) If I is a proper ideal in A , $A_i = I^{-i}$ for $i < 0$ and $A_i = A$ for $i \geq 0$ is a negative filtration of A called the *I -adic filtration* of A .
- (2) A special case of (1), if $A = \mathbb{C}[X]$ and $\lambda \in \mathbb{C}$; $A_i = (X - \lambda)^{-i} \mathbb{C}[X]$ for $i < 0$ and $A_i = \mathbb{C}[X]$ for $i \geq 0$. So $A_i = \{\text{functions with a zero of order } \geq -i\}$.
- (3) If $A = \mathbb{C}(X)$ and $\lambda \in \mathbb{C}$; $A_i = \{\text{functions with a pole of order at most } i \text{ at } \lambda\}$.
- (4) If $A = \mathbb{Z}$ and p prime, $A_i = (p^{-i})$ for $i < 0$ and $A_i = \mathbb{Z}$ for $i \geq 0$.
- (5) If $A = \mathbb{C}[x_1, \dots, x_n]$, $A_i = 0$ for $i < 0$ and A_i is the span of all monomials of total degree at most i for $i \geq 0$.

Definitions. Given a filtered ring A and an A -module M , a *filtration* on M is a set of additive subgroups $(M_i)_{i \in \mathbb{Z}}$ of M such that

- $M_i \subset M_{i+1}$ for all $i \in \mathbb{Z}$;
- $A_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$;
- $\bigcup_{i \in \mathbb{Z}} M_i = M$.

The filtration is said to be *separated* if $\bigcap_{i \in \mathbb{Z}} M_i = 0$.

Example. If X is a generating set for M and $M_i = A_i X$ for all $i \in \mathbb{Z}$ is a filtration of M . In particular A is a filtered A -module.

Definition. We say a filtration is *good* if there are $m_1, \dots, m_r \in M$ and $k_1, \dots, k_r \in \mathbb{Z}$ such that $M_j = \sum_{i=1}^r A_{j-k_i} m_i$ for all $j \in \mathbb{Z}$.

Lemma. If M_n and M'_n are two good filtrations of M then there is $n_0 \in \mathbb{N}$ such that

$$M_{n-n_0} \subset M'_n \subset M_{n+n_0}$$

for all $n \in \mathbb{Z}$.

Proof. Let $m_1, \dots, m_r \in M$ and $k_1, \dots, k_r \in \mathbb{Z}$ such that $M_n = \sum_{i=1}^r A_{n-k_i} m_i$ for all $n \in \mathbb{Z}$. Now we can find $t_1, \dots, t_r \in \mathbb{N}$ such that $m_i \in M'_{k_i+t_i}$. Then if $t = \max(t_1, \dots, t_r)$ then $m_i \in M'_{k_i+t}$ for all i and $M_n \subset M'_{n-t} = \sum A_{n-k_i} m_i$ for all $n \in \mathbb{Z}$.

By symmetry also there is a t' such that $M'_n \subset M_{n-t'}$ for all $n \in \mathbb{Z}$. Finally taking $n_0 = \max(t, t')$ gives the result. \square

This result should be viewed as analogous to the statement that any two norms on a finite dimensional vector-space are Lipschitz equivalent.

Definition. A \mathbb{Z} -graded ring is a ring A together with a decomposition

$$A = \bigoplus_{i \in \mathbb{Z}} A_i$$

with A_i additive subgroups of A such that $A_i A_j \subset A_{i+j}$ for all $i, j \in \mathbb{Z}$.

Notice A_0 is a subring of A . We say $a \in A$ is *homogeneous* (hgs) if $a \in A_i$ for some i .

Example. $A = \mathbb{C}[x_1, \dots, x_n]$ and A_i is the span of all monomials of degree i .

Definition. Given a graded ring A , a *graded A -module* is an A -module M that decomposes as $M = \bigoplus_{i \in \mathbb{Z}} M_i$ such that $A_i M_j \subset M_{i+j}$ for all i, j .

Proposition. If A is a graded ring with $A_n = 0$ for all $n > 0$ then the following are equivalent:

- (i) A is Noetherian
- (ii) A_0 is Noetherian and A is a finitely generated A_0 -algebra.

Proof. ((ii) \Rightarrow (i)) follows from Hilbert's basis Theorem.

((i) \Rightarrow (ii)) If A is Noetherian then $A_0 \cong A / \bigoplus_{i < 0} A_i$ is Noetherian and $\bigoplus_{i < 0} A_i$ is a finitely generated ideal in A . Suppose x_1, \dots, x_k generate $\bigoplus_{i < 0} A_i$ as an ideal, then without loss of generality we may assume that the x_i are hgs of strictly negative degree. Let $B = A_0[x_1, \dots, x_k]$ a graded subring of A .

We claim that $B = A$. For this it suffices to show that $A_n \subset B_n$ for all $n \leq 0$. We do this by induction on $-n$. The case $n = 0$ is clear.

Suppose $n < 0$ and $A_{n+i} = B_{n+i}$ for all $i > 0$ and $a \in A_n$. Then there are $a_i \in A_{n - \deg x_i}$ such that $\sum a_i x_i = a$. Since $\deg x_i < 0$ for all i , each a_i is in $B_{n - \deg x_i}$ by the induction hypothesis. Thus $a \in B$ as required. \square

Definition. If A is a filtered ring, the *Rees ring* of A is the graded ring

$$\tilde{A} := \bigoplus_{i \in \mathbb{Z}} A_i t^i \subset A[t, t^{-1}].$$

If A is filtered I -adically then $\tilde{A} = B[t]$ where $B = \bigoplus_{i \leq 0} I^{-i} t^i$.

Then if A is Noetherian and $\{x_1, \dots, x_n\}$ is a generating set for I as an ideal then B is Noetherian since it is generated as an $B_0 = A$ -algebra by $\{x_i t\}$. Thus $\tilde{A} = B[t]$ is Noetherian.

Definition. If A is a filtered A -module, the *Rees module* of M is the graded \tilde{A} -module

$$\tilde{M} := \bigoplus_{i \in \mathbb{Z}} M_i t^i \subset M[t, t^{-1}].$$

Lemma. If A is a filtered ring and M is a filtered A -module then the filtration is good if and only if \tilde{M} is a finitely generated \tilde{A} -module.

Proof. (\Rightarrow) Suppose $m_1, \dots, m_r \in M$ and $k_1, \dots, k_r \in \mathbb{Z}$ are such that $M_n = \sum A_{n-k_i} m_i$. Then $m_i \in M_{k_i}$ and so $m_i t^{k_i} \in \tilde{M}$.

Let $N = \tilde{A} \cdot \{m_i t^{k_i}\} \subset \tilde{M}$. Then $N_n = \sum_{i=1}^r A_{n-k_i} m_i t^n = M_n$ and so $\tilde{N} = \tilde{M}$ and the latter is finitely generated.

(\Leftarrow) Suppose that \widetilde{M} is a finitely generated \widetilde{A} -module. Let $m_1 t^{k_1}, \dots, m_r t^{k_r}$ be a hgs generating set with $m_i \in M$. Then $\widetilde{M}_n = \sum A_{n-k_i} t^{n-k_i} m_i t^{k_i}$, ie $M_n = \sum A_{n-k_i} m_i$ as required. \square

Definitions. If M is a filtered A -module and $N \leq M$ is a submodule then

- $N_n := M_n \cap N$ defines the *submodule filtration* on N .
- $(M/N)_n := M_n + N/N$ defines the *quotient filtration* on M/N .

Theorem. *If A is a filtered ring with \widetilde{A} Noetherian and M is an A -module with a good filtration. Whenever N is a submodule of M the submodule filtration on N and the quotient filtration on M/N are also good.*

Proof. If N and M/N have the submodule/quotient filtrations respectively, then \widetilde{N} is an \widetilde{A} -submodule of \widetilde{M} and $(\widetilde{M/N}) \cong \widetilde{M}/\widetilde{N}$ as \widetilde{A} -modules.

Since the filtration on M is good, \widetilde{M} is a finitely generated \widetilde{A} -module and so \widetilde{N} and $(\widetilde{M/N})$ as finitely generated \widetilde{A} -modules. It follows that the filtrations are good. \square

Lemma. *Suppose that A is filtered I -adically and M is an A -module with a good filtration. Then $IM_n = M_{n-1}$ for all $n < 0$.*

Proof. Since the filtration is good \widetilde{M} is a finitely generated \widetilde{A} -module. If n is smaller than the degrees of all the elements in such a generating set then $\widetilde{M}_{n-1} = t^{-1}I\widetilde{M}_n$. The result follows. \square

Corollary (Artin-Rees Lemma). *Let A be a Noetherian ring, I an ideal of A and N a submodule of a finitely generated A -module M . Then there is $k \geq 0$ such that*

$$(I^{n+k}M) \cap N = I^n(I^kM \cap N)$$

for all $n \geq 0$.

Proof. If we define $M_n = I^{-n}M$ for $n < 0$ and $M_n = M$ for $n \geq 0$ then this gives a good filtration on M . It follows by the Theorem that the subspace filtration on N is good. By the Lemma it follows that for $k \gg 0$, $I^{k+1}M \cap N = I(I^kM \cap N)$. The result follows. \square

Definition. If A is a filtered ring and M is a filtered A -module we may define a topology on M by declaring a subset to be open precisely if it is a union of cosets $m + M_i$ with $m \in M$ and $i \in \mathbb{Z}$. We call this the *filtration topology*.

Note that if a filtration is positive the filtration topology is just the discrete topology.

A filtration is separated if and only if the filtration topology is Hausdorff — if the filtration is separated and $x, y \in M$ are distinct then there is an $i \in \mathbb{Z}$ such that $x + M_i \neq y + M_i$; conversely if the filtration is not separated every open neighbourhood of 0 will also contain any other element of $\bigcap_{i \in \mathbb{Z}} M_i$.

We may define a distance function on M by taking a real constant $c > 1$ and setting $d(x, y) = 0$ if $x - y \in \bigcap_{i \in \mathbb{Z}} M_i$ and $d(x, y) = c^{\inf\{i | x - y \in M_i\}}$ otherwise.

Exercise. Show that

- $d(x, y) = d(y, x)$ for all $x, y \in M$;
- $d(x, z) \leq \max(d(x, y), d(y, z))$ for all $x, y, z \in M$;

- $d(x, x) = 0$ for all x ;
- $(d(x, y) = 0 \Rightarrow x = y)$ if and only if the filtration is separated.

Example. If \mathbb{Z} is filtered p -adically and $c = p$ this is called the p -adic norm.

Lemma. Any two good filtrations M_n and M'_n of M have the same topology.

Proof. We've seen that there exists n_0 such that $M_{n-n_0} \subset M'_n \subset M_{n+n_0}$. □

Definition. Suppose M is a filtered A -module. A *Cauchy sequence* in M is a sequence $(m_n)_{n \geq 0}$ such that for every open set U containing 0 , there exists $c(U) \in \mathbb{N}$ such that if $i, j > c(U)$ then $m_i - m_j \in U$.

We say two Cauchy sequences (m_n) and (m'_n) are *equivalent* if for every open set U containing 0 , there exists $c(U) \in \mathbb{N}$ such that if $n > c(U)$ then $m_n - m'_n \in U$.

Exercise.

- (1) Show that equivalence in this sense is an equivalence relation.
- (2) Moreover if we define $[(m_n)]$ to be the equivalence class of (m_n) , and \widehat{M} to be the set of all equivalence classes, show that \widehat{M} is an abelian group under $[(m_n)] + [(m'_n)] = [(m_n + m'_n)]$.
- (3) Also show \widehat{A} becomes a ring under $[(x_n)][(y_n)] = [(x_n y_n)]$, \widehat{M} becomes an \widehat{A} -module under $[(x_n)][(m_n)] = [(x_n m_n)]$, and \widehat{A} is an A -algebra under $a \mapsto [(a)]$ with kernel $\bigcap_{n \in \mathbb{Z}} A_n$.
- (4) Finally show that \widehat{M} is an A -module and $M \rightarrow \widehat{M}: m \mapsto [(m)]$ is an A -module map with kernel $\bigcap_{n \in \mathbb{Z}} M_n$.

Definition. We call \widehat{A} the *completion* of A and \widehat{M} the *completion* of M . We say that A is *complete* if $A \rightarrow \widehat{A}$ is an isomorphism.

Remark. The completion of an A -module does not depend on the choice of good filtration.

Suppose that we have a Cauchy sequence $(x_n) \in M$. Then for every $i \in \mathbb{Z}$, $x_n + M_i$ is eventually constant. Thus we can define

$$\begin{aligned} \{\text{Cauchy sequences in } M\} &\rightarrow \prod_{i \leq 0} M/M_i \\ (x_n) &\mapsto (\text{final value of } x_n + M)_{i \leq 0}. \end{aligned}$$

It is easy to see this induces a map

$$\begin{aligned} \widehat{M} &\rightarrow \prod_{i \leq 0} M/M_i \\ (x_n) &\mapsto (\xi_i)_{i \leq 0}. \end{aligned}$$

where ξ_i denotes the eventually value of $x_n + M_i$. Thus $\xi_i + M_{i+1} = \xi_{i+1}$ for all i . Moreover, if $(\xi_i)_{i \leq 0} \in \prod_{i \leq 0} M/M_i$ with $\xi_i + M_{i+1} = \xi_{i+1}$ and for each i we choose $x_i \in M$ with $x_i + M_i = \xi_i$ then (x_i) will be Cauchy and its image is $(\xi_i)_{i \leq 0}$.

More generally,

Definition. We call a sequence $(X_n)_{n \leq 0}$ of abelian groups together with maps $\theta_n : X_n \rightarrow X_{n+1}$ ($n \leq -1$) an *inverse system*. The *inverse limit* of X is then

$$\varprojlim X_n := \{(x_n) \in \prod_{n \leq 0} X_n \mid \theta_n(x_n) = x_{n+1}\}$$

If we have another system $(Y_n)_{n \leq 0}$ and $\phi_n : Y_n \rightarrow Y_{n+1}$, $n \leq -1$, a morphism from X to Y is a sequence of maps $f_n \in \text{Hom}_{\mathbb{Z}}(X_n, Y_n)$ such that $f_{n+1}\theta_n = \phi_n f_n$, $\forall n \leq -1$. This will induce a map

$$\begin{aligned} \widehat{f}: \varprojlim X_n &\rightarrow \varprojlim Y_n \\ (x_n) &\mapsto (f_n(x_n)). \end{aligned}$$

If X, Y and Z are inverse systems and $(f_n) \in \text{Hom}(X, Y)$ and $(g_n) \in \text{Hom}(Y, Z)$ then we say $X \xrightarrow{(f_n)} Y \xrightarrow{(g_n)} Z$ is exact if $X_n \xrightarrow{f_n} Y_n \xrightarrow{g_n} Z_n$ is exact for all n .

Proposition. *If $0 \rightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \rightarrow 0$ is a short exact sequence of inverse systems then*

$$0 \rightarrow \varprojlim X_n \xrightarrow{\widehat{f}} \varprojlim Y_n \xrightarrow{\widehat{g}} \varprojlim Z_n$$

is exact. Moreover \widehat{g} is surjective if each map $\theta_n : X_n \rightarrow X_{n+1}$ is surjective.

Proof. Let $d_X : \prod X_n \rightarrow \prod X_n$ be given by $(x_n) \mapsto (x_n - \theta_{n-1}(x_{n-1}))$ so $\varprojlim X_n = \ker d_X$. Also define d_Y and d_Z similarly. Then

Exercise. Show that $0 \rightarrow \ker d_x \rightarrow \ker d_Y \rightarrow \ker d_Z \rightarrow \text{coker } d_x \rightarrow \text{coker } d_Y \rightarrow \text{coker } d_z \rightarrow 0$ is exact. (cf Ex Sheet 4 Q3)

Given this it suffices to prove that if θ_n is surjective then $\text{coker } d_X = 0$ i.e. d_X is surjective.

If $y_0 = 0$ then $\theta_{n-1}(y_{n-1}) = y_n - x_n$ has a solution inductively on $-n$ then $d_X(y_n) = x_n$. \square

Now suppose that M and N are A -modules with a good filtration and \widetilde{A} is Noetherian. Whenever $f \in \text{Hom}_A(M, N)$ the subspace filtration on $f(M)$ is good and so is topologically equivalent to the quotient filtrations from $M \rightarrow f(M)$. Thus $f : M \rightarrow f(M)$ and $f(M) \hookrightarrow N$ are continuous and so $f : M \rightarrow N$ is continuous. In particular f induces $\widehat{f} : \widehat{M} \rightarrow \widehat{N}$.

Theorem. *Let A be a filtered ring with \widetilde{A} Noetherian and let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be a short exact sequence of filtered A -modules with good filtrations. Then the induced sequence $0 \rightarrow \widehat{L} \rightarrow \widehat{M} \rightarrow \widehat{N} \rightarrow 0$ is exact.*

Proof. WLOG L has the subspace filtration from M and N has the quotient filtration from M . Then we get a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L/L_n & \longrightarrow & M/M_n & \longrightarrow & N/N_n \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L/L_{n+1} & \longrightarrow & M/M_{n+1} & \longrightarrow & N/N_{n+1} \longrightarrow 0 \end{array}$$

with exact rows. The result then follows from the proposition. \square

Proposition. *Suppose that A is a filtered ring with \widetilde{A} Noetherian and M is an A -module with a good filtration (in particular M is finitely generated) then $\widehat{A} \otimes_A M \rightarrow \widehat{M}$ given by $(a_n, m) \mapsto (a_n m)$ is an isomorphism..*

Proof. Let $0 \rightarrow K \rightarrow A^n \rightarrow M \rightarrow 0$ be a s.e.s of \widehat{A} -modules. Since $A \cong \widetilde{A}/(t-1)$, A is Noetherian and so K is a finitely generated A -module.

Give A^n , K and M good filtrations then

$$\begin{array}{ccccccc} \widehat{A} \otimes_A K & \longrightarrow & \widehat{A} \otimes_A A^n & \longrightarrow & \widehat{A} \otimes_A M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \widehat{K} & \longrightarrow & \widehat{A}^n & \longrightarrow & \widehat{M} \longrightarrow 0 \end{array}$$

commutes and has exact rows. Moreover the middle vertical arrow is an isomorphism. Since the composite $\widehat{A} \otimes_A A^n \rightarrow \widehat{A}^n \rightarrow \widehat{M}$ is surjective, the map $\widehat{A} \otimes_A M \rightarrow \widehat{M}$ must also be surjective.

But then also $\widehat{A} \otimes_A K \rightarrow \widehat{K}$ is also surjective since K satisfies the same conditions as M did. The result then follows from the five lemma (Ex Sheet 4 Q2). \square

Theorem (Krull's Theorem). *If A is Noetherian and filtered I -adically and M is a f.g. A -module with a good filtration then*

$$\ker(M \rightarrow \widehat{M}) = \{x \in M \mid ax = 0 \text{ some } a \in 1 + I\}.$$

Proof. With loss of generality M is filtered by $M_n = I^{-n}M$ for $n \leq 0$ and $M_n = M$ for $n > 0$. Then $\ker(M \rightarrow \widehat{M}) = \bigcap_{n \geq 0} I^n M$.

Now by the Artin-Rees Lemma $\widehat{I} \cdot (\bigcap I^n M) = \bigcap I^n M$. Since this module is finitely generated we may choose a generating set x_1, \dots, x_r . Then we may find $(a_{ij}) \in I$ such that $x_i = \sum a_{ij} x_j$ for all i . Now $\det(\delta_{ij} - a_{ij}) x_i = 0$ by Cramer's rule. Since $\det(\delta_{ij} - a_{ij}) \in 1 + I$ we get $\ker(M \rightarrow \widehat{M}) \subset \{x \in M \mid ax = 0 \text{ some } a \in 1 + I\}$. But if $(1 - a)x = 0$ some $a \in I$ then $a^n x = x$ for all n and so $x \in \bigcap I^n M$. \square

Corollary. *If $S = 1 + I \subset A$ is a m.c. subset then $\ker(M \rightarrow \widehat{M}) = \ker(M \rightarrow M_S)$. Moreover $A_S \hookrightarrow \widehat{A}$ so $\ker(M_S \rightarrow \widehat{M}) = 0$.*

We say \widehat{A} is *faithfully flat* over A_S .

Corollary. *If A is a Noetherian integral domain and I is an ideal in A then $\bigcap I^n = 0$.*

Proof. $1 + I$ has no zero-divisors so $\ker(A \rightarrow \widehat{A}) = 0$. \square

Corollary. *If A is Noetherian and $I \subset \text{Jac}(A)$ and M is a f.g. A -module then the I -adic topology on M is Hausdorff.*

Proof. $1 + I \subset A^\times$. \square