Homogeneity and Rigidity of the Brookes–Groves invariant for the non-commutative torus

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1 Introduction

In [2] Bieri and Strebel defined a geometric invariant Σ for finitely generated modules over the group algebras of finitely generated abelian groups. They used this to define a criterion for when metabelian groups are finitely presented. This invariant was further developed by Bieri, Strebel and Groves and has many interesting applications. In [1] Bieri and Groves showed that when the group algebra is defined over a Dedekind domain the complement of Σ must be a closed rational spherical polyhedral cone.

In [6] and [7] Brookes and Groves defined a similar invariant Δ for modules over the crossed product of a division ring by a free finitely generated abelian group. Such a crossed product is often known as the (co-ordinate ring of) the non-commutative torus since in the special case where it is commutative it is the coordinate ring of an algebraic torus. If in the commutative case we take the complement of Δ and identify points that differ by a positive scalar multiple we obtain Σ. Brookes and Groves were unable to prove that their invariant must be a rational polyhedral cone, although using the methods of [1] they do prove a weaker version of the result; they show that for any finitely generated module M, $\Delta(M)$ must contain a rational polyhedral cone $\Delta^*(M)$ of dimension equal to the Gelfand–Kirillov dimension of M and moreover that the complement $\Delta(M)\backslash\Delta^*(M)$ must be contained inside a rational polyhedral cone of strictly smaller dimension.

In [18] we proved that

Theorem A (Theorem A of [18]). *If DA is a crossed product of a division ring D by a free finitely generated abelian group A, then, for all finitely generated DA-modules M, $\Delta(M)$ is a closed rational polyhedral cone in $Hom(A,\mathbb{R})$.*

Given a subset S of $\mathbb{R}^n$ and a point x of S, the *local cone* of S at x is defined as

$$LC_x(S) = \{y \in \mathbb{R}^n \,|\, \exists \epsilon_0 > 0 \text{ s.t. } \forall \epsilon \in (0, \epsilon_0]\ x + \epsilon y \in S\}.$$

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We say that a S is *concave* at $x \in S$ if the convex hull of the local cone of S at x is a linear subspace of $\mathbb{R}^n$. We say that S is *totally concave* if it is concave all every point $x \in S$.

In [18] we showed

Theorem B (Theorem B of [18]). *If DA is a crossed product of a division ring D by a free finitely generated abelian group A, then, for all finitely generated DA-modules M and for each $\chi \in \Delta(M)$*

$$LC_\chi(\Delta(M)) = \Delta(\mathrm{gr}^\chi(M)).$$

for each natural χ-filtration of M.

This means that if M is a module such that the Gelfand–Kirillov dimension of M is equal to the Gelfand–Kirillov dimension of $\mathrm{gr}^\chi(M)$ for each $\chi \in \Delta(M)$ then $\Delta(M)$ is a homogeneous polyhedron and so $\Delta(M) = \Delta^*(M)$. We use this idea to prove

Theorem C. *If M is a finitely generated pure DA-module of dimension m then $\Delta(M)$ is a closed totally concave homogeneous rational polyhedron of dimension m.*

This leads to a simplification of many of the results of Brookes and Groves in [7], [8] and [9] since for pure modules we have $\Delta(M) = \Delta^*(M)$. It also provides a genuinely new proof of this result in commutative case.

In addition, we can deduce Theorem B of [8] in a way that more closely follows the proof of its commutative analogue by Bieri and Groves in [4] than Brookes and Groves were able to provide.

In order to prove Theorem C we prove

Theorem D. *Suppose DA be a crossed product of a division ring by a finitely generated free abelian group. Then DA is Cohen–Macaulay; that is for all finitely generated DA-modules M the sum of the Gelfand–Kirillov dimension of M and the grade of M is equal to the Gelfand–Kirillov dimension of DA.*

This theorem was independently proved by Ingalls in [13] for the special case where D is an algebraically closed field that is central in DA. In that case he was also able to describe which subrings that arise as subcrossed products of DA for submoniods S of A are also Cohen–Macaulay; in particular he showed that the result is the same as for the commutative case which was dealt with by Hochster in [11]. It would be interesting to find out whether or not the same holds in the greater generality we deal with here.

2 Preliminaries

We will say that a ring is *Noetherian* if it is both left and right Noetherian, i.e. if every left/right ideal is finitely generated as a left/right ideal.

We say that a ring is a *domain* if it contains no zero-divisors, i.e. the product of a pair of elements in the ring is zero only if one of the elements is zero.

Often we will state and prove a result for left modules that has an obvious analogue for right modules and *vice versa*. When the proof of the analogous result is exactly similar we will often use it without comment.

2.1 Filtrations and Gradings

Suppose that D is a division ring. By a *D-algebra* we will mean a ring R equipped with a ring homomorphism from D to R, giving R a natural left D-module structure.

By an *$\mathbb{R}$-filtration* of a D-algebra R, we will mean a set

$$\{F_\mu R | \mu \in \mathbb{R}\}$$

such that $D \subseteq F_0 R$, $F_\mu R \subseteq F_\nu R$ whenever $\nu \leq \mu$,

$$R = \bigcup_{\mu \in \mathbb{R}} F_\mu R$$

and

$$F_\mu R.F_\nu R \subseteq F_{\mu+\nu} R$$

for each $\mu, \nu \in \mathbb{R}$. We will write $F_\mu^+ R$ for $\bigcup_{\nu > \mu} F_\nu R$.

Given a filtered D-algebra R and a left R-module M, an *$\mathbb{R}$-filtration* of M is a set

$$\{F_\mu M | \mu \in \mathbb{R}\}$$

such that each $F_\mu M$ is a D-submodule of M, $F_\mu M \subseteq F_\nu M$ whenever $\nu \leq \mu$,

$$M = \bigcup_{\mu \in \mathbb{R}} F_\mu M$$

and

$$F_\mu R.F_\nu M \subseteq F_{\mu+\nu} M$$

for each $\mu, \nu \in \mathbb{R}$. Again we write $F_\mu^+ M$ for $\bigcup_{\nu > \mu} F_\nu M$. When it will not cause confusion as to which filtration we are referring, we will write M_λ for $F_\mu M$ and M_λ^+ for $F_\mu^+ M$.

We define the *associated graded ring* of an $\mathbb{R}$-filtered ring R by

$$\mathrm{gr}^F(R) = \bigoplus_{\mu \in \mathbb{R}} F_\mu R / F_\mu^+ R.$$

The multiplication in $\mathrm{gr}^F(R)$ is given on homogeneous elements by

$$(x_1 + F_{\mu_1}^+ R)(x_2 + F_{\mu_2}^+ R) = x_1 x_2 + F_{\mu_1+\mu_2}^+ R$$

and extended linearly. Given $x \in R$ we write $\sigma^F(x) = x + F_\mu^+ R \in \mathrm{gr}^F(R)$, the symbol of x, when $x \in F_\mu R$ but $x \notin F_\mu^+ R$.

Similarly we define the *associated graded module* of an $\mathbb{R}$-filtered R-module M

$$\mathrm{gr}^F(M) = \bigoplus_{\mu \in \mathbb{R}} F_\mu M / F_\mu^+ M,$$

and $\sigma^F(m) = m + F_\mu^+ M \in \mathrm{gr}^F(M)$, the symbol of m, for $m \in F_\mu M \backslash F_\mu^+ M$. This is naturally a $\mathrm{gr}^F(R)$ module with action on homogeneous elements given by

$$(x + F_{\mu_1}^+ R)(m + F_{\mu_2}^+ M) = x.m + F_{\mu_1+\mu_2}^+ M$$

for $x \in F_{\mu_1} R$ and $m \in F_{\mu_2} M$.

Given a monoid G and a ring R we say that R is *G-graded* if R decomposes as a direct sum of additive subgroups

$$R = \bigoplus_{x \in G} R_x$$

with $R_x R_y \subseteq R_{xy}$ for all $x, y \in G$.

Notice that the associated graded ring of an $\mathbb{R}$-filtered D-algebra is $\mathbb{R}$-graded when we think of $\mathbb{R}$ as a monoid with its usual addition.

2.2 Gelfand–Kirillov dimension

Suppose that R is a finitely generated D-algebra with finite generating set X such that $Dx = xD$ for each $x \in X$. We set $V \subset R$ to be the D-vector space spanned by X. Then we may define

$$d_X(n) = \dim_D \left(\sum_{i=0}^{n} V^i \right)$$

We then define the *GK-dimension* of R over D by

$$GK_D(R) = \overline{\lim} \log_n d_X(n).$$

The proof of Lemma 1.1 of [12] tells us that this definition is independent of the choice of generating set X.

Similarly given a finitely generated left R-module M with finite generating set F, we may define

$$d_{X,F}(n) = \dim_D \left(\sum_{i=0}^{n} V^i F \right)$$

and the *GK-dimension* of M over D by

$$GK_D(M) = \overline{\lim} \log_n d_{X,F}(n)$$

again this is independent of the choice of F and X by the proof of Lemma 1.1 of [12].

It is possible for the GK-dimension of a finitely generated algebra to be infinite; consider the free associative algebra on two generators for example. However in the rings we consider it always will be finite. For commutative algebras it agrees with the usual dimension function.

2.3 Localisation

Given a domain R, we will say that a multiplicatively closed subset S of R not containing zero is an *Ore set* in R if it satisfies both the left and right Ore conditions, i.e. if given any pair of elements $s \in S$ and $r \in R$ there exist elements $s', s'' \in S$ and $r', r'' \in R$ such that $rs' = sr'$ and $s''r = r''s$.

Given an Ore set S in R we may form R_S, the *localisation of R at S*. Also, in this case, given an R-module M we may construct the *module of quotients* $M_S \cong M \otimes_R R_S$. See Chapter 2 of [14] for more details.

2.4 Global dimension and Ext

Given a ring R and an R-module M, we may construct a projective resolution of M as an R-module:

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

The projective dimension of M, $\mathrm{pd}(M)$ is defined to be the smallest n such that there is a projective resolution

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

or ∞ if no such exists.

The global dimension of R is then defined to be

$$\sup\{\mathrm{pd}(M) | M \text{ is an } R\text{-module}\}$$

Lemma 2.1. *If S is an Ore set in R, then* $\mathrm{gldim}(R_S) \leq \mathrm{gldim}(R)$.

Proof. See Corollary 7.4.3 of [14]. □

Given a projective resolution of M as above we define

$$E^i_R(M) = \mathrm{Ext}^i_R(M, R)$$

to be the i^{th} homology group of the complex

$$0 \to \mathrm{Hom}_R(P_0, R) \to \mathrm{Hom}_R(P_1, R) \to \cdots \to \mathrm{Hom}_R(P_i, R) \to \cdots.$$

This definition is independent of the choice of resolution, see [17] for example.

Notice that, since R is an R-bimodule, if M is a left (right) R-module then $E^i_R(M)$ is right (left) R-module.

Given a ring R and an R-module M, the *grade* of M, $j_R(M)$ is defined by

$$j_R(M) = \min\{i \geq 0 | E^i(M) \neq 0\}$$

or ∞ if no such i exists.

If R is a Noetherian ring with finite global dimension then for all non-zero finitely generated R-modules M, $j_R(M) \leq \mathrm{gldim}(R)$, see [10]. It is easy to see in this case that $E^i_R(M)$ is zero for all $i > \mathrm{gldim}(R)$.

Also note that if R is Noetherian and M is a Noetherian R-module, then M has a projective resolution consisting of Noetherian free modules. It follows that $E^i_R(M)$ is also a Noetherian R-module, as it is a section of $\mathrm{Hom}_R(P_i, R)$ for some finitely generated free module P_i.

A finitely generated module M over a ring R is said to satisfy *Auslander's condition* if for every $i \geq 0$ and for every submodule N of $E^i_R(M)$ we have $j_R(N) \geq i$.

A ring is said to be *Auslander regular* if it has finite global dimension and all finitely generated modules satisfy Auslandder's condition.

Lemma 2.2. *A short exact sequence*

$$0 \to N \to M \to L \to 0$$

induces a long exact sequence

$$\cdots \to E^{i-1}_R(N) \to E^i_R(L) \to E^i_R(M) \to E^i_R(N) \to \cdots$$

Proof. See Theorem 7.5 of [17] for example. ☐

Lemma 2.3. *If R and M are Noetherian, and S is an Ore set in R, then $E^i_R(M)_S \cong E^i_{R_S}(M_S)$ for each i.*

Proof. Let $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M as an R-module consisting of finitely generated free modules. Then

$$\cdots \to (P_n)_S \to (P_{n-1})_S \to \cdots \to (P_0)_S \to M_S \to 0$$

is a projective resolution of M_S as an R_S-module consisting of finitely generated free R_S-modules. For each i, $(\mathrm{Hom}_R(P_i, R))_S \cong \mathrm{Hom}_{R_S}((P_i)_S, R_S)$. Since localisation is an exact functor the result follows. ☐

Lemma 2.4. *Given a homomorphism $R \to S$ of Noetherian rings such that S is both free as a left R-module and free as a right R-module and M is a Noetherian right R-module then*

$$E^i_S(M \otimes_R S) \cong S \otimes_R E^i_R(M)$$

for each $i \geq 0$.

Proof. Again, let $\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be a projective resolution of M as an R-module consisting of finitely generated free modules. Then

$$\cdots \to P_n \otimes_R S \to P_{n-1} \otimes_R S \to \cdots \to P_0 \otimes_R S \to \quad M \otimes_R S$$

is a projective resolution of $M \otimes_R S$ as an S-module consisting of Noetherian free modules since $- \otimes_R S$ is an exact functor.

For each i, $\mathrm{Hom}_S(P_i \otimes_R S, S) \cong S \otimes_R \mathrm{Hom}_R(P_i, R)$. The result follows as $S \otimes_R -$ is an exact functor. ☐

2.5 Polyhedral cones

We say a subset of $\mathbb{R}^n$ is a *convex polyhedral cone* if it can be written as the intersection of finitely many closed or open linear half spaces in $\mathbb{R}^n$. The *dimension* of a convex polyhedral cone S is defined to be the dimension of the subspace spanned by S and written $\dim(S)$. A convex polyhedral cone is said to be *rational* if each of the half spaces have boundaries induced from a subspace of $\mathbb{Q}^n$.

A subset Δ of $\mathbb{R}^n$ is said to be a *rational polyhedral cone* if it can be written as a finite union

$$\Delta = S_1 \cup \cdots \cup S_k$$

of rational convex polyhedral cones. The dimension of Δ, $\dim(\Delta)$ is defined to be $\max(\dim(S_i))$.

Notice that if Δ is a polyhedral cone then at each point $x \in \Delta$, $LC_x(\Delta)$ is also a polyhedral cone of dimension at most $\dim(\Delta)$. We say that a polyhedral cone Δ is *homogeneous* if $\dim(LC_x(\Delta)) = \dim(\Delta)$ at each point $x \in \Delta$.

2.6 Crossed products

We say that a G-graded ring R is *strongly G-graded* if $R_x R_y = R_{xy}$ for all $x, y \in G$.

If G is a group with identity element e, then we say that a G-graded ring is a *crossed product of R_e by G*, written $R_e G$, if R_x contains a unit $\bar{x}$ for each $x \in G$.

Given a crossed product of a ring R by a group G a typical element α of RG may be written uniquely as a finite sum

$$\alpha = \sum_i \bar{g}_i r_i$$

with r_i non-zero elements of R, and g_i distinct elements of G. The set $\{g_i\}$ is called the *support* of α, and is written $\mathrm{supp}(\alpha)$.

Given a subgroup H of G, $RH = \{\alpha \in RG | \mathrm{supp}(\alpha) \subseteq H\}$ is a crossed product of R by H. If H is normal in G then we may consider RG as a crossed product of RH by G/H.

We now review some well known results about a particular sort of crossed products. We suppose that DA is a crossed product of a division ring D by a finitely generated free abelian group A.

Lemma 2.5. *DA is a Noetherian domain. Its global dimension is at most* $\mathrm{rk}(A)$.

Proof. That DA is Noetherian follows from Theorem 1.5.12 of [14].

We may impose a total order on A compatible with the group structure. Then given a pair of non-zero elements x, y in DA we see that the support of the product of x and y contains the product of the maximal elements in their respective supports; in particular xy is not zero, and DA is a domain.

The statement about global dimension follows from Corollary 7.5.6 of [14]. $\square$

Given any crossed product of the form DA and any subgroup B of A we will write S_B for the subset $DB \backslash 0$ of DA. The point of this definition is the following lemma:

Lemma 2.6. *If B is a subgroup of A then S_B is an Ore set in DA and DA_{S_B} is a crossed product of DB_{S_B} by A/B.*

Proof. Same as proof of Lemma 37.7 in [16]. $\square$

We now recall the definition by Brookes and Groves of an invariant for modules over rings of the form DA and some of their results.

Given an group homomorphism χ from A to $\mathbb{R}$ we may define $F^{\chi}_{\mu}DA$ to be the D-linear span of $\{a \in A | \chi(a) \geq \mu\}$. This defines an $\mathbb{R}$-filtration of DA. We call this the *χ-filtration* of DA.

We say that an $\mathbb{R}$-filtration $\{F_\mu M\}$ of a left DA-module M with respect to the χ-filtration of DA is a *χ-filtration* of M.

A χ-filtration $\{F_\mu M\}$ of a DA-module M is said to be *trivial* if $M = F_\mu M$ for some $\mu \in \mathbb{R}$.

A χ-filtration $\{F_\mu M\}$ of a DA-module M is said to be *natural* if there is a finite generating set X of M such that $F_\mu M = F^{\chi}_{\mu} DA.X$ for each $\mu \in \mathbb{R}$.

Definition. *Given a DA-module M, let $\Delta_A(M) = \Delta(M)$ be the subset of* $\mathrm{Hom}(A, \mathbb{R})$ *such that $\chi \in \Delta(M)$ precisely if there is a non-trivial χ-filtration of M or $\chi = 0$.*

Proposition 2.7 (Proposition 2.1 of [6]). *Suppose that M is a left DA-module with finite generating set X. The following are equivalent for $\chi \in$* $\mathrm{Hom}(A, \mathbb{R})\backslash 0$.

1. $\chi \notin \Delta(M)$;
2. *the natural χ-filtration of M given by $F_\mu M = F_\mu^\chi DA.X$ is trivial;*
3. *M is generated by X over a Noetherian subring of $F_0^\chi DA$;*
4. *M is generated by X over $F_0^\chi DA$;*
5. *for each $x \in X$, there exists $\alpha \in DA$ such that $\alpha.x = 0$ and $\sigma^{F^\chi}(\alpha) = 1$.*

Lemma 2.8 (Corollary 2.2 of [6]). *Suppose that*

$$0 \to L \to M \to N \to 0$$

is a short exact sequence of finitely generated DA-modules. Then

$$\Delta(M) = \Delta(L) \cup \Delta(N)$$

Brookes and Groves showed in the remarks following Proposition 4.2 and in Theorem 4.4 of [7] that the following is true:

Lemma 2.9. *If M is a finitely generated DA-module then the following equalities hold and we may call this* $\dim_A M$ *or the* dimension *of M as a DA-module.*

$$\begin{aligned} \mathrm{GKdim}_D(M) &= \max\{\mathrm{rk}(B) | B \leq A \text{ and } M_{S_B} \neq 0\} \\ &= \min\{\mathrm{rk}(B) | B \leq A \text{ and } M \text{ is a f. g. } DB\text{-module}\}. \end{aligned}$$

This dimension is also equal to the dimension of the maximal convex polyhedral cone contained in $\Delta(M)$.

Lemma 2.10. *If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated DA-modules then*

$$\dim_A M = \max\{\dim_A L, \dim_A N\}.$$

Proof. This follows from Lemmas 2.8 and 2.9. ☐

We say that a finitely generated DA-module M is *pure* if every non-zero submodule has the same dimension.

Lemma 2.11. *If M is a pure DA-module of dimension m and B is an isolated subgroup of A such that M is finitely generated over DB then M is pure as a DB-module of dimension m.*

Proof. First note that it suffices to prove the result in the case where A/B is infinite cyclic since we may then complete by induction.

Let N be a critical DB-submodule of M. By Lemma 2.4 of [7] either $N.DA \cong N \otimes_{DB} DA$ or $\dim_A N.DA = \dim_B N$. Since M is a Noetherian DB-module and $N.DA$ is a submodule of M the former cannot hold and so $\dim_B(N) = \dim_A(N.DA) = \dim_A(M)$, since M is a pure DA-module.

We have now proved that every critical DB-submodule of M has dimension $\dim_A(M)$. But also $\dim_B(M)$ has dimension m by Lemma 2.7 of [7]. The result follows. □

We say that M is *critical* if every proper quotient has strictly smaller dimension.

Notice that every non-zero submodule of a critical module is critical with the same dimension.

Given a subgroup B of A we write $\pi_B^A = \pi_B$ for the restriction map from $\mathrm{Hom}(A, \mathbb{R})$ to $\mathrm{Hom}(B, \mathbb{R})$.

3 The Cohen–Macaulay condition

Recall that given a ring R and an R-module M we write $E_R^i(M)$ to denote $\mathrm{Ext}_R^i(M, R)$ and $j_R(M)$ to denote the grade of M, that is the least i such that $E_R^i(M) \neq 0$.

An algebra R is said to be *Cohen–Macaulay* if for all finitely generated non-zero R-modules M,

$$j_R(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(R)$$

In this section we let DA be a crossed product of a division ring D by a finitely generated free abelian group A. We aim to prove that DA is Cohen–Macaulay.

We begin with the following Lemma:

Lemma 3.1. *If M is a non-zero finitely generated DA-module, then*

$$j_{DA}(M) + \dim_A(M) \leq \mathrm{rk}(A).$$

Proof. Lemma 2.9 tells us that there exists an isolated $B \leq A$ such that $M_{S_B} \neq 0$ and $\mathrm{rk}(B) = \dim_A(M)$. Let $S = S_B$.

Now, by Lemma 2.3 $j_{DA}(M) \leq j_{(DA)_S}(M_S)$. A remark in section 2.4 tells us that $j_{(DA)_S}(M_S) \leq \mathrm{gldim}((DA)_S)$. Finally, Lemma 2.6 tells us that $\mathrm{gldim}((DA)_S) \leq \mathrm{rk}(A/B) = \mathrm{rk}(A) - \mathrm{rk}(B)$. □

Lemma 3.2. *If M is a non-zero finitely generated DA-module, then*

$$\dim_A(E_{DA}^i(M)) \leq \min\{\dim_A(M), \mathrm{rk}(A) - i\}$$

for $0 \leq i \leq \mathrm{rk}(A)$.

Proof. Suppose $B \leq A$ is an isolated subgroup such that

$$\mathrm{rk}(B) > \min\{\mathrm{rk}(A) - i, \dim_A(M)\}.$$

Let $S = S_B$. There are now two cases.

Firstly, if $\mathrm{rk}(B) > \mathrm{rk}(A) - i$ then $\mathrm{gldim}((DA)_S) \leq \mathrm{rk}(A/B) < i$ and so $E^i_{DA}(M) = 0$.

Secondly, if $\mathrm{rk}(B) > \dim_A(M)$ then Lemma 2.9 tells us that $M_S = 0$ and Lemma 2.3 gives

$$E^i_{DA}(M)_S \cong E^i_{(DA)_S}(M_S) = 0$$

The result now follows by applying Lemma 2.9 with the finitely generated DA-module $E^i_{DA}(M)$ in the place of M. $\square$

Next we prove a technical result that will be used later.

Lemma 3.3. *If $B \leq A$ with $A/B \cong \mathbb{Z}$ and N is a finitely generated right DB-module, then every non-zero DA-submodule of $N \otimes_{DB} DA$ has dimension at least* 1.

Proof. Lemma 2.9 tells us that it is sufficient to prove that if C is a complement to B in A and $0 \neq x \in N \otimes_{DB} DA$ then $x.DC \cong DC$.

Now if $0 \neq x \in N \otimes_{DB} DA$, we may write it uniquely as

$$x = \sum_{i \in \mathbb{Z}} n_i \otimes \bar{c}^i$$

with $n_i \in N$ and $C = \langle < c \rangle >$ and at least one $n_i \neq 0$.

Then if $x. \sum \lambda_j \bar{c}^j = 0$ with at least one $\lambda_i \neq 0$ we have

$$y = \sum_{i,j \in \mathbb{Z}} n_i \lambda_j^{(i)} \otimes \bar{c}^{i+j} = 0$$

where $\lambda_j^{(i)} \bar{c}^i = \bar{c}^i \lambda_j$. Now if i, j are maximal such that $n_i \neq 0$ and $\lambda_j \neq 0$, the $\bar{c}^{i+j}$ coefficient of y is $n_i \lambda_j^{(i)} \neq 0$, a contradiction. So $x.DC \cong DC$. $\square$

Theorem 3.4. *DA is Cohen–Macaulay.*

Proof. We prove this by induction on $\mathrm{rk}(A)$.

Due to Lemma 3.1 the result is trivial for $\mathrm{rk}(A) \leq 1$ as if $\mathrm{rk}(A) = 1$ then only torsion modules can have dimension 0 and these must have grade at least 1 since $\mathrm{Hom}_{DA}(M, DA) = 0$.

Suppose that A is a minimal counterexample.

By symmetry it is enough to prove the result for right modules. So we pick M a finitely generated right DA-module of maximal dimension such that $j_{DA}(M) + \dim_A(M) < \mathrm{rk}(A)$. We let $m = \dim_A(M)$ and $k = \mathrm{rk}(A) - m$. Clearly $m < \mathrm{rk}(A)$.

We now aim to prove that for each $i < k$, that if $E^i_{DA}(M) \neq 0$ then $\dim_A(E^i_{DA}(M)) = 0$. Suppose that $B \leq A$ is isolated and has rank 1, and $S = S_B$. By Lemma 2.9, it is enough to show that $E^i_{DA}(M)$ must be DB-torsion for any such B.

By Lemma 2.9, $\mathrm{GKdim}_{DB_S}(M_S) \leq m - 1$, since if $C/B \leq A/B$ with M_S not torsion over $DB_S(C/B)$ then M is not torsion over DC.

By the induction hypothesis on $\mathrm{rk}(A)$,

$$j_{DA_S}(M_S) = \mathrm{rk}(A/B) - \mathrm{GKdim}_{DB_S}(M_S) \geq (\mathrm{rk}(A) - 1) - (m - 1) = k.$$

It follows that if $i < k$ then $E^i_{DA_S}(M_S)$ is zero. But by Lemma 2.3 we have

$$(E^i_{DA}(M))_S = E^i_{DA_S}(M_S)$$

and so $E^i_{DA}(M)_S = 0$. Thus $E^i_{DA}(M)$ is DB-torsion as required.

Since $\dim_A(M) < \mathrm{rk}(A)$, Lemma 2.9 ensures that there exists $B \leq A$ with $A/B \cong \mathbb{Z}$ such that M is finitely generated as a DB-module. Write N for M considered as a DB-module.

The map $N \otimes_{DB} DA \to M$; $n \otimes \alpha \mapsto n\alpha$ induces a short exact sequence

$$0 \to L \to N \otimes_{DB} DA \to M \to 0$$

Since $\dim_A(N \otimes_{DB} DA) = m + 1 > m$ and

$$\dim_A(N \otimes_{DB} DA) = \max\{\dim_A(M), \dim_A(L)\}$$

by Lemma 2.10, we have $\dim_A(L) = m + 1$. The maximality of the dimension of M as a counterexample implies that

$$j_{DA}(L) = \mathrm{rk}(A) - \dim_A(L) = \mathrm{rk}(A) - (m + 1) = k - 1$$

and similarly that $j_{DA}(N \otimes_{DB} DA) = k - 1$.

For $i \leq k - 1$ there is an exact sequence

$$E^{i-1}_{DA}(L) \to E^i_{DA}(M) \to E^i_{DA}(N \otimes_{DB} DA)$$

and $E^{i-1}_{DA}(L) = 0$.

For $i < k - 1$, $E^i_{DA}(N \otimes_{DB} DA) = 0$, and so $E^i_{DA}(M) = 0$.

For $i = k - 1$, using Lemma 2.4, the exact sequence becomes

$$0 \to E^{k-1}_{DA}(M) \to DA \otimes_{DB} E^{k-1}_{DB}(N).$$

We have already shown that $\dim_A(E^{k-1}_{DA}(M)) = 0$. By the left module version of Lemma 3.3 every non-zero submodule of $DA \otimes_{DB} E^{k-1}_{DB}(N)$ has dimension at least 1. It follows that $E^{k-1}_{DA}(M) = 0$, and $j_{DA}(M) \geq k$, a contradiction. □

Corollary 3.5. *DA is Auslander regular.*

Proof. Let M be a finitely generated DA-module and let $i \geq 0$. By Lemma 3.2, $\dim_A(E^i_{DA}(M)) \leq \mathrm{rk}(A) - i$ and so $\dim_A(N) \leq \mathrm{rk}(A) - i$ for each DA-submodule N of $E^i_{DA}(M)$. It now follows from Theorem 3.4 that $j_{DA}(N) \geq i$ as required. □

4 Ext and $\Delta(M)$

We now aim to produce another condition for when $\chi \notin \Delta$ to go along with those found in Proposition 2.7. We begin by making a definition.

Definition. *If $\chi \in \mathrm{Hom}(A, \mathbb{R})$ we define*

$$S = S_\chi = \{\alpha \in DA | \sigma^{F^\chi}(\alpha) = 1\}$$

where $\sigma^{F^\chi}(\alpha)$ denotes the symbol of α with respect to the χ-filtration F^χ defined in section 2.6

The following lemma was proved in [7] in the case when χ is discrete. Here we extend that result with a broadly similar proof.

Lemma 4.1. *For all $\chi \in \mathrm{Hom}(A, \mathbb{R})$, S_χ is an Ore set.*

Proof. Firstly note that given non-zero $\alpha, \beta \in DA$, $\sigma^{F^\chi}(\alpha\beta) = \sigma^{F^\chi}(\alpha)\sigma^{F^\chi}(\beta)$ so S is multiplicatively closed.

Now observe that $S \subseteq F_0^\chi DA$, and that it is sufficient to prove that S is an Ore set in the ring $F_0^\chi DA$ since it is invariant under conjugation by the units in $\overline{A} \subseteq DA$.

Now let $s \in S$ and $r \in F_0^\chi DA$. Write $X = \mathrm{supp}(s) \cup \mathrm{supp}(r)$ a finite set and let R be the ring generated by D and $\overline{X}$.

By McConnell's extension of the Hilbert basis theorem, Theorem 1.4.5 of [14], R is a Noetherian subring of $F_0^\chi DA$. Also R contains both s and r.

Let $T = S \cap R$, and set $I = T - 1$, an ideal in R. As R is Noetherian I has a finite generating set Y say. We let $Z = \bigcup_{y \in Y} \mathrm{supp}(y)$. Then $\overline{Z} \subset I$ since I is a homogeneous ideal in R with respect to the natural grading by monoid generated by X. Indeed $\overline{Z}$ is itself a finite generating set for I as an ideal in R.

Given $z \in Z$, $\bar{z}R = R\bar{z}$. It follows by Proposition 4.2.6 of [14] that I has the Artin-Rees Property. Now Proposition 4.2.9 of [14] tells us that T is a Ore set in R. Since $s \in T \subseteq S$, and $r \in R \subseteq F_0^\chi DA$ it follows that S is a right Ore set in $F_0^\chi DA$. $\square$

Now we may strengthen Proposition 2.7 to include the statement that $\chi \notin \Delta(M)$ if and only if $M_{S_\chi} = 0$ as the latter is plainly equivalent to condition (5).

This enables us to connect the homological results of the previous section with Δ.

Firstly,

Proposition 4.2. *If M is a finitely generated DA-module then*

$$\Delta(M) = \bigcup_{i=0}^{\mathrm{rk}(A)} \Delta(E^i_{DA}(M))$$

Proof. By Lemma 2.1, for each $\chi \in \mathrm{Hom}(A, \mathbb{R})$, the global dimension of DA_{S_χ} is at most the global dimension of DA which in turn is at most the rank of A by Lemma 2.5. So by a remark in section 2.4 we have $M_{S_\chi} \neq 0$ if and only if $E^i_{DA_{S_\chi}}(M_{S_\chi}) \neq 0$ for some $i \leq \mathrm{rk}(A)$. Thus Lemma 2.3 tells us that $M_{S_\chi} \neq 0$ if and only if $E^i_{DA}(M)_{S_\chi} \neq 0$ for some $i \leq \mathrm{rk}(A)$. The result follows immediately. $\square$

This provides a kind of filtration of Δ as $\dim_A(E^i(M)) \leq \mathrm{rk}\ A - i$ by Lemma 3.2.

Notice that it follows from this that $\dim_A(E_{DA}^{j_{DA}(M)}(M)) = \dim_A(M)$.

Theorem 4.3. *If M is a finitely generated critical DA-module then $\Delta(M) = \Delta\left(E^j_{DA}(M)\right)$ and $\Delta(N) = \Delta(M)$ for all non-zero submodules N of M.*

Proof. Let $j = j_{DA}(M)$. Proposition 2.5 of [5], tells us that as DA is Auslander regular there is an exact sequence

$$0 \to M \to E^j_{DA}(E^j_{DA}(M)) \to Q \to 0$$

with $j_{DA}(Q) \geq j+2$.

So using Proposition 4.2 and Lemma 2.8 we deduce that

$$\Delta(E^j_{DA}(M)) \subseteq \Delta(M) \subseteq \Delta(E^j_{DA}(E^j_{DA}(M))) \subseteq \Delta(E^j_{DA}(M))$$

and we get equalities throughout.

Now if $0 \neq N \leq M$, then $\dim_A(M/N) < \dim_A(M)$ and Theorem 3.4 tells us that $j_{DA}(M/N) > j$. So the long exact sequence, Lemma 2.2, implies that $E^j_{DA}(M)$ embeds into $E^j_{DA}(N)$. Lemma 2.8 therefore gives

$$\Delta(N) \subseteq \Delta(M) = \Delta(E^j_{DA}(M)) \subseteq \Delta(E^j_{DA}(N)) = \Delta(N)$$

and equalities hold throughout. $\square$

We now show that $E^{j_{DA}(-)}_{DA}(-)$ provides a kind of duality between right DA-module and left DA-modules.

Definition. *A critical composition series for a DA-module M of length n is a chain of submodules*

$$0 = M_0 < M_1 < M_2 < \cdots < M_n = M$$

such that M_i/M_{i-1} is critical with $\dim_A(M_i/M_{i-1}) \leq \dim_A(M_{i+1}/M_i)$ *for each i.*

Lemma 4.4. *Every DA-module M has a critical composition series.*

Proof. By Proposition 2.5 of [7] every finitely generated DA-module has a critical submodule. We construct a critical composition series for M by letting M_i/M_{i-1} be a maximal critical submodule of M/M_{i-1} of minimal dimension for $i \geq 1$. The process must stop as M is Noetherian. We claim that at each stage every non-zero submodule of M/M_i has dimension at least $d := \dim_A(M_i/M_{i-1})$ so we may continue.

To prove our claim we suppose that N/M_i is a non-zero submodule of M/M_i with $\dim_A(N/M_i) < d$. By Lemma 2.10 $\dim_A(N/M_{i-1}) = d$. We aim to show that N/M_{i-1} is critical of dimension d contradicting the maximality of M_i. To that end suppose that L/M_{i-1} is a non-zero submodule of N/M_{i-1} with $\dim_A(N/L) = d$. Then as

$$\dim_A(N/(M_i + L)) \leq \dim_A(N/M_i) < d$$

and

$$\dim_A(N/L) = \max\{\dim_A(N/(L + M_i)), \dim_A((L + M_i)/L)$$

we must have $\dim_A(M_i/(L \cap M_i)) = \dim_A((L + M_i)/L) = d$. Since M_i/M_{i-1} is critical of dimension d we can deduce that $L \cap M_i = M_{i-1}$ and so L/M_{i-1} embeds in N/M_i but has strictly bigger dimension, a contradiction. $\square$

Definition. *We say that a submodule N of a module M is* essential in *M if for every non-zero submodule L of M we have $L \cap N \neq 0$.*

Notice that if M is a critical module then any non-zero submodule is essential in M. Since if N and L are two non-zero submodules of M with trivial intersection then L is isomorphic to a submodule of M/N but has strictly bigger dimension.

Definition. *We say two modules M_1 and M_2 are* similar *if they have non-zero essential submodules N_1 and N_2 respectively such that $N_1 \cong N_2$.*

Lemma 4.5. *Any two critical composition series for M have equal length. Indeed, up to permutation their composition factors must be similar.*

Proof. This is identical to the proof of Proposition 6.2.21 of [14]. □

Definition. *Given any finitely generated DA-module M, there is a unique largest submodule N with $\dim_A(N) < \dim_A(M)$. M/N is pure and we define the* length *of M, $l(M)$, to be the length of a critical composition series for M/N.*

Lemma 4.6. *If $0 \to N \to M \to L \to 0$ is exact then*

1. *If $\dim_A(N) = \dim_A(L) = \dim_A(M)$ then $l(M) = l(N) + l(L)$;*
2. *if $\dim_A(N) < \dim_A(L) = \dim_A(M)$ then $l(M) = l(L)$;*
3. *if $\dim_A(L) < \dim_A(N) = \dim_A(M)$ then $l(M) = l(N)$.*

Proof. This is straightforward. □

Proposition 4.7. *For all finitely generated DA-modules,*

$$l(M) = l(E_{DA}^{j_{DA}(M)}(M)).$$

Proof. Let $j = j_{DA}(M)$. By Proposition 2.5 of [5] we know that there is an exact sequence

$$0 \to N \to M \to E_{DA}^{j}(E_{DA}^{j}(M)) \to Q \to 0$$

with $\dim_A(N) < \dim_A(M)$ and $\dim_A(Q) < \dim_A(M)$. So Lemma 4.6 tells us that $l(M) = l(E_{DA}^{j}(E_{DA}^{j}(M)))$.

It now suffices to prove that $l(E_{DA}^{j}(M)) \geq l(M)$ for all finitely generated left or right modules M.

We prove this by induction on $l(M)$.

If $l(M) = 1$ the result is clear since every module has length at least one.

Otherwise, there is an exact sequence

$$0 \to N \to M \to L \to 0$$

with $\dim_A(N) = \dim_A(M) = \dim_A(L)$. It follows from Lemma 2.2 that there is an exact sequence

$$0 \to E_{DA}^{j}(L) \to E_{DA}^{j}(M) \to E_{DA}^{j}(N) \to E_{DA}^{j+1}(L).$$

By Lemma 3.2 and the remark following Proposition 4.2,

$$\dim_A(E_{DA}^{j+1}(L)) < \mathrm{rk}(A) - j = \dim_A(E_{DA}^{j}(N)),$$

and we deduce from Lemma 4.6 and the induction hypothesis that

$$l(E_{DA}^{j}(M)) = l(E_{DA}^{j}(L)) + l(E_{DA}^{j}(N)) \geq l(L) + l(N) = l(M).$$

□

We see in particular from this that the correspondance $M \leftrightarrow E^{j(M)}_{DA}(M)$ defines a bijection between simple left DA-modules of minimal dimension and simple right DA-modules of minimal dimension.

5 Homogeneity of $\Delta(M)$

In this section we explain how to prove Theorem C. We adopt the following strategy: to prove that a polyhedron Δ is homogeneous is to prove that at each point $x \in \Delta$, $\dim LC_x(\Delta) = \dim \Delta$. If Δ is a *closed rational* polyhedron then it is the closure of its rational points. It follows that in this case it suffices to prove $\dim LC_x(\Delta) = \dim \Delta$ for each rational point in Δ. By Theorems A and B this reduces us to proving that $\dim_A(\mathrm{gr}^\chi(M)) = \dim_A(M)$ for each rational character $\chi \in \Delta(M)$. By Theorem 3.4 this is equivalent to proving that $j_{DA}(\mathrm{gr}^\chi(M)) = j_{DA}(M)$ for each rational character $\chi \in \Delta(M)$. Once we have proved the polyhedron is homogeneous we can slightly strengthen a projection result of Brookes and Groves and then use it to prove that Δ is totally concave.

We quote a result of Björk and Ekström:

Proposition 5.1 (Corollary 5.8 of [5]). *Let R be a filtered ring whose Rees ring is left and right Noetherian and such that R and its associated graded ring are both Auslander regular. If M is a pure R-module with a good filtration whose associated graded module is non-zero, then $j_R(\mathrm{gr}(M)) = j_R(M)$.*

This is the result we need but we must spend some time interpreting it in our language before we can apply it.

By a filtered ring Björk and Ekström mean a $\mathbb{Z}$-filtered ring which, practically speaking, coincides with our notion of $\mathbb{R}$-filtered ring in the case where $F^+_\mu R = F_\mu R$ for each $\mu \notin \mathbb{Z}$; except for the fact that their filtrations also go in the opposite direction to ours – that is they insist that $F_\mu R \subset F_\nu R$ whenever $\mu \leq \nu$. This latter problem can be dealt with by simple re-indexing and is not at all serious.

The Rees ring $\tilde{R}$ of a $\mathbb{Z}$-filtered ring R is the subring of $R[t, t^{-1}]$ given by

$$\tilde{R} = \bigoplus_{n \in \mathbb{Z}} F_n R t^n.$$

If χ is a rational character then for our purposes we do not lose generality by assuming that $\chi(A) = \mathbb{Z}$. In this case it is not difficult to see to that the Rees ring of $F^\chi DA$ is Noetherian since

$$\widetilde{F^\chi DA} = D(ker\chi)[t^{-1}, (\bar{a}t)^{\pm 1}]$$

where $a \in A$ is chosen such that $\chi(a) = 1$.

For the filtrations that we are interested in DA is isomorphic to its associated graded ring and so these rings are Auslander regular by Corollary 3.5.

By a pure module Björk and Ekström mean a module M such that each non-zero submodule M' of M satisfies $j_R(M') = j_R(M)$. By Theorem 3.4 this coincides with our definition of pure for DA-modules.

It now just remains to point out that by Remark 4.13 of [5] that, for χ rational, a natural χ-filtration is always good.

At last we can see that Proposition 5.1 tells us precisely what we wanted to know, namely that for rational characters $\dim_A(\mathrm{gr}^\chi(M)) = \dim_A(M)$. This completes the proof of

Theorem 5.2. *If M is a pure finitely generated DA-module of dimension m, then $\Delta(M)$ is a closed homogeneous rational polyhedral cone of dimension m.*

It now just requires a little work to complete the proof of Theorem C.

Lemma 5.3. *Let M be a finitely generated critical DA-module and let B be a subgroup of A. Then for any critical DB-submodule N of M of minimal dimension we have*

$$\pi_B(\Delta(M)) = \Delta(N).$$

Proof. Just run the proof of Theorem 5.5 of [7] remembering that $\Delta^*(N) = \Delta(N)$ and $\Delta^*(M) = \Delta(M)$. □

Theorem 5.4. *If M is a finitely generated pure DA-module of dimension m then $\Delta(M)$ is a closed totally concave homogeneous rational polyhedron of dimension m.*

Proof. We have already proved everything except that $\Delta(M)$ is totally concave.

Recall Theorem B, that is that for each $\chi \in \Delta(M)$ we have $LC_\chi(\Delta(M)) = \Delta(\mathrm{gr}^\chi M)$. Since $\mathrm{gr}^\chi M$ is a finitely generated DA-module it is sufficient to prove that the convex hull of $\Delta(N)$ is a linear subspace of $\mathrm{Hom}(A, \mathbb{R})$ for all finitely generated DA-modules N.

Let N be any finitely generated DA-module. Suppose that X is the linear subspace of $\mathrm{Hom}(A, \mathbb{R})$ spanned by $\Delta(N)$ and that $\hat{\Delta} \subset X$ is the convex hull of $\Delta(N)$ in $\mathrm{Hom}(A, \mathbb{R})$. If $\hat{\Delta}$ were contained in a closed halfspace of X then as $\Delta(N)$ is a rational polyhedral cone it is also contained in a rational half space of X, that is a half space of the form

$$H = \{\chi \in X | \chi(a) \geq 0\} \text{ for some } a \in A.$$

Now suppose that N' is a composition factor in a critical composition series for N and let L be a critical $D < a >$ submodule of N' of minimal dimension. By Lemma 5.3, $\pi_{<a>}(\Delta(N')) = \Delta(L)$, but

$$\pi_{<a>}(\Delta(N')) \subset \pi_{<a>}(\Delta(N)) \subset \mathbb{R}^{\geq 0}$$

and so $\Delta(L) \subset \mathbb{R}^{\geq 0}$. It follows that $\Delta(L) = 0$ and so that $\pi_{<a>}(\Delta(N')) = 0$. Now $\Delta(N)$ is the union of $\Delta(N')$ as N' ranges over all the composition factors in the critical composition series for N, so $\pi_{<a>}(\Delta(N)) = 0$, and $\Delta(N) \subseteq \{\chi \in X | \chi(a) = 0\}$ a proper subspace of X, contradicting the definition of X. □

6 Rigidity of $\Delta(M)$

In this section we show how the ideas of Bieri and Groves in [4] can be used to produce an alternative proof of Theorem B of [8]

If DA is a crossed product of a division ring D by a finitely generated free abelian group A we say a DA-module M is *impervious* if it contains no non-zero submodule of the form $N \otimes_{DB} DA$ for B a subgroup of A of infinite index.

Recall that two DA-modules are said to be similar if they have isomorphic essential submodules or, equivalently, if they have isomorphic injective hulls. We will write $[M]$ for the equivalence class containing M.

Given a DA-module M we define $\mathcal{F}_M(A)$ for the subgroup of A consisting of those a in A such that M has an essential submodule N that is $D\langle a\rangle$-torsion. This group depends only on the similarity class of M.

Given a ring automorphism γ of DA and a DA-module M we may define a new DA-module γM, by composing the action of DA on M with γ. We define the stabiliser of $[M]$, $\mathrm{Stab}_{DA}([M])$ to be the subgroup of $\mathrm{Aut}(DA)$ consisting of those γ such that $[M] = [\gamma M]$.

An automorphism of DA induces an automorphism of A that leaves $\mathcal{F}_M(A)$ invariant.

Theorem 6.1 (Theorem B of [8]). *Let M be a finitely generated impervious DA-module. Then $\mathrm{Stab}_{\mathrm{Aut}(DA)}[M]$ has finite image in $\mathrm{Aut}(A/\mathcal{F}_M(A))$.*

We begin by recalling a lemma from [4] that will prove to be useful.

Lemma 6.2 (Lemma 5.3 of [4]). *Let V be an n-dimensional vector space over $\mathbb{Q}$ and $\mathfrak{C}$ a finite family of $n-1$-dimensional subspaces. If $\mathfrak{C}$ has the property that it contains complements to every one dimensional subspace $L \leq V$, then V is spanned by the 1-dimensional subspaces of the form $X_1 \cap \cdots \cap X_{n-1}$, $X_i \in \mathfrak{C}$.*

We now prove Theorem 6.1 in the case where M is a uniform module; that is where that every non-zero submodule of M is essential in M.

Proposition 6.3. *If M is a finitely generated impervious uniform DA-module. Then $\mathrm{Stab}_{\mathrm{Aut}(DA)}[M]$ has finite image in $\mathrm{Aut}(A/\mathcal{F}_M(A))$.*

Proof. Suppose that N and N' are critical DA-submodules of M. Then as $N \cap N'$ is a non-zero submodule of N and of N' we have by Lemma 4.3

$$\Delta(N) = \Delta(N \cap N') = \Delta(N').$$

So we may define $\Delta^{core}(M) = \Delta(N)$. It is easy to see that $\Delta^{core}(M)$ depends only on the similarity class of M and so if $\gamma \in \mathrm{Stab}_{\mathrm{Aut}(DA)}[M]$ then the automorphism induced on $\mathrm{Hom}(A, \mathbb{R})$ leaves $\Delta = \Delta^{core}(M)$ invariant. Notice also that $\mathcal{F}_N(A) = \mathcal{F}_M(A)$ since M is uniform.

Now the *Bergman carrier* $\mathfrak{C}(\Delta)$ of Δ is defined to be the uniquely determined finite set of rational subspaces $X \leq \mathrm{Hom}(A, \mathbb{R})$ such that $\bigcup\{X | X \in \mathfrak{C}(\Delta)\}$ contains Δ and is minimal with respect to that property. Since, by Theorem C, $\Delta(N)$ is a homogeneous polyhedral cone, the dimension of each $X \in \mathfrak{C}(\Delta)$ is the dimension of N, m say.

By Lemma 2.5 of [8] the intersection $\bigcap\{X | X \in \mathfrak{C}(\Delta)\} = 0$, since N is impervious and critical. Moreover, by Lemma 2.4 of [8] the span of $\bigcup\{X | X \in \mathfrak{C}(\Delta)\}$ is $\ker \pi_{\mathcal{F}_N(A)}$. As $\mathrm{Stab}_{\mathrm{Aut}(DA)}([M])$ fixes Δ it acts on the finite set $\mathfrak{C}(\Delta)$ and so has a subgroup of finite index G, say, that fixes each of the elements of $\mathfrak{C}(\Delta)$ pointwise. If we can show that the 1-dimensional spaces that occur as intersections of spaces in $\mathfrak{C}(\Delta)$ span $\ker \pi_{\mathcal{F}_N(A)}$ then G must fix $\ker \pi_{\mathcal{F}_N(A)} \cong \mathrm{Hom}(A/\mathcal{F}_M(A))$ pointwise and the result will follow.

Now suppose that L is a 1-dimensional rational subspace of some $X \in \mathfrak{C}(\Delta) \subset \mathrm{Hom}(A, \mathbb{R})$. Then there is a subgroup B of A such that $A/B \cong \mathbb{Z}$

and L is the kernel of π_B. By Lemma 5.3, $\pi_B(\Delta) = \Delta(N')$ for some critical DB-submodule of N and so is a homogeneous totally concave polyhedral cone.

If $\dim \pi_B(\Delta) = m - 1$ then $L \subset Y$ for each $Y \in \mathfrak{C}(\Delta)$. But that would mean that L lies in the intersection $\bigcap\{Y | Y \in \mathfrak{C}(\Delta)\}$, a contradiction.

So $\dim \pi_B(\Delta) = m$. As $\pi_B(\Delta)$ is homogeneous and $\dim \pi_B(X) = m - 1$, it follows that $\pi_B(X)$ is contained in $\pi_B(Y)$ for some $Y \in \mathfrak{C}(\Delta)$ with $X \neq Y$. Thus $X \subset Y + L$ and so $X = (Y \cap X) \oplus L$. It follows that the set

$$\mathfrak{C}_X = \{X \cap Y | X \neq Y \in \mathfrak{C}(\Delta)\}$$

of rational subspaces of the rational space X has the property that every rational line in X has a complement in $\mathfrak{C}_X$. So by Lemma 6.2 we have that X is spanned by 1-dimensional intersections of subspaces in the Bergman carrier of X. Since this holds for each $X \in \mathfrak{C}(\Delta)$ we have that all these 1-dimensional spaces span the kernel of $\pi_{\mathcal{F}_M(A)}$ as required. $\square$

We now complete the proof of Theorem 6.1.

Proof. Suppose that M is not a uniform module. Then there is an essential submodule of M isomorphic to $N = M_1 \bigoplus \cdots \bigoplus M_k$ with each M_i a uniform DA-module. Then $[M] = [N]$. Now $\{\Delta^{core}(M_1), \dots, \Delta^{core}(M_k)\}$ is an invariant of $[M]$. It follows that the image of $\mathrm{Stab}_{\mathrm{Aut}(DA)}([M])$ in $\mathrm{Aut}(A)$ acts on this finite set and so has a subgroup G of finite index that fixes each of the elements $\Delta^{core}(M_i)$. By Proposition 6.3 the image of G in $\mathrm{Aut}(A)/\mathcal{F}_{M_i}(A)$ is finite for each i. So the image of G in the direct product of all these groups is finite. If H is the kernel of this last map then it has finite index in G and the image of H in $\mathrm{Aut}(A)/\mathcal{F}_M(A)$ is trivial since $\mathcal{F}_M(A) = \bigcap \mathcal{F}_{M_i}(A)$. The result follows. $\square$

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