Noncommutative Noetherian Rings
Lent 2026

Example Sheet 3

Throughout this sheet, R will denote a ring and k will denote a field.

1. Show that if R is a left noetherian domain then R is uniform as a left R-module. Deduce directly that $R - \{0\}$ is a left Ore set.

2. Let $F_1, F_2, \ldots,$ be a countable family of fields. Show that $I = \bigoplus_{n \in \mathbb{N}} F_n$ is an ideal in $A = \prod_{n \in \mathbb{N}} F_n$. Show that no nonzero principal ideal of $R = A/I$ is indecomposable and deduce that $E(R)$, the injective hull of R as a left R-module, has no indecomposable direct summands.

3. Suppose M is a R-module of finite rank and $N \leqslant M$ is essentially closed. Show that rank M/N is finite and rank $M = $ rank $N + $ rank M/N.

4. Classify all uniform finitely generated $\mathbb{Z}$-modules. Show that a finitely generated uniform $\mathbb{Z}$-module may have a quotient of arbitrarily large finite rank. By considering $\mathbb{Q}/\mathbb{Z}$ find an explicit description of all uniform injective $\mathbb{Z}$-modules.

5. Show that the relation on R-modules $M \sim N$ if and only if M and N have isomorphic essential submodules is an equivalence relation. Show that $E(M)$ is isomorphic to $E(N)$ if and only if $M \sim N$. Deduce that non-isomorphic simple modules never have isomorphic injective hulls.

6. Suppose that $R = A_1(k) = k\langle x, y \rangle/(yx - xy - 1)$. Show that $\{1, x, x^2, \ldots\}$, $\{1, y, y^2, \ldots\}$, $k[x] \backslash 0$ and $k[y] \backslash 0$ are all Ore sets in R.

7. Suppose S is a left Ore set in a ring R and M is a left R-module.

   (a) $t_S(M)$ is an S-torsion module and $M/t_S(M)$ is S-torsionfree.

   (b) If N is a left R-module and $f \in \operatorname{Hom}_R(M, N)$ then $f(t_S(M)) \leqslant t_S(N)$.

   (c) If $N \leqslant M$, then M is S-torsion if and only if both N and M/N are S-torsion.

   (d) If $N_i \leqslant M$ for $i \in I$ and each N_i is S-torsion then $\sum_{i \in I} N_i$ is S-torsion.

   (e) All submodules and products of S-torsionfree modules are S-torsionfree.

   (f) If $N \leqslant_e M$ and N is S-torsionfree then M is S-torsionfree.

   (g) If $N \leqslant M$ with N and M/N both S-torsionfree then M is S-torsionfee.

8. Show that if R is a left noetherian ring and S is a left Ore set consisting of regular elements then $S^{-1}R$ is a left noetherian ring.

9. Show that a ring R has a simple artinian classical left quotient ring if and only if R is a prime left Goldie ring.

10. Suppose that α is ring endomorphism of R.

   (a) Show that there exists a ring $S = R[y; \alpha]$ that can be equipped with the data of an injective ring homomorphism $\phi \colon R \to S$ and an element $y \in S$ such that for every ring T there is a bijection between the set of ring homomorphisms $\psi \colon S \to T$ and the set of pairs (θ, t) such that $\theta \colon R \to T$ is a ring homomorphism and $t \in T$ satifying $t\theta(r) = \theta(\alpha(r))t$ for all $r \in R$ given by $\psi \mapsto (\psi\phi, \psi(y))$.

   (b) Show that if R is left noetherian and α is an automorphism then $R[y; \alpha]$ is left noetherian.

   (c) Show that if $R = k(x)$ is the field of rational functions in one variable over k and α is defined by $\alpha(f(x)) = f(x^2)$ then $R[y; \alpha]$ is a left noetherian domain but is not right noetherian. Show moreover that the non-zero elements of $R[y; \alpha]$ form a left Ore set but not a right Ore set.

11. Show that if $V = k^{\mathbb{N}}$ is a k-vector space with a countable basis then

$$S = \{\alpha \in \operatorname{End}_k(V) : \alpha \text{ injective}\}$$

is a multiplicatively closed subset of $\operatorname{End}_k(V)$ all of whose elements have left inverses. Show that S is a left Ore set that is not left reversible.

12. Suppose that R is a semiprime left Goldie ring, S is the set of its regular elements and M is a left R-module.

   (a) Using Goldie's Regular Element Lemma show that if M is S-torsionfree and S-divisible then M is injective. Deduce that the classical left quotient ring Q of R is an injective hull of R.

   (b) Show that if M is not S-torsion then M has a uniform submodule isomorphic to a left ideal of R.

   (c) Give an example to show that it is not always the case that all S-torsionfree uniform modules are isomorphic to left ideals of R.

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