6.4. **Examples of Noetherian rings.** So far the only rings we can easily prove are Noetherian are principal ideal domains, like $\mathbb{Z}$ and $k[x]$, or finite. Our goal now is to develop theorems that enable us to create new Noetherian rings from old.

Proposition. *If A is a Noetherian ring and $f : A \to B$ makes B an A-algebra so that B is a finitely generated A-module under the multiplication $a.b = f(a)b$, then B is a Noetherian ring.*

Proof. Since every B-submodule of B is also an A-submodule and B is a Noetherian A-module it follows that B satisfies a.c.c. as a B-module and so B is a Noetherian ring. □

Examples.

(1) It can be shown that the ring of integers in any algebraic number field is a finitely generated $\mathbb{Z}$-module so it follows from the proposition that these are all Noetherian.
(2) It also follows that any quotient ring of a Noetherian ring is Noetherian.

Proposition. *If A is a Noetherian ring and S is a m.c. subset of A, then A_S is Noetherian.*

Proof. Suppose that I is an ideal in A_S, then $I = (I \cap A)A_S$. Since A is Noetherian $I \cap A$ is a finitely generated A-module. But then any such generating set is a generating set for I as as A_S-module. Thus all ideals in A_S are finitely generated and A_S is a Noetherian ring. □

The most important result in this section is undoubtedly the following theorem.

Theorem (Hilbert's Basis Theorem (1888))**.** *If A is a Noetherian ring then the polynomial ring $A[x]$ is also Noetherian.*

We will see (easily) from this that it follows that any finitely generated algebra over a Noetherian ring is Noetherian. In particular any finitely generated k-algebra is Noetherian when k is a field.

Corollary. *If A is a Noetherian ring then so is $A[x_1, \ldots, x_n]$.*

Proof. This follows from Hilbert's basis theorem by induction on n. □

Corollary. *If B is a finitely generated A-algebra and A is Noetherian then B is Noetherian.*

Proof. There is a positive integer n such that there is a surjective A-algebra map from $A[x_1, \ldots, x_n]$ to B. By the previous corollary the former is Noetherian and then the latter is a finite (in fact cyclic) module over the former and so also Noetherian. □

Proof of Hilbert's Basis Theorem.
Suppose I is an ideal of $A[X]$ we try to prove that I is finitely generated. First we let

$$J_n = \{a \in A \mid \exists f \in I \text{ st } \deg(f - aX^n) < n\}.$$

So J_n is an ideal in A for each n since if $\deg(f - aX^n) < n$ and $\deg(g - bX^n) < n$ then $\deg((\lambda f + \mu g) - (\lambda a + \mu b)X^n) < n$ for each $\lambda, \mu \in A$.

Also $J_n \subset J_{n+1}$ for each $n \geq 0$ since whenever $\deg(f - aX^n) < n$ we must have $\deg(Xf - aX^{n+1}) < n + 1$. So by the a.c.c. for ideals in A there is an N such that $J_N = J_{N+k}$ for all $k \geq 0$.

Now as A is Noetherian there exist $a_1, \ldots, a_r \in J_N$ generating J_N, and then by definition of J_N there exist $f_1, \ldots, f_r \in I$ such that $\deg(f_i - a_i X^N) < N$.

Claim. *If $f \in I$ then there exists $g \in I$ such that*

- $f = g + \sum_{i=1}^r p_i f_i$ *some* $p_i \in A[X]$
- $\deg(g) < N$.

Proof of Claim. By induction on $\deg(f)$. The result is true, with $p_i = 0$ for all i, if $\deg(f) < N$.

Suppose that $\deg f = k \geq N$. Then there is $a \in A$ such that $\deg(f - aX^k) < k$. So $a \in J_k = J_N$. Then there are $b_1, \ldots, b_r$ in A such that $a = \sum a_i b_i$. So $\deg(f - \sum b_i X^{k-N} f_i) < k$. So by the induction hypothesis we may find $p_i \in A[X]$ such that $\deg(f - \sum (b_i X^{k-N} + p_i) f_i) < N$. □

Now it suffices to prove that $I \cap (A + AX + \cdots AX^{N-1})$ is a finitely generated A-module. But this is true since $A + AX + \cdots AX^{N-1}$ is a Noetherian A-module. □

6.5. **Spec of a Noetherian ring.**

Theorem (The weak Nullstellensatz)**.** *Suppose that k is a field and A is a finitely generated k-algebra which is also a field. Then A is an algebraic extension of k.*

Proof. (For uncountable fields k — general case later)

Suppose for contradiction that $a \in A$ is transcendental over k, and consider the set

$$\{1/(a - \lambda) | \lambda \in k\}.$$

It suffices to show that this set is linearly independent over k since A has a countable spanning set over k given by monomials in the generators.

So suppose that $\sum_{i=1}^n \mu_i . 1/(a - \lambda_i) = 0$. with μ_i, λ_i in k and the λ_is distinct. Then $\sum \mu_i (\prod_{j \neq i} (a - \lambda_j)) = 0$ and as a is transcendental over k this must be an identity of polynomials over k. i.e. $\sum \mu_i (\prod_{j \neq i} (\lambda - \lambda_j)) = 0$ for every λ in k. In particular evaluating at $\lambda = \lambda_k$ we get $\mu_k = 0$ as required. □

Corollary. *If A is a finitely generated k-algbra then*

$$\mathrm{maxSpec}(A) = \{\ker \theta \mid \theta \colon A \to \bar{k} \text{ a } k\text{-algebra map}\}.$$

Proof. If $\mathfrak{m} \in \mathrm{maxSpec}(A)$ then $A/\mathfrak{m}$ is a finitely generated k-algebra and a field and so by the weak Nullstellensatz there is an (injective) k-algebra map from $A/\mathfrak{m}$ to $\bar{k}$.

Conversely if $\theta \colon A \to \bar{k}$ then its image is some finite dimensional k-vector space which must be a field so the kernel is in $\mathrm{maxSpec}(A)$. □

Corollary. *If $k = \bar{k}$ and I is an ideal in $k[X_1, \ldots, X_n]$. Then $V(I) \cap \mathrm{maxSpec}(A)$ corresponds to $\{\mathbf{x} \in k^n \mid f(\mathbf{x}) = 0 \quad \forall f \in I\}$ under $\ker \theta \mapsto (\theta(X_1), \ldots, \theta(X_n))$. Moreover Hilbert's Basis Theorem says finitely many f in I suffice to describe $V(I)$.*

Exercise. What can we say if $k = \mathbb{R}$?

Recall that $\mathrm{Jac}(A)$ is the intersection of all maximal ideals in A and $\mathrm{Jac}(A) = \{x \in A | 1 - xy \in A^\times \text{ for all } y \in A\}$.

Theorem (Nullstellensatz)**.** *If A is a finitely generated k-algebra then* $\mathrm{Jac}(A) = N(A)$.

Proof. We already know that $N(A) \subset \mathrm{Jac}(A)$. So suppose a is not in $N(A)$ and let I be the ideal in $A[x]$ generated by $1 - ax$. By Q2 of example sheet 1, $1 - ax$ is not a unit (as a is not nilpotent) and so is contained in a maximal ideal $\mathfrak{m}$ of $A[x]$. Since $A[x]/\mathfrak{m}$ is a field and a finitely generated k-algebra, it is isomorphic to a finite algebraic extension of k by the weak Nullstellensatz. Thus the composite $A \to A[x] \to A[x]/\mathfrak{m}$ defines an algebra map from A onto $k' \leq \bar{k}$ whose kernel does not contain a since the image of a is a unit. Since k' is a field the kernel is a maximal ideal. It follows that a is not in every maximal ideal, and so a is not in $\mathrm{Jac}(A)$ as required. □

Corollary (Hilbert 1893)**.** *If $k = \bar{k}$ and A is a finitely generated k-algebra then*

$$\{\textit{closed subsets of } \mathrm{maxSpec}(A)\} \leftrightarrow \{\textit{radical ideals of } A\}.$$

Proof. By the Nullstellensatz, if $I = \sqrt{I}$ then $\mathrm{Jac}(A/I) = 0$, so $I = \bigcap_{\mathfrak{m} \in V(I)} \mathfrak{m}$.

The converse is straightforward. □

Proposition. *Suppose that A is a Noetherian ring and I is an ideal not equal to A. Then*

(1) There exist $P_1, \ldots, P_n$ containing I (not necessarily distinct) such that $P_1 \cdots P_n \subset I$.

(2) The minimal primes over I are those minimal primes in $\{P_1, \ldots, P_n\}$ in (1). In particular there are only finitely many of them.

Proof. Suppose that there is a proper ideal I of A not satisfying (1). By the Noetherian condition we can choose a maximal such I.

Suppose further that J, K are ideals properly containing I. By the choice of I we may find $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_m$ such that $P_1 \cdots P_n \subset J$ and $Q_1 \cdots Q_m \subset K$. But now, $P_1 \cdots P_n Q_1 \cdots Q_m \subset JK$, and so JK cannot be contained in I by the choice of I. It follows that I is itself prime, which is absurd.

Suppose now that we have an I and $\{P_1, \ldots, P_n\}$ as in (1), and let $\{X_1, \ldots, X_m\}$ be the minimal elements of the latter.

Suppose now, that Q is a prime containing I, we wish to show that Q contains some X_j. But we can find indices $i_1, \ldots, i_n$ such that

$$X_{i_1} \cdots X_{i_n} \subset P_1 \cdots P_n \subset I \subset Q.$$

So by primality of Q we are done. □

Corollary. *If A is a Noetherian ring it has only finitely many minimal primes. Moreover $N(A)$ is nilpotent.*

Remark. Geometrically this means that if A is Noetherian then $\mathrm{Spec}(A)$ is a finite union of irreducible closed subsets (where irreducible means not a proper union of two closed subsets).

Exercise. Prove it!

Proof of Corollary. The first part follows immediately. For the second part let $P_1, \ldots, P_n$ be the minimal primes then $N(A) = P_1 \cap \ldots P_n$, and so $N(A)^{nk} \subset (P_1 \cdots P_n)^k \subset 0$ for some k by part (2) of the proposition. □

Remark. The second part can be proved directly without too much difficulty.

6.6. **Support and Associated primes.**

Definition. Given an A-module M, we define the *support* of M, $\mathrm{Supp}(M)$, to be the set of prime ideals P in A such that $M_P \neq 0$.

Proposition. *Suppose that A is a ring and L, M and N are A-modules.*

(i) $\mathrm{Supp}(M) = \emptyset$ *if and only if* $M = 0$

(ii) If $0 \to L \to M \to N \to 0$ *is a short exact sequence then*

$$\mathrm{Supp}(M) = \mathrm{Supp}(L) \cup \mathrm{Supp}(N).$$

(iii) If M is finitely generated then $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ *and so is a closed subset of* $\mathrm{Spec}(A)$.

Proof. (i) Just says that being 0 is a local property of modules.

(ii) Since localisation is exact we have for every prime P the sequence

$$0 \to L_P \to M_P \to N_P \to 0$$

is exact. It follows that M_P is zero if and only if both L_P and N_P are zero. In other words, $P \notin \mathrm{Supp}(M)$ if and only if $P \notin \mathrm{Supp}(L) \cap \mathrm{Supp}(N)$ i. e.

$$\mathrm{Supp}(M) = \mathrm{Supp}(N) \cup \mathrm{Supp}(L).$$

(iii) Suppose first that $P \in \mathrm{Supp}(M)$, i. e. $M_P \neq 0$. Then there is an $m \in M$ such that $\mathrm{Ann}(m) \subseteq P$ so $\mathrm{Ann}(M) \subseteq P$ and $P \in V(\mathrm{Ann}(M))$.

Suppose now that $m_1, \ldots, m_n$ are a generating set for M and $P \in V(\mathrm{Ann}(M))$ i. e. $P \supseteq \bigcap \mathrm{Ann}(m_i)$. Since $\bigcap \mathrm{Ann}(m_i) \supseteq \prod \mathrm{Ann}(m_i)$, and P is prime it follows that $P \supseteq \mathrm{Ann}(m_i)$ for some i. Thus $sm_i \neq 0$ for some $s \in A \backslash P$ and $M_P \neq 0$ as required. □

Exercise. If M is a finitely generated A-module and N is any A-module such that $\mathrm{Supp}(M) \cap \mathrm{Supp}(N) = \emptyset$ then $\mathrm{Hom}_A(M, N) = 0$.

Related to the support of a module is the set of associated primes:

Definition. If M is an A-module, and P is a prime ideal in A, then we say that P is an *associated prime* of M if there is an $m \in M$ such that $\mathrm{Ann}(m) = P$. We write $\mathrm{Ass}(M)$ for the set of associated primes of M.

Notice that if $\mathrm{Ann}(m) = P$ then $A.m$ is isomorphic to A/P as an A-module. So a prime P is in $\mathrm{Ass}(M)$ precisely if A/P is isomorphic to a submodule of M.

Lemma.

(i) If P is a prime ideal of A then $\mathrm{Ass}(A/P) = \{P\}$.

(ii) If I is a maximal element of the set $\{\mathrm{Ann}(m) | m \in M \backslash 0\}$ *then* $I \in \mathrm{Ass}(M)$. *In particular I is prime.*

(iii) If A is Noetherian and $M \neq 0$ *then* $\mathrm{Ass}(M) \neq \emptyset$.

Proof. (i) If $a + P \in A/P$ is non-zero then $b.(a+P) = P$ if and only if $ba \in P$ if and only if $b \in P$ since P is prime. So every non-zero element of A/P has annihilator P and the result follows.

(ii) Suppose I is a maximal element of the set $\{\mathrm{Ann}(m) | m \in M \backslash 0\}$, say $I = \mathrm{Ann}(m)$. Suppose further that a, b are elements of A and ab is in I. We have $abm = 0$. If $bm = 0$ then $b \in I$. Otherwise $bm \neq 0$ and $a \in \mathrm{Ann}(bm) \supseteq I$. By the maximality of I it follows that $\mathrm{Ann}(bm) = I$ and $a \in I$. Thus I is prime.

(iii) Since A is Noetherian and $\{\mathrm{Ann}(m) | m \in M \backslash 0\}$ is non-empty if $M \neq 0$, it must have a maximal element. We are then done by (ii). □

Proposition. *Suppose that A is a Noetherian ring and M is a finitely generated A-module, then* $\mathrm{Ass}(M)$ *is a subset of* $\mathrm{Supp}(M)$ *and the minimal elements of* $\mathrm{Supp}(M)$ *are all in* $\mathrm{Ass}(M)$. *In particular* $\overline{\mathrm{Ass}(M)} = \mathrm{Supp}(M)$.

Proof. If P is in $\mathrm{Ass}(M)$ then A/P is isomorphic to a submodule of M and so as localisation is exact $(A/P)_P$ is isomorphic to a submodule of M_P. Since the former is non-zero the latter is too and so $P \in \mathrm{Supp}(M)$.

Suppose now that P is a minimal element of $\mathrm{Supp}(M)$. We show that P is an associated prime of M in two steps. First we show that P_P is an associated prime of M_P as an A_P-module and then lift this result back up to M and A.

By assumption $M_P \neq 0$ and so $\mathrm{Ass}(M_P) \neq \emptyset$. Suppose that Q is a prime in A_P then $\iota^*(Q)$ is a prime in A contained in P. It follows (by minimality of P) that $M_{\iota^*(Q)} = 0$. But $M_{\iota^*(Q)} \cong (M_P)_Q$ and so $\mathrm{Supp}(M_P) = \{P_P\}$. By the first part, $\mathrm{Ass}(M_P)$ is contained in $\mathrm{Supp}(M_P)$ so must also be just $\{P_P\}$, and step one is complete.

So we have some (non-zero) $m/s \in M_P$ whose annihilator is P_P. We consider the element $m \in M$. Certainly $m/1$ is not zero in M_P so tm is not zero in M for every $t \in A \backslash P$. We claim that $\mathrm{Ann}(tm) = P$ for some such choice of t.

It is already clear from the previous remarks that $\mathrm{Ann}(tm)$ is contained in P for every choice of t. Suppose that f is in P but not $\mathrm{Ann}(m)$, then as $f/1.m/s = 0$, there is a t in $A \backslash P$ such that $ftm = tfm = 0$. So by careful choice of t we can ensure any given $f \in P$ lives in $\mathrm{Ann}(tm)$. We need to get them all in at once. But A is Noetherian and so P is a finitely generated A-module. If $f_1, \ldots, f_n$ is a generating set and $t_1, \ldots, t_n$ are corresponding elements of $A \backslash P$ such that $f_i t_i m = 0$ then setting $t = \prod t_i$ we get $\mathrm{Ann}(tm) = P$ as required. $\square$

7. Projective modules

7.1. **Local properties.** Recall X is a projective A-module precisely if $\mathrm{Hom}_A(X,-)$ is exact which occurs if and only if there is an A-module Y such that $X \oplus Y$ is free.

We should think of finitely generated projective A-modules as modules that are locally free. This is partly due to the following lemma.

Lemma. *If A is a local ring with maximal ideal $\mathfrak{m}$, then every finitely generated projective module is free. In particular if X is a finitely generated projective module then $X \cong A^n$, with $n = \dim_{A/\mathfrak{m}}(X/\mathfrak{m}X)$.*

In fact Kaplansky (1958) proved this is true without the requirement that X be finitely generated.

Proof. By Nakayama's Lemma we may find $e_1, \ldots, e_n \in X$ that generated X such that their images in $X/\mathfrak{m}X$ is a basis over $k = A/\mathfrak{m}$.

Thus we have a surjective A-linear map $\phi\colon A^n \to X$ such that $\phi((a_i)) = \sum a_i e_i$. Since X is projective, ϕ splits and we get $A^n \cong X \oplus \ker\phi$.

Now

$$\frac{A^n}{\mathfrak{m}A^n} \cong \frac{X}{\mathfrak{m}X} \oplus \frac{\ker\phi}{\mathfrak{m}\ker\phi},$$

so as $\dim_k \frac{X}{\mathfrak{m}X} = n$ we see $\frac{\ker\phi}{\mathfrak{m}\ker\phi} = 0$. But $\ker\phi$ is a finitely generated A-module so by Nakayama's Lemma again we get $\ker\phi = 0$. □

This enables us to show that finitely generated projectives are locally free. In fact we can show more.

Proposition. *If X is a finitely generated projective A-module, and $P \in \mathrm{Spec}(A)$ then X_P is isomorphic as an A_P-module to A_P^n for some $n \geq 0$. Indeed, we may find an $s \in A \backslash P$ such that $X_s \cong A_s^n$ as A_s-modules. It follows that $X_Q \cong A_Q^n$ for every $Q \in D(s) \subset \mathrm{Spec}(A)$.*

Proof. If $X \oplus Y \cong A^m$ then $X_P \oplus Y_P = A_P^m$ and so X_P is a f.g. projective A_P-module. Since A_P is local it follows from the lemma that X_P is free. Now if x_i/s_i for $i = 1, \ldots, n$ is a free generating set for X_P. Defining $f : A^m \to X$ by $f(e_i) = x_i$ we get an A-module map whose cokernel M is finitely generated and satisfies $M_P = 0$. But P is not an element of $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$, so there is $s \in A \backslash P$ such that $M_s = 0$.

Now $f_s : A_s^n \to X_s$ is surjective and X_s is projective (by the same argument as before) so there is an A_s-module map $g_s : X_s \to (A_s)^n$ such that $f_s g_s = \mathrm{id}$, and so $A_s^n \cong X_s \oplus N$ for some finitely generated A_s-module N such that $N_P = 0$ (by Nakayma's Lemma).

Since N is finitely generated and $N_P = 0$ by the argument above there is $t \in A \backslash P$ such that $N_t = 0$. Then $X_{st} \cong A_{st}^n$ as required. □

Definition. Suppose that A is a ring, and X is a f.g. A-module. We can define a rank function rk_X from $\mathrm{Spec}(A)$ to $\mathbb{N}$ that sends $P \in \mathrm{Spec}\, A$ to the dimension of $X_P/P_P X_P \cong X \otimes_{A/P} Q(A/P)$ as an $Q(A/P)$-vector space — the *fibre* of X at P.

Notice that the proposition we just proved shows that if X is a f.g. projective module then rk_X is a continuous function on $\mathrm{Spec}(A)$ (where $\mathbb{N}$ has the discrete topology). In particular, if $\mathrm{Spec}(A)$ is connected then rk_X is constant.

Exercise. Find a f.g. $\mathbb{Z}$-module without constant rank.

Definition. We say an A-module X is *locally free* if for every P in $\mathrm{Spec}(A)$ there is a basic open set $D(s)$ containing P such that X_s is a free module.

Exercise. Show that if M is a finitely presented A-module (i.e. there exists an exact sequence $A^m \to A^n \to M \to 0$) and $S \subset A$ is m.c. then

$$\mathrm{Hom}_{A-mod}(M,N)_S \cong \mathrm{Hom}_{A_S-mod}(M_S,N_S)$$

as A_S-modules.

Remarks.

(1) If A is Noetherian then every finitely generated module is finitely presented.
(2) In general every finitely generated projective module is finitely presented.

Theorem. *If A is a Noetherian ring and M is a finitely generated A-module then the following are equivalent:*

(1) M is projective.
(2) M is locally free.
(3) M_P is a free A_P-module for every P in $\mathrm{Spec}(A)$.

Proof. (1) implies (2) is above and (2) implies (3) is trivial.

For (3) implies (1): Since M is finitely generated, there is a surjective A-module map $\epsilon\colon A^n \to M$ for some $n \geq 0$. We claim that $\epsilon_* : \mathrm{Hom}(M,A^n) \to \mathrm{Hom}(M,M)$ is surjective.

If the claim holds then we can find $f \in \mathrm{Hom}(M,A^n)$ such that $\epsilon_*(f) = \mathrm{id}_M$, i.e. ϵ splits and $M \oplus \ker\epsilon \cong A^n$, and M is projective as required.

To prove the claim we show that ϵ_* is locally surjective, that is

$$(\epsilon_*)_P\colon \mathrm{Hom}(M,A^n)_P \to \mathrm{Hom}(M,M)_P$$

is surjective for each P in $\mathrm{Spec}(A)$. Now the lemma above tells us that there are natural isomorphisms

$$\mathrm{Hom}(M,A^n)_P \cong \mathrm{Hom}_{A_P}(M_P,A_P^n) \text{ and } \mathrm{Hom}(M,M)_P \cong \mathrm{Hom}_{A_P}(M_P,M_P)$$

and so $(\epsilon_*)_P$ naturally induces $(\epsilon_P)_*\colon \mathrm{Hom}_{A_P}(M_P,A_P^n) \to \mathrm{Hom}_{A_P}(M_P,M_P)$ that we must show is surjective. But we are assuming that M_P is a free A_P-module and $A_P^n \to M_P$ is surjective so this last map is surjective and we're done. $\square$