6.4. **Examples of Noetherian rings.** So far the only rings we can easily prove
are Noetherian are principal ideal domains, like $Z$ and $k[x]$, or finite. Our goal now
is to develop theorems that enable us to create new Noetherian rings from old.

**Proposition.** If $A$ is a Noetherian ring and $f : A \rightarrow B$ makes $B$ an $A$-algebra so
that $B$ is a finitely generated $A$-module under the multiplication $a.b = f(a)b$, then $B$ is a Noetherian ring.

**Proof.** Since every $B$-submodule of $B$ is also an $A$-submodule and $B$ is a Noetherian
$A$-module it follows that $B$ satisfies a.c.c. as a $B$-module and so $B$ is a Noetherian ring. □

**Examples.**

(1) It can be shown that the ring of integers in any algebraic number field is
a finitely generated $Z$-module so it follows from the proposition that these
are all Noetherian.

(2) It also follows that any quotient ring of a Noetherian ring is Noetherian.

**Proposition.** If $A$ is a Noetherian ring and $S$ is a m.c. subset of $A$, then $A_S$ is
Noetherian.

**Proof.** Suppose that $I$ is an ideal in $A_S$, then $I = (I \cap A)A_S$. Since $A$ is Noetherian
$I \cap A$ is a finitely generated $A$-module. But then any such generating set is a
generating set for $I$ as as $A_S$-module. Thus all ideals in $A_S$ are finitely generated and $A_S$ is a Noetherian ring. □

The most important result in this section is undoubtedly the following theorem.

**Theorem (Hilbert’s Basis Theorem (1888)).** If $A$ is a Noetherian ring then the
polynomial ring $A[x]$ is also Noetherian.

We will see (easily) from this that it follows that any finitely generated algebra
over a Noetherian ring is Noetherian. In particular any finitely generated $k$-algebra
is Noetherian when $k$ is a field.

**Corollary.** If $A$ is a Noetherian ring then so is $A[x_1, \ldots, x_n]$.

**Proof.** This follows from Hilbert’s basis theorem by induction on $n$. □

**Corollary.** If $B$ is a finitely generated $A$-algebra and $A$ is Noetherian then $B$ is
Noetherian.

**Proof.** There is a positive integer $n$ such that there is a surjective $A$-algebra map
from $A[x_1, \ldots, x_n]$ to $B$. By the previous corollary the former is Noetherian and
then the latter is a finite (in fact cyclic) module over the former and so also Noe-
therian. □

**Proof of Hilbert’s Basis Theorem.**

Suppose $I$ is an ideal of $A[X]$ we try to prove that $I$ is finitely generated. First
we let

$$J_n = \{ a \in A \mid \exists f \in I \text{ st } \deg(f - aX^n) < n \}.$$  

So $J_n$ is an ideal in $A$ for each $n$ since if $\deg(f - aX^n) < n$ and $\deg(g - bX^n) < n$
then $\deg((\lambda f + \mu g) - (\lambda a + \mu b)X^n) < n$ for each $\lambda, \mu \in A$. 

Also \( J_n \subset J_{n+1} \) for each \( n \geq 0 \) since whenever \( \deg(f - aX^n) < n \) we must have \( \deg(Xf - aX^{n+1}) < n + 1 \). So by the a.c.c. for ideals in \( A \) there is an \( N \) such that \( J_N = J_{N+k} \) for all \( k \geq 0 \).

Now as \( A \) is Noetherian there exist \( a_1, \ldots, a_r \in J_N \) generating \( J_N \), and then by definition of \( J_N \) there exist \( f_1, \ldots, f_r \in I \) such that \( \deg(f_i - a_iX^N) < N \).

**Claim. If \( f \in I \) then there exists \( g \in I \) such that**

- \( f = g + \sum_{i=1}^r p_if_i \) some \( p_i \in A[X] \)
- \( \deg(g) < N \).

**Proof of Claim.** By induction on \( \deg(f) \). The result is true, with \( p_i = 0 \) for all \( i \), if \( \deg(f) < N \).

Suppose that \( \deg(f) = k \geq N \). Then there is \( a \in A \) such that \( \deg(f - aX^k) < k \). So \( a \in J_k = J_N \). Then there are \( b_1, \ldots, b_r \) in \( A \) such that \( a = \sum b_ib_i \). So \( \deg(f - \sum b_iX^{k-N}f_i) < k \). So by the induction hypothesis we may find \( p_i \in A[X] \) such that \( \deg(f - \sum (b_iX^{k-N} + p_i)f_i) < N \).

Now it suffices to prove that \( \operatorname{I} \cap (A + AX + \cdots AX^{N-1}) \) is a finitely generated \( A \)-module. But this is true since \( A + AX + \cdots AX^{N-1} \) is a Noetherian \( A \)-module. □

### 6.5. Spec of a Noetherian ring.

**Theorem (The weak Nullstellensatz).** Suppose that \( k \) is a field and \( A \) is a finitely generated \( k \)-algebra which is also a field. Then \( A \) is an algebraic extension of \( k \).

**Proof.** (For uncountable fields \( k \) — general case later)

Suppose for contradiction that \( a \in A \) is transcendental over \( k \), and consider the set

\[
\{ 1/(a - \lambda) | \lambda \in k \}.
\]

It suffices to show that this set is linearly independent over \( k \) since \( A \) has a countable spanning set over \( k \) given by monomials in the generators.

So suppose that \( \sum_{i=1}^n \mu_i 1/(a - \lambda_i) = 0 \), with \( \mu_i, \lambda_i \) in \( k \) and the \( \lambda_i \)'s distinct. Then \( \sum \mu_i (\prod_{j \neq i} (a - \lambda_j)) = 0 \) and as \( a \) is transcendental over \( k \) this must be an identity of polynomials over \( k \). i.e. \( \sum \mu_i \prod_{j \neq i} (\lambda - \lambda_j) = 0 \) for every \( \lambda \) in \( k \). In particular evaluating at \( \lambda = \lambda_k \) we get \( \mu_k = 0 \) as required. □

**Corollary.** If \( A \) is a finitely generated \( k \)-algebra then

\[
\operatorname{maxSpec}(A) = \{ \ker \theta \mid \theta : A \to \bar{k} \text{ a } k \text{-algebra map} \}.
\]

**Proof.** If \( m \in \operatorname{maxSpec}(A) \) then \( A/m \) is a finitely generated \( k \)-algebra and a field and so by the weak Nullstellensatz there is an (injective) \( k \)-algebra map from \( A/m \) to \( k \).

Conversely if \( \theta : A \to \bar{k} \) then its image is some finite dimensional \( k \)-vector space which must be a field so the kernel is in \( \operatorname{maxSpec}(A) \). □

**Corollary.** If \( k = \bar{k} \) and \( I \) is an ideal in \( k[X_1, \ldots, X_n] \). Then \( V(I) \cap \operatorname{maxSpec}(A) \) corresponds to \( \{ x \in k^n \mid f(x) = 0 \text{ for all } f \in I \} \) under \( \ker \theta \mapsto (\theta(X_1), \ldots, \theta(X_n)) \). Moreover Hilbert’s Basis Theorem says finitely many \( f \) in \( I \) suffice to describe \( V(I) \).

**Exercise.** What can we say if \( k = \mathbb{R} \)?

Recall that \( \operatorname{Jac}(A) \) is the intersection of all maximal ideals in \( A \) and \( \operatorname{Jac}(A) = \{ x \in A \mid 1 - xy \in A^\times \text{ for all } y \in A \} \).
A maximal ideal. It follows that a is not a unit (as does not contain a finite algebraic extension of k by the weak Nullstellensatz. Thus the composite \( A \to A[x] \to A[x]/m \) defines an algebra map from A onto \( k' \leq k \) whose kernel does not contain a since the image of a is a unit. Since \( k' \) is a field the kernel is a maximal ideal. It follows that a is not in every maximal ideal, and so a is not in \( \text{Jac}(A) \) as required.

**Corollary** (Hilbert 1893). If \( k = k' \) and A is a finitely generated k-algebra then
\[
\{ \text{closed subsets of } \text{maxSpec}(A) \} \leftrightarrow \{ \text{radical ideals of } A \}.
\]

**Proof.** By the Nullstellensatz, if \( I = \sqrt{J} \) then \( \text{Jac}(A/I) = 0 \), so \( I = \bigcap_{m \in V(I)} m \).

The converse is straightforward.

**Proposition.** Suppose that A is a Noetherian ring and I is an ideal not equal to A. Then

1. There exist \( P_1, \ldots, P_n \) containing I (not necessarily distinct) such that \( P_1 \cdots P_n \subseteq I \).

2. The minimal primes over I are those minimal primes in \( \{ P_1, \ldots, P_n \} \) in (1). In particular there are only finitely many of them.

**Proof.** Suppose that there is a proper ideal I of A not satisfying (1). By the Noetherian condition we can choose a maximal such I.

Suppose further that J, K are ideals properly containing I. By the choice of I we may find \( P_1, \ldots, P_n \) and \( Q_1, \ldots, Q_m \) such that \( P_1 \cdots P_n \subseteq J \) and \( Q_1 \cdots Q_m \subseteq K \). But now, \( P_1 \cdots P_n Q_1 \cdots Q_m \subseteq JK \), and so \( JK \) cannot be contained in I by the choice of I. It follows that I is itself prime, which is absurd.

Suppose now that we have an I and \( \{ P_1, \ldots, P_n \} \) as in (1), and let \( \{ X_1, \ldots, X_m \} \) be the minimal elements of the latter.

Suppose now, that Q is a prime containing I, we wish to show that Q contains some \( X_j \). But we can find indices \( i_1, \ldots, i_n \) such that
\[
X_{i_1} \cdots X_{i_n} \subseteq P_1 \cdots P_n \subseteq I \subseteq Q.
\]
So by primality of Q we are done.

**Corollary.** If A is a Noetherian ring it has only finitely many minimal primes. Moreover \( N(A) \) is nilpotent.

**Remark.** Geometrically this means that if A is Noetherian then Spec(A) is a finite union of irreducible closed subsets (where irreducible means not a proper union of two closed subsets).

**Exercise.** Prove it!

**Proof of Corollary.** The first part follows immediately. For the second part let \( P_1, \ldots, P_n \) be the minimal primes then \( N(A) = P_1 \cap \ldots P_n \), and so \( N(A)^{nk} \subset (P_1 \cdots P_n)^k \subset 0 \) for some k by part (2) of the proposition.

**Remark.** The second part can be proved directly without too much difficulty.
6.6. Support and Associated primes.

**Definition.** Given an $A$-module $M$, we define the *support* of $M$, $\text{Supp}(M)$, to be the set of prime ideals $P$ in $A$ such that $M_P \neq 0$.

**Proposition.** Suppose that $A$ is a ring and $L$, $M$ and $N$ are $A$-modules.

(i) $\text{Supp}(M) = \emptyset$ if and only if $M = 0$

(ii) If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is a short exact sequence then

$$\text{Supp}(M) = \text{Supp}(L) \cup \text{Supp}(N).$$

(iii) If $M$ is finitely generated then $\text{Supp}(M) = V(\text{Ann}(M))$ and so is a closed subset of $\text{Spec}(A)$.

**Proof.** (i) Just says that being 0 is a local property of modules.

(ii) Since localisation is exact we have for every prime $P$ the sequence

$$0 \rightarrow L_P \rightarrow M_P \rightarrow N_P \rightarrow 0$$

is exact. It follows that $M_P$ is zero if and only if both $L_P$ and $N_P$ are zero. In other words, $P \notin \text{Supp}(M)$ if and only if $P \notin \text{Supp}(L) \cap \text{Supp}(N)$ i. e.

$$\text{Supp}(M) = \text{Supp}(N) \cup \text{Supp}(L).$$

(iii) Suppose first that $P \in \text{Supp}(M)$, i. e. $M_P \neq 0$. Then there is an $m \in M$ such that $\text{Ann}(m) \subseteq P$ so $\text{Ann}(M) \subseteq P$ and $P \in V(\text{Ann}(M))$.

Suppose now that $m_1, \ldots, m_n$ are a generating set for $M$ and $P \in V(\text{Ann}(M))$ i. e. $P \supseteq \bigcap \text{Ann}(m_i)$. Since $\bigcap \text{Ann}(m_i) \supseteq \prod \text{Ann}(m_i)$, and $P$ is prime it follows that $P \supseteq \text{Ann}(m_i)$ for some $i$. Thus $sm_i \neq 0$ for some $s \in A \setminus P$ and $M_P \neq 0$ as required.

**Exercise.** If $M$ is a finitely generated $A$-module and $N$ is any $A$-module such that $\text{Supp}(M) \cap \text{Supp}(N) = \emptyset$ then $\text{Hom}_A(M, N) = 0$.

Related to the support of a module is the set of associated primes:

**Definition.** If $M$ is an $A$-module, and $P$ is a prime ideal in $A$, then we say that $P$ is an *associated prime* of $M$ if there is an $m \in M$ such that $\text{Ann}(m) = P$. We write $\text{Ass}(M)$ for the set of associated primes of $M$.

Notice that if $\text{Ann}(m) = P$ then $A.m$ is isomorphic to $A/P$ as an $A$-module. So a prime $P$ is in $\text{Ass}(M)$ precisely if $A/P$ is isomorphic to a submodule of $M$.

**Lemma.**

(i) If $P$ is a prime ideal of $A$ then $\text{Ass}(A/P) = \{P\}$.

(ii) If $I$ is a maximal element of the set $\{\text{Ann}(m) | m \in M \setminus \emptyset\}$ then $I \in \text{Ass}(M)$.

In particular $I$ is prime.

(iii) If $A$ is Noetherian and $M \neq 0$ then $\text{Ass}(M) \neq \emptyset$.

**Proof.** (i) If $a + P \in A/P$ is non-zero then $b.(a + P) = P$ if and only if $ba \in P$ if and only if $b \in P$ since $P$ is prime. So every non-zero element of $A/P$ has annihilator $P$ and the result follows.

(ii) Suppose $I$ is a maximal element of the set $\{\text{Ann}(m) | m \in M \setminus \emptyset\}$, say $I = \text{Ann}(m)$. Suppose further that $a, b$ are elements of $A$ and $ab$ is in $I$. We have $abm = 0$. If $bm = 0$ then $b \in I$. Otherwise $bm \neq 0$ and $a \in \text{Ann}(bm) \supseteq I$. By the maximality of $I$ it follows that $\text{Ann}(bm) = I$ and $a \in I$. Thus $I$ is prime.

(iii) Since $A$ is Noetherian and $\{\text{Ann}(m) | m \in M \setminus \emptyset\}$ is non-empty if $M \neq 0$, it must have a maximal element. We are then done by (ii).
Proposition. Suppose that $A$ is a Noetherian ring and $M$ is a finitely generated $A$-module, then $\text{Ass}(M)$ is a subset of $\text{Supp}(M)$ and the minimal elements of $\text{Supp}(M)$ are all in $\text{Ass}(M)$. In particular $\text{Ass}(M) = \text{Supp}(M)$.

Proof. If $P$ is in $\text{Ass}(M)$ then $A/P$ is isomorphic to a submodule of $M$ and so as localisation is exact $(A/P)_P$ is isomorphic to a submodule of $M_P$. Since the former is non-zero the latter is too and so $P \in \text{Supp}(M)$.

Suppose now that $P$ is a minimal element of $\text{Supp}(M)$. We show that $P$ is an associated prime of $M$ in two steps. First we show that $P_P$ is an associated prime of $M_P$ as an $A_P$-module and then lift this result back up to $M$ and $A$.

By assumption $M_P \neq 0$ and so $\text{Ass}(M_P) \neq \emptyset$. Suppose that $Q$ is a prime in $A_P$ then $\iota^*(Q)$ is a prime in $A$ contained in $P$. It follows (by minimality of $P$) that $M_{\iota^*(Q)} = 0$. But $M_{\iota^*(Q)} \cong (M_P)_Q$ and so $\text{Supp}(M_P) = \{P_P\}$. By the first part, $\text{Ass}(M_P)$ is contained in $\text{Supp}(M_P)$ so must also be just $\{P_P\}$, and step one is complete.

So we have some (non-zero) $m/s \in M_P$ whose annihilator is $P_P$. We consider the element $m \in M$. Certainly $m/1$ is not zero in $M_P$ so $tm$ is not zero in $M$ for every $t \in A \setminus P$. We claim that $\text{Ann}(tm) = P$ for some such choice of $t$.

It is already clear from the previous remarks that $\text{Ann}(tm)$ is contained in $P$ for every choice of $t$. Suppose that $f$ is in $P$ but not $\text{Ann}(m)$, then as $f/1 \cdot m/s = 0$, there is a $t$ in $A \setminus P$ such that $f tm = tfm = 0$. So by careful choice of $t$ we can ensure any given $f \in P$ lives in $\text{Ann}(tm)$. We need to get them all in at once. But $A$ is Noetherian and so $P$ is a finitely generated $A$-module. If $f_1, \ldots, f_n$ is a generating set and $t_1, \ldots, t_n$ are corresponding elements of $A \setminus P$ such that $f_i t_i m = 0$ then setting $t = \prod t_i$ we get $\text{Ann}(tm) = P$ as required. \hfill \Box
7. Projective modules

7.1. Local properties. Recall $X$ is a projective $A$-module precisely if $\text{Hom}_A(X, -)$ is exact which occurs if and only if there is an $A$-module $Y$ such that $X \oplus Y$ is free.

We should think of finitely generated projective $A$-modules as modules that are locally free. This is partly due to the following lemma.

Lemma. If $A$ is a local ring with maximal ideal $\mathfrak{m}$, then every finitely generated projective module is free. In particular if $X$ is a finitely generated projective module then $X \cong A^n$, with $n = \dim_{A/\mathfrak{m}}(X/\mathfrak{m}X)$.

In fact Kaplansky (1958) proved this is true without the requirement that $X$ be finitely generated.

Proof. By Nakayama’s Lemma we may find $e_1, \ldots, e_n \in X$ that generated $X$ such that their images in $X/\mathfrak{m}X$ is a basis over $k = A/\mathfrak{m}$.

Thus we have a surjective $A$-linear map $\phi: A^n \to X$ such that $\phi((a_i)) = \sum a_i e_i$. Since $X$ is projective, $\phi$ splits and we get $A^n \cong X \oplus \ker \phi$.

Now

$$\frac{A^n}{\mathfrak{m}A^n} \cong \frac{X}{\mathfrak{m}X} \oplus \frac{\ker \phi}{\mathfrak{m} \ker \phi},$$

so as $\dim_{A/\mathfrak{m}}(X/\mathfrak{m}X) = n$ we see $\frac{\ker \phi}{\mathfrak{m} \ker \phi} = 0$. But $\ker \phi$ is a finitely generated $A$-module so by Nakayama’s Lemma again we get $\ker \phi = 0$. □

This enables us to show that finitely generated projectives are locally free. In fact we can show more.

Proposition. If $X$ is a finitely generated projective $A$-module, and $P \in \text{Spec}(A)$ then $X_P$ is isomorphic as an $A_P$-module to $A^n_P$ for some $n \geq 0$. Indeed, we may find an $s \in A \setminus P$ such that $X_s \cong A^n_s$ as $A_s$-modules. It follows that $X_Q \cong A^n_Q$ for every $Q \in D(s) \subset \text{Spec}(A)$.

Proof. If $X \oplus Y \cong A^n$ then $X_P \oplus Y_P = A^n_P$ and so $X_P$ is a f.g. projective $A_P$-module. Since $A_P$ is local it follows from the lemma that $X_P$ is free. Now if $x_i/s_i$ for $i = 1, \ldots, n$ is a free generating set for $X_P$. Defining $f: A^n \to X$ by $f((e_i)) = x_i$ we get an $A$-module map whose cokernel $M$ is finitely generated and satisfies $M_P = 0$. But $P$ is not an element of $\text{Supp}(M) = V(\text{Ann}(M))$, so there is $s \in A \setminus P$ such that $M_s = 0$.

Now $f_s: A^n_s \to X_s$ is surjective and $X_s$ is projective (by the same argument as before) so there is an $A_s$-module map $g_s: X_s \to (A_s)^n$ such that $f_s g_s = \text{id}$, and so $A^n_s \cong X_s \oplus N$ for some finitely generated $A_s$-module $N$ such that $N_P = 0$ (by Nakayama’s Lemma).

Since $N$ is finitely generated and $N_P = 0$ by the argument above there is $t \in A \setminus P$ such that $N_t = 0$. Then $X_{st} \cong A^n_{st}$ as required. □

Definition. Suppose that $A$ is a ring, and $X$ is a f.g. $A$-module. We can define a rank function $\text{rk}_X$ from $\text{Spec}(A)$ to $\mathbb{N}$ that sends $P \in \text{Spec}(A)$ to the dimension of $X_P/\text{Ann}_P X_P$ as an $A(P)$-vector space — the fibre of $X$ at $P$.

Notice that the proposition we just proved shows that if $X$ is a f.g. projective module then $\text{rk}_X$ is a continuous function on $\text{Spec}(A)$ (where $\mathbb{N}$ has the discrete topology). In particular, if $\text{Spec}(A)$ is connected then $\text{rk}_X$ is constant.

Exercise. Find a f.g. $\mathbb{Z}$-module without constant rank.
Definition. We say an $A$-module $X$ is locally free if for every $P$ in Spec($A$) there is a basic open set $D(s)$ containing $P$ such that $X_s$ is a free module.

Exercise. Show that if $M$ is a finitely presented $A$-module (i.e. there exists an exact sequence $A^n \to A^m \to M \to 0$) and $S \subset A$ is m.c. then
\[ \text{Hom}_{A_{mod}}(M, N)_S \cong \text{Hom}_{A_S_{mod}}(M_S, N_S) \]
as $A_S$-modules.

Remarks.
(1) If $A$ is Noetherian then every finitely generated module is finitely presented.
(2) In general every finitely generated projective module is finitely presented.

Theorem. If $A$ is a Noetherian ring and $M$ is a finitely generated $A$-module then the following are equivalent:
1. $M$ is projective.
2. $M$ is locally free.
3. $M_P$ is a free $A_P$-module for every $P$ in Spec($A$).

Proof. (1) implies (2) is above and (2) implies (3) is trivial.
For (3) implies (1): Since $M$ is finitely generated, there is a surjective $A$-module map $\epsilon : A^n \to M$ for some $n \geq 0$. We claim that $\epsilon_* : \text{Hom}(M, A^n) \to \text{Hom}(M, M)$ is surjective.
If the claim holds then we can find $f \in \text{Hom}(M, A^n)$ such that $\epsilon_*(f) = \text{id}_M$, i.e. $\epsilon$ splits and $M \oplus \ker \epsilon \cong A^n$, and $M$ is projective as required.
To prove the claim we show that $\epsilon_*$ is locally surjective, that is
\[(\epsilon_*)_P : \text{Hom}(M, A^n)_P \to \text{Hom}(M, M)_P\]
is surjective for each $P$ in Spec($A$). Now the lemma above tells us that there are natural isomorphisms
\[ \text{Hom}(M, A^n)_P \cong \text{Hom}_{A_P}(M_P, A^n_P) \text{ and } \text{Hom}(M, M)_P \cong \text{Hom}_{A_P}(M_P, M_P) \]
and so $(\epsilon_*)_P$ naturally induces $(\epsilon_*)_P : \text{Hom}_{A_P}(M_P, A^n_P) \to \text{Hom}_{A_P}(M_P, M_P)$ that we must show is surjective. But we are assuming that $M_P$ is a free $A_P$-module and $A^n_P \to M_P$ is surjective so this last map is surjective and we're done. $\square$