Suppose that $\rho$ is a character of an irreducible representation for each $1 \leq r \leq n/2$. What happens for $r > n/2$?

3. Suppose $\rho: G \to GL(V)$ is an irreducible representation of $G$ with character $\chi$. By considering $V \otimes V$, $S^2V$ and $\Lambda^2V$ show that

$$\frac{1}{|G|} \sum_{g \in G} \chi(g^2) = \begin{cases} 0 & \text{if } \chi \text{ is not real-valued} \\ \pm 1 & \text{if } \chi \text{ is real valued.} \end{cases}$$

Deduce that if $|G|$ is odd then $G$ has only one real-valued irreducible character.

4. Let $\rho: G \to GL(V)$ be a representation of $G$ of dimension $d$.

(a) Compute $\dim S^nV$ and $\dim \Lambda^nV$ for all $n$.

(b) Let $g \in G$ and $\lambda_1, \ldots, \lambda_d$ be the eigenvalues of $\rho(g)$. What are the eigenvalues of $g$ on $S^nV$ and $\Lambda^nV$?

(c) Let $f(t) = \det(tI - \rho(g))$ be the characteristic polynomial of $\rho(g)$. What is the relationship between the coefficients of $f$ and $\chi_{\Lambda^nV}$?

(d) What is the relationship between $\chi_{S^nV}(g)$ and $f^2$? (Hint: start with case $d = 1$).

5. Recall the character table of $D_{10}$ from sheet 2. Explain how to view $D_{10}$ as a subgroup of $A_5$ and then use induction from $D_{10}$ to $A_5$ to reconstruct the character table of $A_5$.

6. Obtain the character table of the dihedral group $D_{2m}$ by using induction from the cyclic group $C_m$; you will want to split into two cases according as $m$ is odd or even.

7. Find all the characters of $S_5$ obtained by inducing irreducible representations of $S_4$. Use these to reconstruct the character table of $S_5$. Then repeat, replacing $S_4$ by the subgroup $\langle (12345), (2354) \rangle$ of $S_5$ of order 20.

8. Prove that if $H$ is a subgroup of a group $G$, and $K$ is a subgroup of $H$, and $W$ is a representation of $K$ then $\text{Ind}^G_K W \cong \text{Ind}^H_K \text{Ind}^G_H W$.

9. Let $H$ be a subgroup of a group $G$. Show that for every irreducible representation $(\rho, V)$ of $G$ there is an irreducible representation $(\rho', W)$ of $H$ such that $\rho$ is an irreducible component of $\text{Ind}^G_H W$.

Deduce that if $A$ is an abelian subgroup of $G$ then every irreducible representation of $G$ has dimension at most $|G/A|$.

10. Suppose that $G$ is a Frobenius group with Frobenius kernel $K$. Show that if $V$ is a non-trivial irreducible representation of $K$ then $\text{Ind}^G_K V$ is also irreducible. Hence, explain how to construct the character table of $G$ given the character tables of $K$ and $G/K$.

11. Suppose that $V$ is a faithful representation of a group $G$ such that $\chi_V$ takes $r$ distinct values. Show that each irreducible representation of $G$ is a summand of $V^\otimes n$ for some $n < r$.

12. Suppose $G$ is a finite group of odd order and with $k$ conjugacy classes. Show that $|G| \equiv k \pmod{16}$.

Comments and corrections to S.J.Wadsley@dpmms.cam.ac.uk.