Representation Theory — Examples Sheet 3

On this sheet all groups are finite and all representations are complex representations

1. Calculate S^2V and Λ^2V for the two-dimensional irreducible representations of D_8 and of Q_8. Which has the trivial representation as a subrepresentation in each case?

2. Let $G = S_n$ act naturally on the set $X = \{1, \dots, n\}$. For each non-negative integer r, let X_r be the set of all r-element subsets of X equipped with the natural action of G, and π_r be the character of the corresponding permutation represention. If $0 \leq l \leq k \leq n/2$, show that
$$\langle \pi_k, \pi_l \rangle_G = l + 1.$$
Deduce that $\pi_r - \pi_{r-1}$ is a character of an irreducible representation for each $1 \leq r \leq n/2$. What happens for $r > n/2$?

3. Suppose $\rho\colon G \to GL(V)$ is an irreducible representation of G with character χ. By considering $V \otimes V$, S^2V and Λ^2V show that
$$\frac{1}{|G|}\sum_{g\in G}\chi(g^2) = \begin{cases} 0 & \text{if } \chi \text{ is not real-valued} \\ \pm 1 & \text{if } \chi \text{ is real valued.} \end{cases}$$
Deduce that if $|G|$ is odd then G has only one real-valued irreducible character.

4. Let $\rho\colon G \to GL(V)$ be a representation of G of dimension d.
   (a) Compute $\dim S^nV$ and $\dim \Lambda^nV$ for all n.
   (b) Let $g \in G$ and $\lambda_1, \dots, \lambda_d$ be the eigenvalues of $\rho(g)$. What are the eigenvalues of g on S^nV and Λ^nV?
   (c) Let $f(t) = \det(tI - \rho(g))$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of f and $\chi_{\Lambda^n V}$?
   (d) What is the relationship between $\chi_{S^nV}(g)$ and f? (Hint: start with case $d = 1$).

5. Recall the character table of D_{10} from sheet 2. Explain how to view D_{10} as a subgroup of A_5 and then use induction from D_{10} to A_5 to reconstruct the character table of A_5.

6. Obtain the character table of the dihedral group D_{2m} by using induction from the cyclic group C_m; you will want to split into two cases according as m is odd or even.

7. Find all the characters of S_5 obtained by inducing irreducible representations of S_4. Use these to reconstruct the character table of S_5. Then repeat, replacing S_4 by the subgroup $\langle (12345), (2354) \rangle$ of S_5 of order 20.

8. Prove that if H is a subgroup of a group G, and K is a subgroup of H, and W is a representation of K then $\mathrm{Ind}_K^G W \cong \mathrm{Ind}_H^G \mathrm{Ind}_K^H W$.

9. Let H be a subgroup of a group G. Show that for every irreducible representation (ρ, V) of G there is an irreducible representation (ρ', W) of H such that ρ is an irreducible component of $\mathrm{Ind}_H^G W$.
   Deduce that if A is an abelian subgroup of G then every irreducible representation of G has dimension at most $|G/A|$.

10. Suppose that G is a Frobenius group with Frobenius kernel K. Show that if V is a non-trivial irreducible representation of K then $\mathrm{Ind}_K^G V$ is also irreducible. Hence, explain how to construct the character table of G given the character tables of K and G/K.

11. Suppose that V is a faithful representation of a group G such that χ_V takes r distinct values. Show that each irreducible representation of G is a summand of $V^{\otimes n}$ for some $n < r$.

12. Suppose G is a finite group of odd order and with k conjugacy classes. Show that $|G| \equiv k \mod 16$.

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