

Noncommutative Noetherian Rings

Lent 2026

Example Sheet 1

Throughout this sheet, k denotes a commutative ring.

1. Suppose that $\alpha: R \rightarrow S$ is a ring homomorphism show that $\alpha(R^\times) \leq S^\times$ but that $\alpha(Z(R)) \subset Z(S)$ does not necessarily hold.
2. Suppose that R is a k -algebra.
 - (a) Show that $\text{Der}_k(R)$ is a k -Lie algebra under the commutator bracket.
 - (b) Show that for all $a \in R$, $\text{ad}_a: R \rightarrow R$; $\text{ad}_a(b) = [a, b]$ is a derivation of R .
 - (c) Show that $\text{ad}: R \rightarrow \text{Der}_k(R)$; $\text{ad}(x)(y) = [x, y]$ is a Lie algebra homomorphism.
 - (d) When $R = k[x_1, \dots, x_d]$, show that for every $\delta \in \text{Der}_k(R)$ there are unique $f_1, \dots, f_d \in R$ such that $\delta(y) = \sum_{i=1}^d f_i \partial y / \partial x_i$ for all $y \in R$.
3. Let R be a ring containing an element x . By using the binomial theorem, or otherwise, prove that

$$x^n r = \sum_{i=0}^n \binom{n}{i} \text{ad}_x^i(r) x^{n-i} \text{ for all } r \in R.$$

4. Suppose that R is a k -algebra and $\delta: R \rightarrow R \in \text{Der}_k(R)$, let $R^{\mathbb{N}}$ be the k -module of infinite sequences $(v_i) = (v_0, v_1, v_2, \dots)$ of elements of R under pointwise operations. Define $\psi: R \rightarrow \text{End}_k(R^{\mathbb{N}})$ by $\psi(r)((v_i)) = (rv_i)$, and let $\sigma \in \text{End}_k(R^{\mathbb{N}})$ be given by $\sigma((v_i)) = (v_{i-1} + \delta(v_i))$ with the convention $v_{-1} := 0$.

- (a) Prove that ψ is an injective k -algebra homomorphism, and that

$$\sigma\psi(r) = \psi(r)\sigma + \psi(\delta(r)) \text{ for all } r \in R.$$

- (b) Show that the k -subalgebra S of $\text{End}_k(R^{\mathbb{N}})$ generated by $\text{Im}(\psi)$ and σ is a free R -module on $\{\sigma^i : i \geq 0\}$ under $r \cdot s = \psi(r)s$ for $r \in R$ and $s \in S$.
- (c) Deduce that for any k -algebra T , the set of k -algebra homomorphism $\beta: S \rightarrow T$ is in bijection with the set of pairs (α, y) such that $\alpha: R \rightarrow T$ is a k -algebra homomorphism and y is an element of T such that $y\alpha(a) - \alpha(a)y = \alpha(\delta(a))$ under the map $\beta \mapsto (\beta \circ \psi, \beta(\sigma))$. Show that this property uniquely determines S up to unique isomorphism in a suitable sense.¹
- (d) Show that if R is left noetherian then S is left noetherian.
- (e) Show that if $R = k[x]$ and $\delta = d/dx$ then S is isomorphic to $A_1(k)$.

¹ S is known as a skew-polynomial ring and is sometimes denoted $R[x; \delta]$.

5. Suppose that k is a field.
- Show that the Weyl-algebra $A_n(k)$ is left noetherian.
 - Prove that $\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_n^{\beta_n} : \alpha, \beta \in \mathbb{N}_0^n\}$ is a basis for $A_n(k)$ as a k -vector space.
 - Deduce that $A_n(k) \rightarrow \text{End}_k(k[x_1, \dots, x_n])$ is injective if and only if $\text{char}(k) = 0$.
6. (a) Let R be a ring, and let N be a submodule of an R -module M . Prove that M is noetherian if and only if N and M/N are noetherian. Show that if M is the sum of its noetherian submodules M_1, \dots, M_n then M is also noetherian. Deduce that R is left noetherian if and only if every finitely generated R -module is noetherian.
- (b) Suppose that R has left noetherian subring S and R is finitely generated as a left S -module. Prove that R is also left noetherian.
7. Show that the free algebra $k\langle x, y \rangle$ is neither left nor right noetherian. Deduce that $k\langle x_1, \dots, x_n \rangle$ is left noetherian if and only if it is right noetherian if and only if k is noetherian and $n < 2$.
8. Show that the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right, but not left, noetherian. Find an example of a ring which is left noetherian but not right noetherian.
9. (a) Let G be a group which has a polycyclic subgroup of finite index. Prove that if k is noetherian then kG is noetherian.
- (b) Show that if G is a group with subgroup H then the free k -module kG/H is naturally a cyclic left kG -module. Deduce that if G has a strictly ascending chain of subgroups then kG isn't left noetherian.
10. Let R_1, \dots, R_n be rings. Show that the *direct product* $R = R_1 \times \cdots \times R_n$ is a ring with componentwise multiplication and addition. Show that R is left noetherian if and only if each R_i is left noetherian. Show that the same result holds with noetherian replaced by artinian.
11. Let R be a ring and let M be a noetherian left R -module. Suppose $\varphi : M \rightarrow M$ is an R -module homomorphism. Show that if φ is surjective, then it is also injective. Deduce that if R is a left noetherian ring with $ab = 1$ for some $a, b \in R$ then $ba = 1$.
12. Suppose that k is a field of characteristic 0.
- Show that $k[x]$ is a simple $\Delta(k[x])$ -module under the natural action. Show that $k[x, x^{-1}]$ is also naturally a $\Delta(k[x])$ -module. What are its submodules and quotient modules?
 - Consider the k -Lie algebra \mathfrak{sl}_2 of 2×2 -matrices with trace 0 and its basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$
 Show that for each $\lambda \in k$ there is a unique k -algebra homomorphism $\theta_\lambda : U(\mathfrak{sl}_2(k)) \rightarrow \Delta(k[x])$ with $e \mapsto -\partial/\partial x$, $h \mapsto -2\hat{x}\partial/\partial x + \lambda$ and $f \mapsto \hat{x}^2\partial/\partial x - \lambda\hat{x}$.
 - When is $k[x]$ a simple $U(\mathfrak{sl}_2(k))$ -module under the action induced by θ_λ ? In the cases when it is not simple what are its submodules and quotient modules? What about for $k[x, x^{-1}]$?