Representation Theory — Examples Sheet 1

1. Let ρ be a representation of a group G. Show that $\det \rho$ is a representation of G. What is its degree?

2. Let θ be a one-dimensional representation of a group G and $\rho\colon G \to GL(V)$ another representation of G. Show that $\theta \otimes \rho\colon G \to GL(V)$ given by $\theta \otimes \rho(g) = \theta(g) \cdot \rho(g)$ defines a representation of G. If ρ is irreducible, must $\theta \otimes \rho$ also be irreducible?

3. Suppose that N is a normal subgroup of a group G. Given a representation of the quotient group G/N on a vector space V, explain how to construct an associated representation of G on V. Which representations of G arise in the way? Recall that G' is the normal subgroup of G generated by all elements of the form $ghg^{-1}h^{-1}$ with $g, h \in G$. Show that the 1-dimensional representations of G are precisely those that arise from 1-dimensional representations of G/G'.

4. Suppose that (ρ, V) and (σ, W) are representations of a group G. Show that $(\tau, \mathrm{Hom}_k(V, W))$ is a representation of G where $\tau(g)(\alpha) := \sigma(g) \circ \alpha \circ \rho(g^{-1})$ for all $g \in G$ and $\alpha \in \mathrm{Hom}_k(V, W)$.

5. Let $\rho\colon \mathbb{Z} \to GL_2(\mathbb{C})$ be the representation defined by $\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Show that ρ is not completely reducible. By a similar construction, show that if k is a field of characteristic p there is a two dimensional k-representation of C_p that is not completely reducible.

6. Let C_n be the cyclic group of order n. Explicitly decompose the complex regular representation $\mathbb{C}C_n$ as a direct sum of irreducible subrepresentations.

7. Let D_{10} be the dihedral group of order 10. Show that every irreducible $\mathbb{C}$-representation of D_{10} has degree 1 or 2. By describing them explicitly, show that there are precisely four such representations up to isomorphism. Show moreover that for each such representation it is possible to choose a basis so that all the representing matrices have real entries.

8. What are the irreducible real representations $\rho\colon C_n \to GL(V)$ of a cyclic group of order n? Compute $\mathrm{Hom}_G(V, V)$ in each case. How does the real regular representation $\mathbb{R}C_n$ of C_n break up as a direct sum of irreducible representations?

9. Show that (up to isomorphism) there is only one irreducible complex representation of Q_8 of dimension at least two. Show that this representation cannot be realised over $\mathbb{R}$ and deduce that that Q_8 is not isomorphic to a subgroup of $GL_2(\mathbb{R})$. Find a four-dimensional irreducible real representation V of Q_8. Compute $\mathrm{Hom}_G(V, V)$ in this case.

10. Suppose that k is algebraically closed. Using Schur's Lemma, show that if G is a finite group with trivial centre and H is a subgroup of G with non-trivial centre, then any faithful representation of G is reducible after restriction to H. What happens for $k = \mathbb{R}$?

11. Let (ρ, V) be an irreducible complex representation of a finite group G. For each $v \in V$, show that the $\mathbb{C}$-linear map $\mathbb{C}G \to V$ given by $\delta_g \mapsto \rho(g)(v)$ is G-linear and deduce that V is isomorphic to a subrepresentation of $\mathbb{C}G$. What is $\dim \mathrm{Hom}_G(\mathbb{C}G, V)$?

12. Suppose that G and H are groups, (ρ, V) is a representation of $G \times H$ and (σ, W) is a representation of G. By considering the restriction of ρ to G along the evident homomorphism $G \to G \times H$; $g \mapsto (g, e_H)$ show that $\mathrm{Hom}_G(V, W)$ is a representation of H via $h \cdot \alpha = \alpha \circ \rho(e_G, h^{-1})$ for all $\alpha \in \mathrm{Hom}_G(V, W)$ and $h \in H$.

13. Let G be the subgroup of the symmetric group S_6 generated by $(123), (456)$ and $(23)(56)$. Show that G has an index two subgroup of order 9 and four normal subgroups of order 3. By considering quotients show that G has two complex representations of degree 1, and four pairwise non-isomorphic irreducible complex representations of degree 2, none of which is faithful. Does G have a faithful irreducible complex representation?

14. Show that if $\rho\colon G \to GL(V)$ is a representation of a finite group G on a real vector space V then there is a basis for V with repect to which the matrix representing $\rho(g)$ is orthogonal for every $g \in G$. Which finite groups have a faithful two-dimensional real representation?

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