THE ABELIAN ARITHMETIC REGULARITY LEMMA

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Abstract. We introduce and prove the arithmetic regularity lemma of Green and Tao in the abelian case. This exposition may serve as an introduction to the general case.

The purpose of this note is to provide a brief, self-contained treatment of the arithmetic regularity lemma of Green and Tao \[GT10\] in the abelian (\(s = 1\)) case, particularly in order to aid the reader of \[EGM13\]. This exposition may also serve as an introduction to the general (\(s > 1\)) case. All the results of this paper can therefore be read out of the more general results expounded in the first two sections of \[GT10\], but to do so would require digesting the higher-order theory as well, which is a little more involved. In particular, while in \[GT10\] the authors rely on the inverse theorem for the \(U^{s+1}\) norm, we need only the inverse theorem for the \(U^2\) norm, which is elementary both to state and to prove.

The results of this paper are also contained in \[Tao12\]. Compared to that treatment, we use slightly different language in a few places, and we absorb the Ratner-type theory into the statement of the regularity lemma.

The arithmetic regularity lemma states, roughly speaking, that an arbitrary function \(f : [N] \to [0, 1]\) is the sum of a structured part \(f_{\text{str}}\), a small part \(f_{\text{sml}}\), and a (Gowers-)uniform part \(f_{\text{unf}}\). Moreover we can buy higher-order uniformity of \(f_{\text{unf}}\) at the cost of more involved structure of \(f_{\text{str}}\), but in this paper we will only be able to afford \(U^2\) uniformity.

We start with the inverse theorem for the \(U^2\) norm. We define the \(U^2(\mathbb{Z}/M\mathbb{Z})\) norm of a function \(f : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}\) as

\[
\|f\|_{U^2(\mathbb{Z}/M\mathbb{Z})} = \left( \mathbb{E}_{a, h_1, h_2 \in \mathbb{Z}/M\mathbb{Z}} f(a)f(a + h_1)f(a + h_2)f(a + h_1 + h_2) \right)^{\frac{1}{4}},
\]

and then the \(U^2([N])\) norm of a function \(f : [N] \to \mathbb{C}\) as

\[
\|f\|_{U^2([N])} = \frac{\|f\|_{U^2(\mathbb{Z}/M\mathbb{Z})}}{\|1_{[N]}\|_{U^2(\mathbb{Z}/M\mathbb{Z})}},
\]

where \(M \geq 2N\) and we define \(f(x) = 0\) if \(x \notin \{1, \ldots, N\}\); one easily checks that this definition is independent of the choice of \(M\). We will often abbreviate \(U^2([N])\) to \(U^2\) when no confusion can arise.
Given \( f : \mathbb{Z}/M\mathbb{Z} \to \mathbb{C} \) we define the Fourier transform \( \hat{f} \) of \( f \) by

\[
\hat{f}(r) = \mathbf{E}_{x \in \mathbb{Z}/M\mathbb{Z}} f(x)e_{M}(-rx)
\]

for \( r \in \mathbb{Z}/M\mathbb{Z} \). The Fourier inversion formula then states

\[
f(x) = \sum_{r \in \mathbb{Z}/M\mathbb{Z}} \hat{f}(r)e_{M}(rx).
\]

Using these formulae one easily proves

\[
\|f\|_{U^2(\mathbb{Z}/M\mathbb{Z})} = \left( \sum_{r \in \mathbb{Z}/M\mathbb{Z}} |\hat{f}(r)|^4 \right)^{\frac{1}{4}}.
\]

**Lemma 1** (Inverse theorem for the \( U^2 \) norm). If \( f : [N] \to [-1,1] \) is a function such that \( \|f\|_{U^2} \geq \delta \), then there exists \( \theta \in \mathbb{R}/\mathbb{Z} \) such that

\[
\left| \mathbf{E}_{n \in [N]} f(n)e(-\theta n) \right| \gg \delta.
\]

**Proof.** The condition \( \|f\|_{U^2([N])} \geq \delta \) implies that \( \|f\|_{U^2(\mathbb{Z}/M\mathbb{Z})} \geq \delta \), where \( M = 2N \) and as usual we extend \( f \) by zero to the rest of \( \mathbb{Z}/M\mathbb{Z} \). We therefore have

\[
\sum_{r \in \mathbb{Z}/M\mathbb{Z}} |\hat{f}(r)|^4 \gg \delta^4.
\]

From Parseval’s theorem and the hypothesis \( |f| \leq 1 \) it then follows that

\[
\delta^4 \ll \sup |\hat{f}|^2 \left( \sum_{r \in \mathbb{Z}/M\mathbb{Z}} |\hat{f}(r)|^2 \right) = \sup |\hat{f}|^2 \left( \mathbf{E}_{x \in \mathbb{Z}/M\mathbb{Z}} |f(x)|^2 \right) \leq \sup |\hat{f}|^2.
\]

Thus \( |\hat{f}(r)| \gg \delta^2 \) for at least one \( r \in \mathbb{Z}/M\mathbb{Z} \), so we may take \( \theta = r/M \). \( \square \)

We need a slightly modified form of the above lemma in order to apply an energy increment argument, but first we need some language. Let \( \mathcal{T}^d = (\mathbb{R}/\mathbb{Z})^d \) denote the \( d \)-dimensional torus, and let us say that \( f : [N] \to \mathbb{R} \) has 1-complexity at most \( M \) if \( f(n) = F(\theta n) \) for some \( F : \mathcal{T}^d \to \mathbb{R} \) and \( \theta \in \mathcal{T}^d \) such that \( d, \|F\|_{\text{Lip}} \leq M \). Here we take the Euclidean metric

\[
d(x, y) = \min_{z \in \mathcal{T}^d} \|x - y - z\|_2
\]

on \( \mathcal{T}^d \), and we define the Lipschitz norm \( \|F\|_{\text{Lip}} \) of \( F : \mathcal{T}^d \to \mathbb{R} \) by

\[
\|F\|_{\text{Lip}} = \sup_{x} |F(x)| + \sup_{x \neq y} \frac{|F(x) - F(y)|}{d(x, y)}.
\]

The Fourier inversion formula shows that every \( f : [N] \to \mathbb{C} \) has finite 1-complexity, but functions of bounded 1-complexity are special.
Our results from now on will be quantified by an arbitrary growth function, by which we mean simply an increasing function \( F: \mathbb{R}^+ \rightarrow \mathbb{R}^+ \). By \( F \ll X \) we will mean that \( F \) is bounded by a function \( \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) depending only on the parameter \( X \); in other words \( F \ll X \) means \( F(M) \ll X,M \).

We say \( f \) is 1-measurable with growth \( F \) if for every \( M > 0 \) there is some function \( f_{str}: [N] \rightarrow \mathbb{R} \) of 1-complexity at most \( F(M) \) such that
\[
\|f - f_{str}\|_2 \leq \frac{1}{M},
\]
where the \( L^2([N]) \) norm of a function \( f: [N] \rightarrow \mathbb{C} \) is defined by
\[
\|f\|_2 = \left( \mathbb{E}_{x \in [N]} |f(x)|^2 \right)^{\frac{1}{2}}.
\]
A set \( E \subset [N] \) is called 1-measurable with growth \( F \) if \( 1_E \) is so. Note that if \( f \) and \( g \) are 1-measurable with growth \( F \) then \( f + g \) and \( fg \) are 1-measurable with growth \( \ll F \), so if \( E \) and \( F \) are 1-measurable with growth \( F \) then \( E \cup F, E \cap F, E \setminus F \), and so on, are all 1-measurable with growth \( \ll F \).

**Lemma 2** (\( U^2 \) inverse theorem, alternative formulation). If \( f: [N] \rightarrow [-1,1] \) is a function such that \( \|f\|_{U^2} \geq \delta \), then there is a 1-measurable set \( E \subset [N] \) with growth \( \ll \delta \) such that
\[
\left| \mathbb{E}_{n \in [N]} f(n) 1_E(n) \right| \gg \delta.
\]

**Proof.** By the previous lemma there is some \( \theta \in \mathbb{T} \) such that \( \phi(n) = e(-\theta n) \) satisfies
\[
\left| \mathbb{E}_{n \in [N]} f(n) \phi(n) \right| \gg \delta.
\]
Now by replacing \( \phi \) with its the real or imaginary part, and then with its positive or negative part, we may assume that \( \phi \) is real and nonnegative (e.g., if we take the real and then positive parts, then \( \phi(n) = (\Re e(-\theta n))^+ \)).

For \( 0 \leq t \leq 1 \), let
\[
E_t = \{ n \in [N] : \phi(n) \geq t \}.
\]
Noting that
\[
\phi(n) = \int_0^1 1_{E_t}(n) \, dt,
\]
it follows that
\[
\int_0^1 \left| \mathbb{E}_{n \in [N]} f(n) 1_{E_t}(n) \right| \, dt \gg \delta 1,
\]
and so
\[
\left| \mathbb{E}_{n \in [N]} f(n) 1_{E_t}(n) \right| \gg \delta 1
\]
for all \( t \) in a set \( \Omega \subset [0,1] \) of measure \( |\Omega| \gg \delta 1 \).
Among these sets $E_t$ with $t \in \Omega$ there must be some $E_t$ which is approximately invariant under small changes in $t$. Indeed, if
\[
M(t) = \sup_{r > 0} \frac{1}{2rN} |\{n \in [N] : |φ(n) - t| \leq r\}|
\]
then the Hardy-Littlewood maximal inequality (see any standard reference, such as [Rud87]) states
\[
|\{t \in [0, 1] : M(t) \geq \lambda\}| \ll \frac{1}{\lambda}.
\]
Since $|\Omega| \gg \delta$ there is some $t \in \Omega$ such that $M(t) \ll \delta_1$.

For any such $t$, $E_t$ is $1$-measurable with growth $\ll \delta_1$. Indeed, note for any $r > 0$
\[
|\{n \in [N] : |φ(n) - t| \leq r\}| \ll \delta rN.
\]
Choosing $\eta : \mathbb{R} \to \mathbb{R}^+$ of Lipschitz norm $\|\eta\|_{Lip} \ll 1/r$ such that $\eta(x) = 0$ if $x < t - r$ and $\eta(x) = 1$ if $x > t + r$, it follows that $\|1_{E_t} - \eta \circ φ\|_2 \ll \delta \sqrt{r}$. Since $φ$ is a function of $θn$ of Lipschitz norm $\ll 1$, this implies that $1_{E_t}$ is $1$-measurable with growth $\ll \delta_1$. □

A factor $B$ of $[N]$ is a ($σ$-)subalgebra of $2^{[N]}$, or equivalently a partition of $[N]$ into cells. We say a factor $B'$ refines another $B$ if every cell of $B$ is a union of cells of $B'$. We call $B$ a $1$-factor with complexity at most $M$ and growth $F$ if $B$ has $M$ cells, each of which is $1$-measurable with growth $F$. Note in this case that every $B$-measurable (i.e., constant on each cell of $B$) function $f : [N] \to [-1, 1]$ is $1$-measurable with growth $\ll_{M,F} 1$.

For $x \in [N]$ we define $B(x)$ to be the unique cell containing $x$, and we define the conditional expectation $\mathbf{E}(f|B)$ of a function $f : [N] \to \mathbb{C}$ by
\[
\mathbf{E}(f|B)(x) = \frac{1}{|B(x)|} \sum_{y \in B(x)} f(y).
\]
Equivalently, $\mathbf{E}(f|B)$ is the orthogonal projection of $f$ onto the subspace of $B$-measurable functions. Finally, with respect to a fixed function $f : [N] \to \mathbb{C}$, the energy of $B$ is $\mathcal{E}(B) = \| \mathbf{E}(f|B) \|_2^2$.

**Corollary 3** (Lack of uniformity allows energy increment). Suppose $B$ is a $1$-factor of complexity $\ll M$ and growth $F$ and $f : [N] \to [-1, 1]$ is a function such that $\|f - \mathbf{E}(f|B)\|_{L^2([N])} \geq \delta$. Then there exists a refinement $B'$ of $B$ of complexity $\ll 2M$ and growth $\ll_{M,\delta,F} 1$ such that
\[
\mathcal{E}(B') - \mathcal{E}(B) \gg_{\delta} 1.
\]
Proof. By the previous corollary there is a 1-measurable set \( E \subset [N] \) with growth \( \ll \delta \) such that
\[
|\langle f - E(f|B), 1_E \rangle| \gg \delta.
\]
Let \( B' \) be the factor generated by \( B \) and \( E \). Then \( B' \) is a 1-factor of complexity \( \leq 2M \) and growth \( \ll_{M, \delta, F} \) 1, and since \( 1_E \) is \( B' \)-measurable we have
\[
|\langle (\mathbf{E}(f|B') - \mathbf{E}(f|B)), 1_E \rangle| \gg \delta.
\]
Now Cauchy-Schwarz and the Pythagorean theorem imply that
\[
\mathbf{E}(f|B') - \mathbf{E}(f|B) = \| \mathbf{E}(f|B') - \mathbf{E}(f|B) \|_2^2 \gg \delta.
\]
\[\Box\]

We can now deduce a weak form of the (non-irrational) regularity lemma.

**Corollary 4 (Weak regularity).** Let \( B \) be a 1-factor of complexity \( M \) and growth \( F \), and let \( f : [N] \to [-1, 1] \) be a function. Then there exists a refinement \( B' \) of \( B \) of complexity \( \ll_{\delta, M} \) and growth \( \ll_{\delta, M, F} \) such that
\[
\| f - \mathbf{E}(f|B') \|_{U^2([N])} \leq \delta.
\]

**Proof.** Repeatedly apply the previous corollary to refine the 1-factor \( B \). Since \( 0 \leq \mathcal{E}(B) \leq 1 \), this process must end after \( \ll \delta \) steps. \[\Box\]

Finally, by iterating this result, we deduce full non-irrational regularity.

**Theorem 5 (Non-irrational regularity).** Let \( f : [N] \to [0, 1] \) be a function, \( F \) a growth function, and \( \varepsilon > 0 \). Then there is a quantity \( M \ll_{\varepsilon, F} 1 \) and a decomposition
\[
f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}}
\]
of \( f \) into functions \( f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : [N] \to [-1, 1] \) such that

1. \( f_{\text{str}} \) has 1-complexity at most \( M \),
2. \( f_{\text{sml}} \) has \( L^2([N]) \) norm at most \( \varepsilon \),
3. \( f_{\text{unf}} \) has \( U^2([N]) \) norm at most \( 1/F(M) \),
4. \( f_{\text{str}} \) and \( f_{\text{str}} + f_{\text{sml}} \) take values in \([0,1]\).

**Proof.** Starting with \( M_0 = 1 \) and \( B_0 = \{\emptyset, [N]\} \), suppose inductively that \( B_i \) is a 1-factor of complexity and growth \( \ll_{i, M_i, F} 1 \). Then there is a function \( f_{\text{str}}^{(i)} : [N] \to \mathbb{R} \) of 1-complexity \( M_{i+1} \ll_{\varepsilon, i, M_i, F} 1 \) such that \( M_{i+1} \geq M_i \) and
\[
\| \mathbf{E}(f|B_i) - f_{\text{str}}^{(i)} \|_2 \leq \varepsilon/2.
\]
Moreover, by truncating $f^{(i)}_{str}$ above and below (which doesn’t increase its 1-complexity) we may assume that $f^{(i)}_{str} : [N] \to [0,1]$. By the previous corollary there is a refinement $B_{i+1}$ of $B_i$ of complexity and growth $\ll_{i,M+1,F} 1$ such that
\[
\|f - E(f|B_{i+1})\|_{U^2([N])} \leq 1/F(M_{i+1}).
\]

Note in the end that $M_i \ll_{\epsilon,F} 1$, and since $(E(B_i))$ is an increasing sequence in $[0,1]$ there is some $i \ll_{\epsilon} 1$ such that
\[
E(B_{i+1}) - E(B_i) = \|E(f|B_{i+1}) - E(f|B_i)\|_2^2 \leq \epsilon^2/4.
\]

Let $M = M_{i+1}$ and let
\[
\begin{align*}
    f_{str} &= f^{(i)}_{str}, \\
    f_{sml} &= E(f|B_{i+1}) - f^{(i)}_{str}, \\
    f_{unf} &= f - E(f|B_{i+1}).
\end{align*}
\]

We have nearly finished proving the regularity lemma. To finish, we will try to make the structure of $f_{str}$ a little more explicit. Specifically, we would like $\theta \in T^d$ to be $(A,N)$-irrational for some large $A$, meaning that if $q \in \mathbb{Z}^d \backslash \{0\}$ and $\|q\|_1 \leq A$ then $\|q \cdot \theta\|_{\mathbb{R}/\mathbb{Z}} \geq A/N$. (This is important in several counting lemmata in which we relate a sum involving $f_{str}$ to an integral over $T^d$.) Of course, there are other possible behaviours of $q \cdot \theta$: it may be that $\theta$ itself is small, in which case $q \cdot \theta$ moves slowly away from 0, or it may be that $\theta$ is rational, in which case $q \cdot \theta$ frequently returns to 0. Nevertheless, it turns out that once these two pollutants are boiled off, the remnant is highly irrational in the above sense.

We say a subtorus $T$ of $T^d$ of dimension $d'$ has complexity at most $M$ if there is some $L \in SL_d(\mathbb{Z})$, all of whose coefficients have size at most $M$, such that $L(T) = T^d \times \{0\}^{d-d'}$. In this case we implicitly identify $T$ with $T^{d'}$ using $L$. For instance, we say $\theta \in T^d$ is $(A,N)$-irrational in $T$ if $L(\theta)$ is $(A,N)$-irrational in $T^{d'}$.

**Theorem 6.** Given $\theta \in T^d$, a positive integer $N$, and a growth function $F$, there is a quantity $M \ll_{d,F} 1$ and a decomposition
\[
\theta = \theta_{smth} + \theta_{rat} + \theta_{irrat}
\]
such that

1. $\theta_{smth}$ is $(M,N)$-smooth, meaning $d(\theta_{smth},0) \leq \frac{M}{N}$,
2. $\theta_{rat}$ is $M$-rational, meaning $q\theta_{rat} = 0$ for some $q \leq M$, and
3. $\theta_{irrat}$ is $(F(M),N)$-irrational in a subtorus of complexity $\leq M$. 

Proof. Starting with $M_0 = 1$, $\theta^{(0)}_{\text{smth}} = \theta^{(0)}_{\text{rat}} =\theta^{(0)}_{\text{irrat}}$, and $T_0 = T^d$, suppose inductively that

$$\theta = \theta^{(i)}_{\text{smth}} + \theta^{(i)}_{\text{rat}} + \theta^{(i)}_{\text{irrat}},$$

where $\theta^{(i)}_{\text{smth}}$ is $(M_i, N)$-smooth, $\theta^{(i)}_{\text{rat}}$ is $M_i$-rational, and $\theta^{(i)}_{\text{irrat}}$ lies in a subtorus $T_i$ of dimension $d - i$ and complexity $\leq M_i$.

If $\theta^{(i)}_{\text{irrat}}$ is $(F(M_i), N)$-irrational in $T_i$ then we are done, so suppose that $L \in \text{SL}_d(Z)$ is a linear map of complexity $\leq M_i$ identifying $T$ with $T^{d-i}$ and such that

$$\|q \cdot L(\theta^{(i)}_{\text{irrat}})\|_{R/Z} \leq \frac{F(M_i)}{N}$$

for some $q \in Z^{d-i}\setminus \{0\}$ such that $\|q\|_1 \leq F(M_i)$. Choose $\theta^{(i)}_{\text{irrat}}' \in T$ so that

$$\|q \cdot L(\theta^{(i)}_{\text{irrat}} - \theta^{(i)}_{\text{irrat}}')\|_{R/Z} = 0$$

and such that $d(L(\theta^{(i)}_{\text{irrat}}'), 0) \leq F(M_i)/N$, so $d(\theta^{(i)}_{\text{irrat}}', 0) \ll M_i, d, x$ and since $\theta^{(i)}_{\text{irrat}}$ is primitive in $Z^d$, suppose $q'$ is primitive in $Z^{d-i}$. Then

$$q' \cdot L(\theta^{(i)}_{\text{irrat}} - \theta^{(i)}_{\text{irrat}}') \in \frac{1}{m} Z.$$

Now using the Euclidean algorithm, choose $\theta^{(i)}_{\text{irrat}}' \in T$ so that

$$q' \cdot L(\theta^{(i)}_{\text{irrat}} - \theta^{(i)}_{\text{irrat}}') = 0$$

and such that $mL(\theta^{(i)}_{\text{irrat}}') = 0$, so that $m\theta^{(i)}_{\text{irrat}} = 0$. Finally, let

$$\theta^{(i+1)}_{\text{smth}} = \theta^{(i)}_{\text{smth}} + \theta^{(i)}_{\text{irrat}}',$$

$$\theta^{(i+1)}_{\text{rat}} = \theta^{(i)}_{\text{rat}} + \theta^{(i)}_{\text{irrat}}',$$

$$\theta^{(i+1)}_{\text{irrat}} = \theta^{(i)}_{\text{irrat}} - \theta^{(i)}_{\text{smth}} - \theta^{(i)}_{\text{rat}}',$$

and choose $M_{i+1} = M_{i+1}, d, x$ so that $\theta^{(i+1)}_{\text{smth}}$ is $(M_{i+1}, N)$-smooth, $\theta^{(i+1)}_{\text{rat}}$ is $M_{i+1}$-rational, and the subtorus $T_{i+1} = \{x \in T_i : q' \cdot L(x) = 0\}$ has complexity $\leq M_{i+1}$.

In the end note that $M_i \ll i, d, x$ and since $T_i$ has dimension $d - i$ we can iterate this argument no more than $d$ times, so for some $i \leq d$ we must have that $\theta^{(i)}_{\text{irrat}}$ is $(F(M_i), N)$-irrational in $T_i$. \hfill \Box

Note by replacing $F(M)$ with $\tilde{F}(M) = F(M + M_0)$ we can assume $M_0 \leq M \ll_{M_0, d, x} 1$ instead of $M \ll_{d, x} 1$; alternatively we can prove this by starting the induction at $M_0$ instead of 1. This modification is sometimes convenient.

We can finally state and prove the full regularity lemma, which improves on Theorem 5 by giving $f_{\text{str}}$ the structure

$$f_{\text{str}}(n) = F(n/N, n(\mod q), \theta n),$$
where

\[ F : [0, 1] \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{T}^d \to \mathbb{R}, \]

\[ q, d, \|F\|_{\text{Lip}} \leq M, \text{ and } \theta \text{ is } (\mathcal{F}(M), N)-\text{irrational}. \]

Here we take the usual Euclidean metrics on \([0, 1]\) and \(\mathbb{T}^d\), the discrete metric on \(\mathbb{Z}/q\mathbb{Z}\), the sum of these metrics on \([0, 1] \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{T}^d\), and then define \(\|F\|_{\text{Lip}}\) as before.

**Theorem 7** (The abelian arithmetic regularity lemma). Let \(f : [N] \to [0, 1]\) be a function, \(\mathcal{F}\) a growth function, and \(\varepsilon > 0\). Then there is a quantity \(M \ll_{\varepsilon, \mathcal{F}} 1\) and a decomposition

\[ f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}} \]

of \(f\) into functions \(f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : [N] \to [-1, 1]\) such that

1. \(f_{\text{str}}(n) = F(n/N, n(\text{mod } q), \theta n)\), where
    \[ F : [0, 1] \times \mathbb{Z}/q\mathbb{Z} \times \mathbb{T}^d \to [0, 1], \]
    \(q, d, \|F\|_{\text{Lip}} \leq M, \text{ and } \theta \in \mathbb{T}^d\) is \((\mathcal{F}(M), N)\)-irrational,
2. \(f_{\text{sml}}\) has \(L^2([N])\) norm at most \(\varepsilon\),
3. \(f_{\text{unf}}\) has \(U^2([N])\) norm at most \(1/\mathcal{F}(M)\),
4. \(f_{\text{str}}\) and \(f_{\text{str}} + f_{\text{sml}}\) take values in \([0, 1]\).

**Proof.** Let \(\mathcal{F}_1\) and \(\mathcal{F}_2\) be growth functions depending on \(\varepsilon\) and \(\mathcal{F}\) in a manner to be determined. By Theorem 5 there exists \(M_1 \ll_{\varepsilon, \mathcal{F}_1} 1\) and a decomposition

\[ f = f_{\text{str}} + f_{\text{sml}} + f_{\text{unf}} \]

of \(f\) into functions \(f_{\text{str}}, f_{\text{sml}}, f_{\text{unf}} : [N] \to [-1, 1]\) such that

1. \(f_{\text{str}}(n) = F(\theta n)\), where \(F : \mathbb{T}^d \to [0, 1]\), \(d, \|F\|_{\text{Lip}} \leq M_1\), and \(\theta \in \mathbb{T}^d\),
2. \(f_{\text{sml}}\) has \(L^2([N])\) norm at most \(\varepsilon\),
3. \(f_{\text{unf}}\) has \(U^2([N])\) norm at most \(1/\mathcal{F}_1(M_1)\), and
4. \(f_{\text{str}}\) and \(f_{\text{str}} + f_{\text{sml}}\) take values in \([0, 1]\).

Now by the previous theorem we can find \(M_2 \ll_{M_1, \mathcal{F}_2} 1\) such that \(M_2 \geq M_1\) and such that \(\theta\) decomposes as

\[ \theta = \theta_{\text{smth}} + \theta_{\text{rat}} + \theta_{\text{irrat}}, \]

where

1. \(\theta_{\text{smth}}\) is \((M_2, N)\)-smooth, meaning \(d(\theta_{\text{smth}}, 0) \leq M_2\),
2. \(\theta_{\text{rat}}\) is \(M_2\)-rational, meaning \(q\theta_{\text{rat}} = 0\) for some \(q \leq M_2\), and
3. \(\theta_{\text{irrat}}\) is \((\mathcal{F}_2(M_2), N)\)-irrational in a subtorus of complexity \(\leq M_2\).

Then

\[ F(\theta n) = F(\theta_{\text{smth}} n + \theta_{\text{rat}} n + \theta_{\text{irrat}} n) = \tilde{F}(n/N, n(\text{mod } q), nL(\theta_{\text{irrat}})), \]
where \( \tilde{F} : [0, 1] \times \mathbb{Z}/q\mathbb{Z} \times T^{d'} \rightarrow [0, 1] \) is defined by
\[
\tilde{F}(x, y, z) = F(N\theta_{\text{smth}}x + \theta_{\text{rat}}y + L^{-1}(z)).
\]

Noting that \( \|\tilde{F}\|_{\text{Lip}} \ll M_2^{-1} \), we can find \( M \ll M_2 \) exceeding both \( M_2 \) and \( \|\tilde{F}\|_{\text{Lip}} \). But since \( M \ll M_2 \), if \( F_2 \) is sufficiently large depending on \( F \) then \( F_2(M_2) \geq F(M) \), and similarly \( M_2 \ll M_1, F_2 \) 1, so if \( F_1 \) is sufficiently large depending on \( F_2 \) then \( F_1(M_1) \geq F_2(M_2) \geq F(M) \). After all these dependencies are fixed we have \( M \ll \epsilon, F \) 1, and the conclusion of the theorem holds. \( \square \)

References


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