Noncommutative Noetherian Rings

Lent 2026

Example Sheet 4

Throughout this sheet, R will denote a ring and k will denote a field.

1. Let $\phi\colon R \to T$ make T into a left ring of fractions with respect to a left denominator set S of R. Suppose that $\theta\colon R \to U$ is any ring homomorphism such that $\theta(S) \subseteq U^\times$. Show that there is a unique ring homomorphism $\psi : T \to U$ such that $\theta = \psi \circ \phi$.

2. Let R be a ring and S a left denominator set in R.

   (a) Show that if M is an S-torsionfree S-divisible R-module then $\iota_M\colon M \to S^{-1}M$ is an isomorphism of R-modules.

   (b) Show that if M, N are left $S^{-1}R$-modules then every R-module homomorphism $M \to N$ is an $S^{-1}R$-module homomorphism.

3. Show that if k is a field and R is the ring of endomorphism of a vector space with a countably infinite basis, and S is the set of injective maps in R then $S^{-1}R = 0$.

4. Recall that, for $r \in R$, ad_r is defined by

$$\mathrm{ad}_r\colon R \to R; a \mapsto ra - ar.$$

   We say that ad_r is *locally nilpotent* if for all $a \in R$ there is $n \geq 1$ such that $\mathrm{ad}_r^n(a) = 0$.

   (a) Suppose that S is a multiplicatively closed subset of R. Show that if for every $s \in S$, ad_s is locally nilpotent then S is a left denominator set in R.

   Suppose now that there is $t \in Z(R)$ such that $t^n = 0$ and R/tR is commutative.

   (b) Deduce that every multiplicatively closed subset $S \subset R$ is a left denominator set.

   (c) Show that if $S \subset R$ is multiplicatively closed and $\overline{S} = S + tR \subset R/tR$ then there is a natural bijection

$$\mathrm{Spec}(\overline{S^{-1}}(R/tR)) \to \mathrm{Spec}(S^{-1}R)$$

5. Show that if R is a filtered ring with Rees ring $\widetilde{R}$ and M is a filtered R-module then the Rees module $\widetilde{M}$ has the following properties

   (a) $\widetilde{M}/t\widetilde{M} \cong \mathrm{gr}\, M$ as graded $\widetilde{R}$-modules.

   (b) $\widetilde{M}/(t-1)\widetilde{M} \cong M$ as left $\widetilde{R}$-modules.

6. Show that the subspace filtration and quotient filtration are filtrations and there is a natural exact sequence of graded $\operatorname{gr} R$-modules

$$0 \to \operatorname{gr} N \to \operatorname{gr} M \to \operatorname{gr} M/N \to 0.$$

7. Show that if A is a commutative noetherian ring and $\operatorname{Kdim}(A) < \infty$ then Kdim is a dimension function for A.

8. Let k be a field of characteristic p and $n \geq 1$

   (a) Show that the nth Weyl algebra $A_n(k)$ has a non-zero module that is finite-dimensional over k.

   (b) Describe all the simple modules of $A_n(k)$ in the case that k is algebraically closed.

9. Let $\mathfrak{g} = \mathfrak{sl}_2$ be the $\mathbb{C}$-Lie algebra $\mathfrak{g} = \{A \in \operatorname{Mat}_2(\mathbb{C}) : \operatorname{tr}(A) = 0\}$ with commutator bracket $[-,-]$ and let $G = GL_2(\mathbb{C})$ act on $\mathfrak{g}$ by conjugation and on $\mathfrak{g}^* = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathbb{C})$ via

$$(g \cdot \theta)(A) = \theta(g^{-1}Ag) \text{ for } g \in G, \theta \in \mathfrak{g}^* \text{ and } A \in \mathfrak{g}.$$

   (a) Show that the bilinear form $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$; $(A, B) \mapsto \operatorname{tr}(AB)$ induces a G-linear bijection $\mathfrak{g} \to \mathfrak{g}^*$.

   (b) What are the G-orbits in $\mathfrak{g}$? What are the G-orbits in $\mathfrak{g}^*$?

   (c) For each $\theta \in \mathfrak{g}^*$ the natural Poisson bracket on $S(\mathfrak{g})$ induces an alternating bilinear form on $\mathfrak{g}$

$$(x, y)_\lambda = \lambda([x, y]).$$

   For a chosen representative λ of each orbit $G \cdot \lambda \subset \mathfrak{g}^*$ describe $(x, y)_\lambda$.

   (d*) Repeat parts (a)-(c) for $\mathfrak{g} = \mathfrak{sl}_3$ and $G = GL_3(\mathbb{C})$.

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