Fixed point theorems for holomorphic maps on Teichmüller spaces and beyond

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When does a holomorphic map from Teichmüller to itself have a fixed point?
The short answer...

Theorem (SA). If a holomorphic map $F : T_{g,n} \to T_{g,n}$ has a recurrent orbit, then it has a fixed point.

In other words, there is a dichotomy: either there is a fixed point, or every orbit diverges.

Proof. Focus on the intrinsic geometry of $T_{g,n}$. □

Remarks:
• I'd like to thank A. Karlsson for asking the question answered by the theorem above.
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There is a vast literature on this topic and I will not attempt to be comprehensive.
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The long answer...

...will takes us through the following list of questions:

• What is Teichmüller space $T_{g,n}$?
• Why care about the existence of fixed points?
• Isn't the theorem true for all bounded domains? or, What's special about $T_{g,n}$?
• Is this really a result in complex analysis? or, How about a theorem for topological manifolds?
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Definition **Teichmüller space** $\mathcal{T}_{g,n}$ is the *universal cover* of the moduli space of Riemann surfaces of genus $g$ and $n$ marked points. It is naturally a complex manifold of dimension $3g - 3 + n$ that is homeomorphic to an open ball.
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- $\mathcal{T}_{g,n}$ can be realized as a **bounded domain** in $\mathbb{C}^{3g-3+n}$. (L. Bers)

- In particular, it is equipped with a **complete, intrinsic** metric: the Teichmüller-Kobayashi metric. (H. Royden)
Definition The intrinsic, or Kobayashi, metric of a bounded domain $\Omega$ in $\mathbb{C}^n$ is characterized by the property: it is the largest metric such that, every holomorphic map $F : \Delta \to \Omega$ is non-expanding: $\|F'(0)\| \leq 1$.
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Example The Kobayashi metric of the unit disk \( \Delta \) is given by \[ \frac{|dz|}{1 - |z|^2}. \]

The following important fact follows readily from the definition:

A holomorphic map between two complex domains is non-expanding for the Kobayashi metrics.
• The study of fixed point theorems for Teichmüller space provides a framework for proving geometrization theorems. The three fundamental theorems of W. Thurston are equivalent to, and are proved by, fixed point theorems for certain holomorphic maps on Teichmüller space $\mathcal{T}_{g,n}$. 

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Recall: Contraction mapping theorem
A strictly contracting self-map of a complete metric space has a fixed point.
• In complex dimension one:

  Theorem (Denjoy-Wolff). A holomorphic map $F: \Delta \to \Delta$ with a recurrent orbit has a fixed point.

  **Dichotomy**: A holomorphic map either has a fixed point, or every orbit diverges.

  **Proof.** Schwarz's lemma (which is simply the fundamental property of the Kobayashi metric in dimension one). □

• In higher dimensions: life is more interesting.

  – The theorem of Denjoy-Wolff remains true for convex domains but,

  – M. Abate et al, constructed a holomorphic self-map of a contractible bounded domain with recurrent orbits and no fixed points.

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The **problem** is that $\mathcal{T}_{g,n}$ is not a convex domain and its boundary is not a smooth manifold.
Although $\mathcal{T}_{g,n}$ is not a convex domain, it is convex from within:
Intrinsically straight complex spaces

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**Examples of straight metric spaces:**

- The unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$
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**Theorem** (SA). A holomorphic self-map of a finite-dimensional complex manifold, whose intrinsic metric is straight, either has a fixed point, or every orbit diverges.
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Combining this lemma with a basic fact from homotopy theory: a finite group cannot act freely on a contractible finite-dimensional CW-complex. We conclude the following proposition.
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Combining this lemma with a basic fact from *homotopy* theory: a finite group cannot act *freely* on a **contractible finite-dimensional** CW-complex. We conclude the following proposition.

**Proposition.** *If a holomorphic map* $F : \mathcal{T}_{g,n} \to \mathcal{T}_{g,n}$ *has a periodic point, then it has a fixed point.*
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**Proposition.** If a holomorphic self-map of a complex manifold (whose intrinsic metric is complete) has a recurrent orbit then the closure of the set of its iterates contains a retraction.
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Finally, a simple *combinatorial* approach (using Ramsey theory) is used to prove a generalisation of this proposition, which can be applied to prove a similar fixed point theorem for non-expanding maps for *straight metrics* on finite-dimensional manifolds.
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Corollary. **A non-expanding map from \( \mathbb{R}^n \) to itself, equipped with the Euclidean metric, either has a fixed point or every orbit diverges.**
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Corollary. A non-expanding map from $\mathbb{R}^n$ to itself, equipped with the Euclidean metric, either has a fixed point or every orbit diverges.

Remark: There are examples of maps from $\mathbb{R}^n$ to itself with bounded orbits, yet having no fixed points!
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Brouwer’s (1912) ‘translation theorem’ and the bounded orbits conjecture in the plane.
Thank you!