7.2. **The Picard group of a ring.**

Definition. A line bundle over a ring A is a finitely generated projective A-module such that the rank function $\operatorname{Spec} A \to \mathbb{N}$ is constant with value 1. We call A itself the trivial line bundle.

We want to put an abelian group structure on the set of isomorphism classes of line bundles over A. The product will be given by tensor product and the inverse by a kind of duality. We know that $\otimes$ is associative and commutative and the trivial line bundle is an identity up to isomophism in each case. But to show that the group operation is well-defined and we have inverses we need to prove a few results.

Definition. If X is a f.g. projective module, we define its dual by

$$X^* := \operatorname{Hom}_A(X, A).$$

Lemma. *Let X and Y be f.g. projective A-modules*

(1) *$X \otimes_A Y$ is a f.g. projective A-module and $\operatorname{rk}_{X\otimes_A Y} = \operatorname{rk}_X . \operatorname{rk}_Y$ pointwise;*
(2) *X^* is a f.g. projective A-module and $\operatorname{rk}_{A^*} = \operatorname{rk}_A$;*
(3) *If X is a line bundle then $X^* \otimes_A X \cong A$.*

Proof. Since X is projective there exists an A-module Z such that $X \oplus Z \cong A^n$ is free. Then $(X \otimes_A Y) \oplus (Z \otimes_A Y) \cong A^n \otimes_A Y \cong Y^n$. Since Y is f.g. projective, Y^n is f.g. projective and so $X \otimes_A Y$ is a direct summand of a f.g projetive and so is a f.g. projective.

Next we compute the ranks: if P in Spec(A), then $X_P \cong A_P^m$ and $Y_P \cong A_P^n$ say, then $(X \otimes_A Y)_P \cong X_P \otimes_{A_P} Y_p \cong A_P^{nm}$ and (1) follows.

Similarly $\operatorname{Hom}_A(X, A) \oplus \operatorname{Hom}_A(Z, A) \cong \operatorname{Hom}_A(A^n, A) \cong A^n$ and so X^* is f.g. projective.

Also as X is finitely presented, $(X^*)_P \cong (X_P)^* \cong A_P^n$ and (2) follows.

Finally if $m = 1$, let $\theta\colon X^* \otimes X \to A; \theta(f, x) = f(x)$. Locally we have

$$\theta_P\colon \operatorname{Hom}_{A_P}(A_P, A_P) \cong \operatorname{Hom}_A(X, A)_P \to A_P$$

is an isomorphism and so θ is an isomorphism. □

Definition. If A is a ring we may define the *Picard group* of A, Pic(A) to be the set of isomorphism classes of line bundles over A, with multiplication given by $\otimes$ and inverses given by $X \mapsto X^*$.

Corollary. (Pic$(A), \otimes$) *is an abelian group.*

Examples.

(1) If A is a local ring then all line bundles are trivial, and so $\operatorname{Pic}(A) \cong 0$.
(2) If $\mathcal{O}$ is the ring of integers of a number field then (as we will see) Pic$(\mathcal{O})$ is just the ideal class group of $\mathcal{O}$

Exercise. Pic is a functor from rings to abelian groups; more specifically if $f\colon B \to C$ is a ring homomorphism then there is a group homomorphism $\operatorname{Pic}(f)\colon \operatorname{Pic}(A) \to \operatorname{Pic}(B)$ given by $X \mapsto C \otimes_B X$ and if also $g\colon A \to B$ is a ring homomorphism then $\operatorname{Pic}(fg) = \operatorname{Pic}(f)\operatorname{Pic}(g)$.

As hinted in Example 2, when A is an integral domain, the Picard group of A is something that arises classically in a familiar way: Since Spec(A) is connected in this case a f.g. projective A-module is a line bundle if and only if $Q(A) \otimes_A X \cong$

$Q(A)$; that is $\mathrm{rk}_X((0)) = 1$. Indeed since projective modules are flat (Ex. Sheet 2 Q2) the exact sequence $0 \to A \to Q(A)$ induces an injection

$$X \cong A \otimes_A X \to Q(A) \otimes_A X \to Q(A).$$

Thus X is isomorphic to a finitely generated submodule of $Q(A)$.

Definition. A *fractional ideal* I of A is a non-zero A-submodule of $Q(A)$ that is contained in a cyclic A-submodule of $Q(A)$.

In particular every finitely generated A-submodule of $Q(A)$ is a fractional ideal. It is easy to see that if I and J are both fractional ideals of A then IJ is also fractional ideal. Thus the set $\mathrm{Frac}(A)$ of fractional ideals in A is a commutative monoid with identity given by A.

Definitions. We call a fractional ideal *invertible* or a *Cartier divisor* if it has an inverse in $\mathrm{Frac}(A)$. We call the set of Cartier divisors $\mathrm{Cart}(A)$ — an abelian group.

If $f \in Q(A)^\times$, then the fractional ideal Af is a Cartier divisor with inverse Af^{-1}. We call these divisors *principal divisors* of A.

Now if $I, J \in \mathrm{Cart}(A)$ with $IJ = A$ then there are $x_1, \dots, x_n$ in I and $y_1, \dots, y_n$ in J such that $\sum x_i y_i = 1$. So we can define maps $f \colon I \to A^n$ by $f(a) = (ay_1, \dots, ay_n)$ and $g \colon A^n \to I$ by $g(e_i) = x_i$ and $gf(a) = a \sum x_i y_i = a$. Thus I is a summand of A^n and so is a f.g. projective A-module. Morover $Q(A) \otimes_A I \cong Q(A)$ and so I is a line bundle. Thus all Cartier divisors are line bundles.

Now suppose I and J are any Cartier divisors in A, we want to show that $IJ \cong I \otimes_A J$ and so the map $\mathrm{Cart}(A) \to \mathrm{Pic}(A)$ sending a Cartier divisor to its isomorphism class is a group homomorphism. Since I is projective it is flat (Q2 of example sheet 2), so $I \otimes_A J \to I \otimes_A Q(A) \cong Q(A)$ is an injection and sends $x_i \otimes y_i$ to $x_i y_i$. Thus its image is the set IJ and $IJ \cong I \otimes_A J$ as claimed.

We've seen that every line bundle is isomorphic to a fractional ideal. In fact the fractional ideal must be invertible: suppose X is a line bundle isomorphic to the fractional ideal I. Then $X^* = \mathrm{Hom}_A(X, A) \cong J$ for some fractional ideal J. Then $A \cong X \otimes X^* \cong IJ$ so $IJ = Af$ some $f \in Q(A)^\times$ is a principal divisor. Thus $I(Jf^{-1}) = A$ and I is in $\mathrm{Cart}(A)$.

Suppose now that $I \in \ker(\mathrm{Cart}(A) \to \mathrm{Pic}(A))$. Then $I \cong A$ and so I is a principal divisor.

Thus we have proven the following proposition.

Proposition. *If A is a integral domain, then every Cartier divisor is a line bundle, moreover there is an exact sequence of abelian groups*

$$1 \to A^\times \to Q(A)^\times \to \mathrm{Cart}(A) \to \mathrm{Pic}(A) \to 0.$$

In particular the first non-trivial map is inclusion, the second maps f to the Cartier divisor fA, and the third just realises a Cartier divisor as a line bundle. Thus $\mathrm{Pic}(A) \cong \mathrm{Cart}(A)/($*principal divisors of* $A)$.

Definition. We say a ring is a *Dedekind domain* if every fractional ideal is invertible.

We'll see other ways to characterise Dedekind domains later.

Examples.

(1) Every principal ideal domain is a Dedekind domain. Clearly, $\mathrm{Pic}(A) = 0$ in this case. In fact a Dedekind domain is a principal ideal domain if and only if $\mathrm{Pic}(A) = 0$.
(2) If K is an algebraic number field and $\mathcal{O}$ is its ring of algebraic integers, then $\mathcal{O}$ is a Dedekind domain and $\mathrm{Pic}(\mathcal{O})$ is known as the *ideal class group.* It is known that this group is always finite. Understanding this group is very important in algebraic number theory.
(3) If X is a smooth affine curve over $\mathbb{C}$, then the coordinate ring $\mathbb{C}[X]$ of X is a Dedekind domain. In particular if X is obtained by removing a single point from a smooth projective curve (Riemann surface) $\overline{X}$, then $\mathrm{Pic}(\mathbb{C}[X])$ is known as the Jacobian variety of $\overline{X}$ and is known to be isomorphic to $(\mathbb{R}/\mathbb{Z})^{2g}$ where g is the genus of $\overline{X}$.
(4) (Claborn 1966) Incredibly every abelian group arises as the Picard group of some Dedekind domain.

8. Integral extensions

8.1. **Integral dependence.** Suppose that A and B are rings, with A a subring of B. We say an element of B is integral over A if x is a root of a *monic* polynomial with coefficients in A.

Examples. If K is an algebraic extension of $\mathbb{Q}$ then the set of integral elements of K over $\mathbb{Z}$ is by definition $\mathcal{O}_K$, the algebraic integers of K. In particular an element of $\mathbb{Q}$ is integral over $\mathbb{Z}$ if and only if it is an integer. (Exercise if you haven't seen this: cf Numbers and Sets sheet 3 2006).

Proposition. *Suppose A is a subring of B and x is an element of B. Write $A[x]$ for the subring of B generated by x and A. The following are equivalent:*

(i) x is integral over A;
(ii) $A[x]$ is a finitely generated A-module;
(iii) $A[x]$ is contained in a subring of B that is a finitely generated A-module;
(iv) There is an $A[x]$-module M with $\mathrm{Ann}_{A[x]}(M) = 0$ which is finitely generated as an A-module.

Proof. (i) $\Longrightarrow$ (ii): if x is a root of monic polynomial f of degree n, then also of $x^r.f$ for each $r \geq 0$. Thus x^{n+r} is in the A-module generated by $1, x, \ldots, x^{n+r-1}$ for each $r \geq 0$. Inductively we see that $A[x]$ is generated by $1, \ldots, x^{n-1}$.

(ii) implies (iii): is clear: $A[x]$ is a subring of B with the required properties.

(iii) implies (iv): Let C be the subring of B given by (iii). We may consider C is a finitely generated A-module by definition. Moreover if $x \in \mathrm{Ann}_{A[x]}(C)$ then $x.1 = 0$ so $x = 0$.

(iv) implies (i): Let $m_1, \ldots, m_n$ be a generating set for M as an A-module. Then certainly we can find a_{ij} in A such that $x.m_i = \sum_{j=1}^n a_{ij}m_j$. Thus the matrix $B = (b_{ij})$ with $A[x]$ coefficients given by $xI - (a_{ij})$ satisfies $\sum_{j=1}^n b_{ij}m_j = 0$, it follows (by Cramer's rule) that $\det B$ acts as 0 on each m_j, and so on the whole of M. But in $A[x]$ only 0 can act as 0 on M so $\det B = 0$. But $\det B$ is a monic polynomial in x with coefficients in A, and so x is integral over A as required. □

Lemma. *If $A \subset B \subset C$ is a chain of subrings such that B is a finitely generated A-module and C is a finitely generated B-module, then C is a finitely generated A-module.*

Proof. If $b_1, \ldots, b_m$ is a generating set for B as an A-module and $c_1, \ldots, c_n$ is a generating set for C as a B-module then it is easy to check that $\{b_i c_j\}$ is a generating set for C as an A-module. □

Corollary. *The set C of elements of B that are integral over A is a subring of B.*

Definitions.

- We call the ring C the *integral closure* of A in B.
- We say A is *integrally closed* in B if $A = C$.
- We say B is *integral over* A if $C = B$.

Proof of Corollary. It suffices to show that if x, y are in C then $x \pm y$ and xy are in C. Suppose that x, y are in C and consider the chain of rings $A \subset A[x] \subset A[x][y] \subset B$. We have $A[x]$ is a finitely generated A-module. Moreover as y is integral over A then also over $A[x]$, and so $A[x][y]$ is finitely generated as an $A[x]$-module. It is now follows from the lemma that $A[x][y]$ is a finitely generated A-module. Thus every element of $A[x][y]$ is integral over A by the lemma. In particular $x \pm y$ and xy are integral. □

Definition. We say that an integral domain A is *integrally closed* or *normal* if it is integrally closed in its field of fractions, $Q(A)$.

In particular the ring of integers $\mathcal{O}_K$ of a number field is integrally closed.

Geometrically, an integral domain A is normal means that the singularities of Spec(A) all lie in codimension 2. In particular if Spec(A) is an algebraic curve then A is normal means that the curve is non-singular.

Exercise. Suppose A is a subring of B and $x_1, \ldots, x_n$ in B are each integral over A, then $A[x_1, \ldots, x_n]$, the subring of B generated by A and each x_i is finitely as an A-module. [Hint the result for $n = 2$ is the content of the proof of the last result].

Corollary. *If $A \subset B \subset C$ is a chain of subrings and C is integral over B and B is integral over A then C is integral over A.*

Proof. Suppose $x \in C$ then as x is integral over B we may find an equation $x^n + b_1 x^{n-1} + \cdots + b_n = 0$ with all the b_i in B. If we let D be the subring of C generated by A, $b_1, \ldots, b_n$ we see that D is a finitely generated A-module by the exercise. Moreover, $D[x]$ is a finitely generated D-module since x is integral over D by the equation. So by the lemma above $D[x]$ is a finitely generated A-module. Thus x is integral over A as required. □

Corollary. *If A is a subring of B then the integral closure of A in B is integrally closed in B.*

Proof. Let C be the integral closure of A in B and suppose x in B is integral over C. Then $C[x]$ is integral over A by the above and so x is in C. □

Exercise. Suppose that A is a subring of B, and B is integral over A. Then

(1) If I is an ideal in B then B/I is integral over $A/(I \cap A)$.
(2) If S is a m.c. subset of A then B_S is integral over A_S.

In fact we can say more than this:

Proposition. *If A is a subring of B, S is a m.c. subset of A and C is the integral closure of A in B then C_S is the integral closure of A_S in B_S.*

Proof. By the exercise we have C_S is integral over A_S.

Suppose that an element x in B_S is integral over A_S so we have an equation

$$x^n + r_1x^{n-1} + \cdots + r_n = 0 \in B_S$$

with $r_i \in A_S$.

Multiplying through by the product s of the denominators of x and the r_i we see that $\iota(y) = sx$ satisfies a monic polynomial with coefficients in $\iota(A)$. So there is a $t \in S$ such that ty satisfies a monic polynomial with coefficients in A. Thus ty is integral over A and so in C and $x = (st)^{-1}\iota(y) \in C_S$ as required. □

Now, we can show that to be integrally closed is a local property:

Theorem. *If A is an integral domain then the following are equivalent:*

(i) A is integrally closed;
(ii) A_P is integrally closed for every prime ideal P in A;
(iii) $A_{\mathfrak{m}}$ is integrally closed for every maximal ideal $\mathfrak{m}$ in A.

Proof. Notice that $Q(A) = Q(A_P)$ for every prime ideal P, so (i) implies (ii) follows from the above proposition.

(ii) implies (iii) is trivial as usual.

For (iii) implies (i), let C be the integral closure of A in $Q(A)$ and let $\iota : A \to C$ be the inclusion. For each maximal ideal $\mathfrak{m}$ in A, $C_{\mathfrak{m}}$ is the integral closure of $A_{\mathfrak{m}}$ in $Q(A)$ so by the above proposition again, we have $\iota_{\mathfrak{m}} : A_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is an isomorphism by assumption. Since isomorphism is a local property of module maps we can deduce that ι is also an isomorphism and $A = C$ as required. □

8.2. **Going-up and Going-down Theorems.**

Proposition. *Suppose that $A \subset B$ are rings with B integral over A. Let $\mathfrak{p} \in$ Spec(B) and $P = \mathfrak{p} \cap A$. Then P is maximal if and only if $\mathfrak{p}$ is maximal. That is the natural map* Spec$(B) \to$ Spec(A) *sends closed points to closed points and the fibres of a closed point are all closed.*

Proof. By a previous exercise we know that $B/\mathfrak{p}$ is integral over A/P. So we may assume that A and B are integral domains, $\mathfrak{p} = 0$ and need to show that A is a field if and only if B is a field.

Suppose that A is a field and $b \in B$ is non-zero. We want to show that b is a unit in B. We know that b satisfies a polynomial

$$b^n + a_1b^{n-1} + \cdots + a_n = 0$$

with $a_i \in A$. Since B is an integral domain by dividing by b if necessary, we may assume that $a_n \neq 0$. But then $(b^{n-1} + a_1b^{n-2} + \cdots a_{n-1})b = -a_n$ is a unit in A thus b is a unit in B.

Conversely, suppose that B is a field, and $x \in A$ is non-zero. Now x is invertible in B so we may write

$$x^{-m} + a_1x^{-m+1} + \cdots + a_m = 0$$

with $a_i \in A$. Thus $x^{-1} = -(a_1 + a_2x + \cdots a_mx^{m-1}) \in A$ as required. □

Corollary. *If $A \subset B$ are rings with B integral over A then the natural map $\iota^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ preserves strict inclusions and is surjective.*

Proof. Let P be in $\mathrm{Spec}(A)$. Then $B_P = A_P \otimes_A B$ is integral over A_P and A_P has a unique maximal ideal P_P. Thus by the proposition above, $\iota_P(\mathfrak{p}_P) = P_P$ if and only if $\mathfrak{p}_P$ is a maximal ideal of B_P. Thus for $\mathfrak{p} \in \mathrm{Spec}(B)$, $\iota^*(\mathfrak{p}) = P$ if and only if $\mathfrak{p}_P$ is a maximal ideal in B_P.

Both parts follow easily since $\mathrm{Spec}(B_P)$ is in natural order-preserving 1-1 correspodance with points of $\mathrm{Spec}(B)$ that meet $A\backslash P$ trivially: for the first, if $\mathfrak{p} \subset \mathfrak{q}$ are in $\mathrm{Spec}(B)$ and both map to P in $\mathrm{Spec}(A)$ then $\mathfrak{p}_P$ and $\mathfrak{q}_P$ are both maximal and so equal. Hence $\mathfrak{p} = \mathfrak{q}$. Finally, for each $P \in \mathrm{Spec}(A)$, B_P has a maximal ideal which lifts to a prime ideal $\mathfrak{p} \in \mathrm{Spec}(B)$ and then $\iota^*(\mathfrak{p}) = P$ by construction. □

Suppose that we have a ring homomorphism $f\colon A \to B$.

Definition. The *going-up theorem* is said to hold for f if for every pair of primes $P \subset Q$ in $\mathrm{Spec}(A)$ and every $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $f^*(\mathfrak{p}) = P$ there is $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $\mathfrak{p} \subset \mathfrak{q}$ and $f^*(\mathfrak{q}) = Q$.

Definition. Similarly the *going-down theorem* is said to hold for f if for every pair of primes $P \subset Q$ in $\mathrm{Spec}(A)$ and every $\mathfrak{q} \in \mathrm{Spec}(B)$ such that $f^*(\mathfrak{q}) = Q$ there is $\mathfrak{p} \in \mathrm{Spec}(B)$ such that $\mathfrak{p} \subset \mathfrak{q}$ and $f^*(\mathfrak{p}) = P$.

Theorem. *If $A \subset B$ are rings and B is integral over A then the going-up theorem holds for the natural inclusion map.*

Proof. Let $P \subset Q$ in $\mathrm{Spec}(A)$ and $\mathfrak{p} \in \mathrm{Spec}(B)$ with $\iota^*(\mathfrak{p}) = P$. By the exercise above $B/\mathfrak{p}$ is integral over A/P. Thus by the corollary above, there is $\mathfrak{q}/\mathfrak{p} \in \mathrm{Spec}(B/\mathfrak{p})$ such that $\iota^*(Q/P) = \mathfrak{q}/\mathfrak{p}$. The result follows easily. □

The proof of the going-down theorem is a little more subtle, and depends on the following fact that we will not prove:

Theorem. *Suppose that A is an integrally closed integral domain, F is a normal field extension of $Q(A)$ (in the sense of Galois theory) and B is the integral closure of A in F. Then $\mathrm{Aut}(F/Q(A))$ acts transitively on $\{\mathfrak{p} \in \mathrm{Spec}(B) | \iota^*(\mathfrak{p}) = P\}$ for each $P \in \mathrm{Spec}(A)$.*

Proof. Omitted □

Theorem. *If $A \subset B$ are rings, A is integrally closed and B is integral over A then the going-down theorem holds for the natural inclusion map.*

Proof. Let $P \subset Q$ in $\mathrm{Spec}(A)$ and $\mathfrak{q} \in \mathrm{Spec}(B)$ with $\iota^*(q) = Q$.

Let F be the normal closure of $Q(B)$ over $Q(A)$ and let C be the integral closure of A in F. Since C is integral over A, $\iota^*\colon \mathrm{Spec}(C) \to \mathrm{Spec}(A)$ is surjective, so there is an $X \in \mathrm{Spec}(C)$ such that $X \cap A = \iota^*(X) = P$. Since also B is integral over A, there is a $Y \in \mathrm{Spec}(C)$ such that $Y \cap B = \mathfrak{q}$. By the going-up theorem for the exentension $A \subset C$ we may also find $Z \in \mathrm{Spec}(C)$ such that $Z \supset X$ and $Z \cap A = Q$.

Now $Z \cap A = Y \cap A = Q$ so by the unproved theorem the is a $\sigma \in \mathrm{Aut}(F/Q(A))$ such that $\sigma(Z) = Y$. Now $\sigma(Z) \supset \sigma(X)$ and so $\mathfrak{q} = \sigma(Z) \cap B \supset \sigma(X) \cap B$. Thus if we take $\mathfrak{p} = \sigma(X) \cap B$ we have $\mathfrak{p} \cap A = \sigma(X) \cap A = X \cap A = P$ and we are done. □

8.3. **Dimension.** We are now ready to discuss the (Krull) dimension of a ring.

Definition. Given a ring A and $P \in Spec(A)$ we define the *height* of P to be

$$\mathrm{ht}(P) = \sup\{n | \text{there is a chain in } \mathrm{Spec}(A)\ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P\}$$

Examples. (1) A minimal prime ideal has height zero.
(2) If A is an integral domain then a minimal non-zero prime has height one.

Definition. We define *Krull dimension* of A by

$$\text{K-dim}(A) := \sup\{\mathrm{ht}(P) | P \in \mathrm{Spec}(A)\}.$$

Remark. Notice that we always have $\mathrm{ht}(P) = \text{K-dim}(A_P)$.

We'll see later that if A is Noetherian then every prime ideal has finite height but that it is not true that every Noetherian ring has finite Krull dimension.

Geometrically, the height of a prime ideal P corresponds to the maximal length of a chain of irreducible subvarities of $\mathrm{Spec}(A)$ containing $V(P)$. Thus we think of $\mathrm{ht}(P)$ as being the codimension of P in the largest irreducible component of $\mathrm{Spec}(A)$ containing P.

Notice that Example Sheet 2 Q11 says a Noetherian ring has Krull dimension 0 if and only if it is Artinian.

Lemma. *If $A \subset B$ are rings and B is integral over A then* $\text{K-dim}(B) = \text{K-dim}(A)$.

Proof. We've already seen that $\iota^*\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is strictly order-preserving when B is an integral extension of A and so $\text{K-dim}(A) \geq \text{K-dim}\, B$. Conversely, it follows from the Going-up theorem and the surjectivity of ι^* for integral extensions that every chain of prime ideals in A can be lifted to a chain of prime ideals in B; so $\text{K-dim}(B) \geq \text{K-dim}(A)$. □