Iwasawa Algebras Examples Sheet 2

1. Let (R,v) be a complete filtered ring and $(r_n)_{n\geq 0}$ be a sequence of elements of R such that $v(r_n)\to\infty$ as $n\to\infty$. Show that
$$\sum_{n\geq 0} r_n = \left(\sum_{n=0}^{m_\lambda} r_n + R_\lambda\right)_{\lambda\geq 0}$$
is a well-defined element of $R\cong\widehat{R}$ when the m_λ are chosen so that $v(r_n)\geq\lambda$ whenever $n>m_\lambda$.

   Show moreover that if (s_n) is another such sequence then (r_n+s_n) and $(\sum_{i+j=n} r_i s_j)$ are also such sequences and that
$$\left(\sum_{n\geq 0} r_n\right)+\left(\sum_{n\geq 0} s_n\right)=\sum_{n\geq 0}(r_n+s_n)$$
and
$$\left(\sum_{n\geq 0} r_n\right)\left(\sum_{n\geq 0} s_n\right)=\sum_{n\geq 0}\left(\sum_{i+j=n} r_i s_j\right).$$

2. Show that if (G,ω) is a p-valued group and $H\leq G$ is a subgroup equipped with the restricted p-valuation $\omega|_H$ then there is a natural inclusion $gr\,H\to gr\,G$ of graded Lie algebras.

   Deduce that if G has finite rank then H has finite rank and that if $p>2$ and $G=GL_n^1(\mathbf{Z}_p)$ then $gr\,H$ is a sub-Lie algebra of $t\mathfrak{gl}_n(\mathbf{F}_p[t])$.

3. Suppose that (G,ω) is a filtered group.

   (a) Show that $\widehat{G}$ has a separated filtration given by
$$\widehat{\omega}((g_\lambda G_\lambda)_{\lambda>0})=\inf\omega(g_\lambda\mid g_\lambda\notin G_\lambda)$$
with respect to which it is complete.

   (b) Show that the natural map $G\to\widehat{G}$ is injective if and only if ω is separated and always induces a natural isomorphism $gr\,G\to gr\,\widehat{G}$.

   (c) Show that if (R,v) is a complete filtered ring then every r in R with $v(r-1)>0$ is a unit. Moreover show that if $G=\{g\in R\mid v(g-1)>0\}$ and $\omega(g)=v(g-1)$ for $g\in G$ then (G,ω) is complete.

4. Suppose that (G,ω) is a complete p-valued group, $x\in G$ and $\lambda\in\mathbf{Z}_p$. Show that $\omega(x^\lambda)=\omega(x)+v_p(\lambda)$ and $\sigma(x^\lambda)=\sigma(x)\cdot\sigma(\lambda)$.

5. Suppose that p is an odd prime.

   (a) Show that if
$$G=\left\{\begin{pmatrix}1&x&z\\0&1&y\\0&0&1\end{pmatrix}:x,y,z\in p\mathbf{Z}_p\right\}$$
with the filtration ω induced by restricting the p-adic filtration on $M_n(\mathbf{Z}_p)$. Find an ordered basis (g_1,g_2,g_3) for G and compute
$$(g_1^{\lambda_1}g_2^{\lambda_2}g_3^{\lambda_3})(g_1^{\mu_1}g_2^{\mu_2}g_3^{\mu_3})=g_1^{\nu_1}g_2^{\nu_2}g_3^{\nu_3}$$
for $\lambda,\mu\in\mathbf{Z}_p^3$.

   (b) Find an ordered basis for $GL_n^1(\mathbf{Z}_p)$ with respect to its usual p-adic filtration.

6. Suppose that (R, v) is a filtered ring and let I be a left ideal of R. For $\lambda \in \mathbf{R}^{\geq 0}$ let $I_\lambda = \{r \in I \mid v(r) \geq \lambda\}$ and $I_{\lambda^+} = \{r \in I \mid v(r) > \lambda\}$. Show that

$$gr\, I = \bigoplus_{\lambda \in \mathbf{R}^{\geq 0}} I_\lambda / I_{\lambda^+}$$

can be viewed as a left ideal in $gr\, R$.

Suppose now that (R, v) is complete and $v(R \backslash 0)$ is a closed discrete subset of $\mathbb{R}^{\geq 0}$. Show that I is finitely generated if $gr\, I$ is finitely generated and deduce that R is left Noetherian if $gr\, R$ is left Noetherian.

7. Show that if (G, ω) is a complete p-valued group then an ordered basis for (G, ω) cannot contain a p-th power in G.

8. Let k be a commutative ring. Show that if $\mathfrak{g}$ is a graded k-Lie algebra then $U(\mathfrak{g})$ may be given the structure of a graded associative k-algebra in such a way that $U(\mathfrak{g})$ is free on $\mathfrak{g}$ with respect to the forgetful functor $\mathbf{grAss}_k \to \mathbf{grLie}_k$

9. Show that if $f\colon A \to B$ is a morphism of commutative rings and M is an A-module then $B \otimes_A M$ is the free B-module on M with respect to the restriction functor $\mathbf{Mod}_B \to \mathbf{Mod}_A$ along f.

10. Let (G, ω) be a p-valued group of finite rank. Show that there is a natural functor from the category of filtered $\mathbf{Z}_p$-algebras and filtered $\mathbf{Z}_p$-algebra homomorphisms to $\mathbf{FiltGp}$ such that $\mathbf{Z}_p[G]$ equipped with the filtration

$$\mathbf{Z}_p[G]_\lambda = \mathbf{Z}_p \cdot \{p^r(g_1 - 1)\cdots(g_s - 1) \mid r + \sum \omega(g_i) \geq \lambda \text{ for } g_1, \ldots, g_s \in G\}$$

is free on (G, ω) with respect to this functor.