Representation Theory — Examples Sheet 2

On this sheet all representations are complex representations

1. Let (ρ, V) be a representation of a finite group G with character χ. Show that $\ker \rho = \{g \in G \mid \chi(g) = \chi(1)\}$. Show further that $|\chi(g)| \leq \chi(1)$ for all $g \in G$, with equality precisely if $\rho(g) = \lambda\, \mathrm{id}_V$ for some $\lambda \in \mathbb{C}^\times$. Explain how the set of normal subgroups of G may be calculated directly from the character table.

2. Let χ be the character of a representation of a group G and let $g \in G$. If g has order 2 show that $\chi(g) \in \mathbb{Z}$ and that $\chi(g) \equiv \chi(1) \mod 2$. Show that if in addition G is a non-cyclic simple group then $\chi(g) \equiv \chi(1) \mod 4$. If instead g has order 3 and is conjugate to g^2 show that $\chi(g) \equiv \chi(1) \mod 3$.

3. Construct the character tables of the dihedral group D_8 and the quaternion group Q_8. What do you notice? Compare the determinants of their respective two dimensional representations.

4. Construct the character tables of the dihedral groups D_{10} and D_{12}. How do the irreducible representations decompose when restricted to the subgroups of rotations?

5. Construct the character tables of A_4, S_4, A_5 and S_5. The action of S_n on A_n by conjugation induces an action on the character table of A_n by permuting the conjugacy classes. Describe what this does to the rows of the character table for $n = 4, 5$.

6. A group of order 720 has 11 conjugacy classes. Two representations of the group are known and have corresponding characters α and β. The table below summarises the sizes of the conjugacy classes and the values of α and β on them. Prove that the group has an irreducible representation of degree 16 and calculate its character.

$\lvert[g]\rvert$	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3 .
β	21	1	-3	-1	1	1	1	0	-1	-3	0

7. A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have characters α, β and γ summarised in the table below. Construct the character table of the group. You may assume if required that $\sqrt{7}$ is not in the field generated by $\mathbb{Q}$ and a primitive 7^{th} root of unity.

$\lvert[g]\rvert$	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

8. By considering the action of a finite group G on itself by conjugation show that the sum of elements in any row of the character table of G is a non-negative integer.

9. Let $G = S_n$ act naturally on the set $X = \{1, \ldots, n\}$. For each non-negative integer r, let X_r be the set of all r-element subsets of X equipped with the natural action of G, and π_r be the character of the corresponding permutation represention. If $0 \leq l \leq k \leq n/2$, show that

$$\langle \pi_k, \pi_l \rangle_G = l + 1.$$

Deduce that $\pi_r - \pi_{r-1}$ is the character of an irreducible representation for each $1 \leq r \leq n/2$. What happens for $r > n/2$?

10. By considering the actions induced on the rows and on the columns of the character table by complex conjugation, show that the number of irreducible characters of G that only take real values is the number of self-inverse conjugacy classes.

11. Let G be a finite group and χ be an irreducible character of G. By beginning with the irreducible representations, show that if (ρ, V) is any representation of G then $\frac{\chi(1)}{|G|}\sum_{g\in G}\overline{\chi(g)}\rho(g)$ is a G-linear projection onto a subspace of V. Deduce that every representation can be decomposed uniquely into isotypical components.

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