Noncommutative Noetherian Rings

Lent 2026

Example Sheet 2

Throughout this sheet, R will denote a ring.

1. An element $e \in R$ is an *idempotent* if $e^2 = e$. Suppose $e \in R$ is a central idempotent.
   (a) Show that $1 - e$ is also a central idempotent.
   (b) Show that eR is a ring with the same operations as in R, but with multiplicative identity element e.
   (c) Show that R is isomorphic to the direct product of rings $eR \times (1-e)R$.

2. (a) Suppose that D is an division ring and $n \geq 1$ and let $R = M_n(D)$. By first considering the set of left ideals of R, show that R has no non-trivial two-sided ideals.
   (b) By directly computing the set of left ideals, show that $R = M_{d_1}(D_1) \times \cdots \times M_{d_n}(D_n)$ is a semi-primitive left artinian ring for any finite collection of division rings $D_1, \ldots, D_n$ and positive integers $d_1, \ldots, d_n$. Deduce that a semi-primitive left artinian ring is right artinian.

3. Let R be a left artinian ring and let $x \in R$ be such that $sx = 0$ implies $s = 0$. Prove that x is a unit in R. Deduce that a left artinian ring without zero-divisors must be a division ring.

4. Show that the following are equivalent for R:
   (a) All left R-modules are semisimple;
   (b) All right R-modules are semisimple;
   (c) R is a semisimple left R-module;
   (d) R is a semisimple right R-module;
   (e) R is semiprimitive and left artinian.

5. Find an example of a left artinian ring which is not right artinian.

6. Use the Jacobson Density Theorem to show that if R is left primitive but not left artinian then there is a division ring D such that for every natural number n there is a subring R_n of R with a surjective ring homomorphism $R_n \to M_n(D)$.

7. Let k be a field, and let $T := \{A \in M_n(k) : A_{ij} = 0 \text{ whenever } i > j\}$ be the ring of upper-triangular matrices with entries in k. Find the Jacobson radical $J(T)$ of T.

8. Let R be a finite dimensional $k-$algebra, where k is an algebraically closed field and let M be a simple $R-$module. Show that $\mathrm{End}_R(M) \cong k$.

9. (Maschke) Let G be a finite group and let k be a field. Show that kG is left and right artinian. Show that kG is semiprimitive if and only if $\mathrm{char}\, k \nmid |G|$.

10. Let G be a finite p-group and let k be a field of characteristic p. Using that $Z(G) \neq 1$ and induction on $|G|$, show that $J(kG) = \ker(\varphi)$ where $\varphi : kG \to k = \mathrm{End}_k(k)$ defines the trivial representation.

11. Let k be a field of characteristic 0 and consider the k-Lie algebra $\mathfrak{g}$ with basis x, y and such that $[yx] = x$.

   (a) Show that $U(\mathfrak{g})$ is both left and right noetherian.

   (b) Show that $\{y^i x^j : i, j \geq 0\}$ is a basis for $U(\mathfrak{g})$ as a k-vector space.

   (c) Show that $U(\mathfrak{g})$ is not a simple ring.

   (d) Show that $U(\mathfrak{g})/U(\mathfrak{g})(x-1)$ is a simple faithful $U(\mathfrak{g})$-module. Deduce that $U(\mathfrak{g})$ is a primitive ring.

12. Show that an R-module E is injective if and only if for every inclusion of R-modules $L \to M$ and every map $\alpha \in \mathrm{Hom}_R(L, E)$ there exists $\beta \in \mathrm{Hom}_R(M, E)$ such that $\beta|_L = \alpha$.

13. Show that if $(E_i)_{i\in I}$ is any family of R-modules then $\prod_{i\in I} E_i$ is an injective R-module if and only if each E_i is an injective R-module. Deduce that finite direct sums of injective modules are always injective and direct summands of injective modules are injective.

14. Suppose M is a left R-module. Prove the following statements.

   (a) If $N \leqslant_e M$ and $L \leqslant_e N$ then $L \leqslant_e M$.

   (b) If $L_1, L_2, N_1, N_2 \leqslant_e M$ with $L_1 \leqslant_e N_1$ and $L_2 \leqslant_e N_2$ then $L_1 \cap L_2 \leqslant_e N_1 \cap N_2$.

   (c) If $N \leqslant_e M$ and $m \in M$ then $Nm^{-1} := \{r \in R : rm \in N\} \leqslant_e R$.

   (d) If $N_i \leqslant_e M_i$ for $i \in I$ then $\bigoplus_{i\in I} N_i \leqslant_e \bigoplus_{i\in I} M_i$.

   (e) If M has no proper essential submodules then M is semisimple.

15. Suppose that A is a commutative domain. Show that its field of fractions $Q(A)$ is an injective hull of A viewed as a (left) A-module.

16. Suppose $N < M$ are R-modules show that $E(M)$ is isomorphic to a submodule of $E(N) \oplus E(M/N)$.

17. Let $A = k[x, y]/(x, y)^2$ for a field k. Show that A viewed as a left A-module has finite rank but is not a direct sum of uniform submodules.

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