Iwasawa Algebras Examples Sheet 1

1. Suppose that R is a ring and I a proper ideal. Show that the function $v\colon R \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ given by $v(a) = \sup\{k \in \mathbb{N}_0 : a \in I^k\}$ is a filtration of R that is separated if and only if $\bigcap_{n\geq 0} I^n = 0$.

2. Suppose that (R, v) is a filtered ring. Show that the function $v_n\colon M_n(R) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ given by $v_n(A) = \min_{1\leq i,j\leq n} v(A_{ij})$ is a filtration on $M_n(R)$ that is separated if and only if v is separated. Show moreover that, with respect to these filtrations, $gr\, M_n(R) \cong M_n(gr\, R)$.

3. Show that if (R, v) is a filtered ring then the set consisting of unions of cosets in $\{r+R_\lambda : r \in R, \lambda \in \mathbb{R}^{\geq 0}\}$ are the open sets of a topology on R. Show moreover that addition and multiplication in R are continuous maps $R \times R \to R$ when R is given this topology and $R \times R$ the product topology.

4. Show that if (R, v) is a filtered ring then the function $\hat{v}\colon \hat{R} \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ given by

$$\hat{v}\left((r_\lambda + R_\lambda)_{\lambda\geq 0}\right) = \inf\{\lambda \mid r_\lambda \notin R_\lambda\} = v(r_\mu) \text{ whenever } r_\mu \notin R_\mu$$

is a separated filtration on the completion $\hat{R}$ of R and that the natural map $\iota_R\colon R \to \hat{R};\ r \mapsto (r + R_\lambda)$ is a filtered ring homomorphism. Show that ι_R is injective precisely if v is separated and that $(\hat{R}, \hat{v})$ is always complete.

Show that there is an isomorphism of graded rings $gr\, R \to gr\, \hat{R}$.

5. Suppose that (S, w) is a complete filtered ring and $f\colon (R, v) \to (S, w)$ is a filtered ring homomorphism. Show that there is a unique filtered ring homomorphism $\hat{f}\colon \hat{R} \to S$ such that $f = \hat{f}\iota_R$.

6. Suppose that G is a group and $x, y, z \in G$. Show that
   (a) $(xy, z) = (x, z)^y(y, z)$;
   (b) $(x, yz) = (x, z)(x, y)^z$;
   (c) $(x^y, (y, z))(y^z, (z, x))(z^x, (x, y)) = e_G$.

7. Recall the lower central series γ_n of a group G. Show that $\omega\colon G \to \mathbb{R}^{>0} \cup \{\infty\}$ given by

$$\omega(x) = \sup\{n : x \in \gamma_n(G)\}$$

defines a filtration on G. Show moreover that $\gamma_{n+1}(G)$ is the smallest normal subgroup of G such that $\gamma_n(G)/\gamma_{n+1}(G)$ is contained in the centre of $G/\gamma_{n+1}(G)$ for all $n \geq 1$.

8. Verify the Hall-Petrescu formula for $n \leq 4$. That is show that for all $x, y \in G$ there are elements $c_i \in \gamma_i(G)$ for $2 \leq i \leq 4$ such that

$$x^n y^n = (xy)^n c_2^{\binom{n}{2}} c_3^{\binom{n}{3}} \cdots c_n^{\binom{n}{n}}.$$

9. (a) Let R be a commutative ring and let G denote the group of 3×3 upper-unitriangular matrices with entries in R

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in R \right\}$$

equipped with the filtration coming from the lower central series as in question **??**. Show that the filtration is separated and $gr\, G$ is an R-Lie algebra $RX \oplus RY \oplus RZ$ (free of rank 3 as an R-module) with

$$gr_1\, G = RX \oplus RY \text{ and } gr_2\, G = RZ,$$

$[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$.

(b) Suppose now that $R = \mathbb{Z}_p$ and

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in pR \right\}$$

with the filtration ω induced by restricting the p-adic filtration on $M_n(\mathbb{Z}_p)$. Show that

$$gr\, G = \oplus_{n \in \mathbf{N}}\, gr_n\, G$$

is a $gr\, \mathbb{Z}_p = \mathbb{F}_p[t]$-Lie algebra,

$$gr_n\, G = \mathbb{F}_p t^n X \oplus \mathbb{F}_p t^n Y \oplus \mathbb{F}_p t^n Z \text{ for } n \geq 1$$

with tX, tY and tZ free generators all of degree 1, $[tX, tY] = t^2 Z$ and $[tX, tZ] = [tY, tZ] = 0$.

10. Suppose that ω is a p-valuation on a group G and $g \in \gamma_n(G)$. Show that $\omega(g) > n/(p-1)$. Moreover show that if $g \in \gamma_n(< x, y >)$ and $n \geq 2$ then $\omega(g) > \frac{n-1}{p-1} + \max(\omega(x), \omega(y))$.

11. Show that if (G, ω) is a p-valued group of finite rank then $gr\, G$ is free as a graded $\mathbb{F}_p[t]$-module i.e. that there is a finite generating set of $gr\, G$ as an $\mathbb{F}_p[t]$-module consisting of homogeneous elements.