

Representation Theory — Examples Sheet 2

On this sheet all representations are complex representations

1. Let (ρ, V) be a representation of a finite group G with character χ . Show that $\ker \rho = \{g \in G \mid \chi(g) = \chi(1)\}$. Show further that $|\chi(g)| \leq \chi(1)$ for all $g \in G$, with equality precisely if $\rho(g) = \lambda \text{id}_V$ for some $\lambda \in \mathbb{C}^\times$. Explain how the set of normal subgroups of G may be calculated directly from the character table.
2. Let χ be the character of a representation of a group G and let $g \in G$. If g has order 2 show that $\chi(g) \in \mathbb{Z}$ and that $\chi(g) \equiv \chi(1) \pmod{2}$. Show that if in addition G is a non-cyclic simple group then $\chi(g) \equiv \chi(1) \pmod{4}$. If instead g has order 3 and is conjugate to g^2 show that $\chi(g) \equiv \chi(1) \pmod{3}$.
3. Construct the character tables of the dihedral group D_8 and the quaternion group Q_8 . What do you notice? Compare the determinants of their respective two dimensional representations.
4. Construct the character tables of the dihedral groups D_{10} and D_{12} . How do the irreducible representations decompose when restricted to the subgroups of rotations?
5. Construct the character tables of A_4, S_4, A_5 and S_5 . The action of S_n on A_n by conjugation induces an action on the character table of A_n by permuting the conjugacy classes. Describe what this does to the rows of the character table for $n = 4, 5$.
6. Show that there is only one non-abelian group of order 21 up to isomorphism. Construct its character table.
7. A group of order 720 has 11 conjugacy classes. Two representations of the group are known and have corresponding characters α and β . The table below summarises the sizes of the conjugacy classes and the values of α and β on them. Prove that the group has an irreducible representation of degree 16 and calculate its character.

$ [g] $	1	15	40	90	45	120	144	120	90	15	40
α	6	2	0	0	2	2	1	1	0	-2	3
β	21	1	-3	-1	1	1	1	0	-1	-3	0

8. A group of order 168 has 6 conjugacy classes. Three representations of this group are known and have characters α, β and γ summarised in the table below. Construct the character table of the group. You may assume if required that $\sqrt{7}$ is not in the field generated by \mathbb{Q} and a primitive 7th root of unity.

$ [g] $	1	21	42	56	24	24
α	14	2	0	-1	0	0
β	15	-1	-1	0	1	1
γ	16	0	0	-2	2	2

9. Consider the action of a finite group G by conjugation. What is the character of the corresponding permutation representation $\mathbb{C}G$? Prove that the sum of elements in any row of the character table of G is a non-negative integer.
10. Show that the character table of a finite group G is invertible when viewed as a matrix.
By considering the actions induced on the rows and on the columns of the character table by complex conjugation, show that the number of irreducible characters of G that only take real values is the number of self-inverse conjugacy classes.
11. Let G be a finite group and χ be an irreducible character of G . By beginning with the irreducible representations, show that if (ρ, V) is any representation of G then $\frac{\chi(1)}{|G|} \sum_{g \in G} \chi(\overline{g})\rho(g)$ is a G -linear projection onto a subspace of V . Deduce that every representation can be decomposed uniquely into isotypical components.

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