Mich 2019 SJW

Representation Theory — Examples Sheet 4

On this sheet all representations are complex representations unless stated otherwise

1. Show that character values of S_n are always integers.

2. Let $G = SU(2)$ and V_n be the vector space of complex homogeneous polynomials of degree n in the variables x and y.

   (a) Describe how to view V_n as an irreducible representation of $SU(2)$. What is its character?

   (b) Show that V_n is isomorphic to its dual V_n^*.

   (c) Decompose the representations $V_4 \otimes V_3$, $V_3 \otimes V_3$, S^2V_3 and $\Lambda^2 V_3$ into irreducibles.

   (d) How do $V_1^{\otimes n}$, S^nV_1, S^2V_n and Λ^2V_n decompose into irreducibles for $n \geq 1$. What about S^3V_2?

3. Let $SU(2)$ act on the space $M_3(\mathbb{C})$ of 3×3 complex matrices by
$$A\colon X \mapsto A_1 X A_1^{-1},$$
where A_1 is the 3×3 block diagonal matrix with block diagonal entries $A, 1$. Show that this defines a representation of $SU(2)$ and decompose it into irreducibles.

4. Let χ_n be the character of the irreducible representation of $SU(2)$ of dimension $n+1$. Show that
$$\frac{1}{2\pi}\int_0^{2\pi} K(z)\overline{\chi_n}\chi_m \,\mathrm{d}\theta = \delta_{nm},$$
where $z = e^{i\theta}$ and $K(z) = -\frac{1}{2}(z - z^{-1})^2$.

5. Let G be a compact group. Show that there is a continuous faithful group homomorphism from G to the orthogonal group $O(n)$ if and only if G has an n-dimensional faithful representation over $\mathbb{R}$.

   By considering the action of $SU(2)$ by conjugation on the 2×2 complex matrices A such that $A = -\overline{A}^T$ and $\operatorname{tr} A = 0$, construct a continuous group homomorphism $SU(2) \to SO(3)$. Deduce that $SU(2)/\{\pm I\} \cong SO(3)$ as topological groups.

6. Write down a Haar measure on $SU(2)$ and prove that it is translation invariant and normalised correctly.

7. The *Heisenberg group* is the group G of order p^3 of upper unitriangular matrices over the field with p elements. Show that G has p conjugacy classes of size 1 and $p^2 - 1$ conjugacy classes of size p. Find p^2 characters of G of degree 1.

   Find an abelian subgroup H of G of order p^2. By induction of characters from H to G show that G has $p-1$ irreducible characters of degree p. Write down the character table of G.

8. Let $G = PSL_2(\mathbb{F}_7) = SL_2(\mathbb{F}_7)/Z(SL_2(\mathbb{F}_7))$. Starting with the character table of $GL_2(\mathbb{F}_7)$, calculate the character table of G. Deduce that G is simple. By considering the structure constants of $Z(\mathbb{C}G)$, and only using information in the character table, show that G has elements of order 2 and 3 whose product has order 7. Deduce that G is generated by two of its elements.

*9. Let $\mathbb{F}$ be the field with 2^n elements for some $n \geq 1$. Construct the character table of $GL_2(\mathbb{F})$. Deduce that $PGL_2(\mathbb{F}) = GL_2(\mathbb{F})/Z(GL_2(\mathbb{F}))$ is simple for $n \geq 2$. What can you say about $PGL_2(\mathbb{F})$ when $n = 1$?

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