INDEPENDENCE OF ℓ FOR FROBENIUS CONJUGACY CLASSES ATTACHED TO ABELIAN VARIETIES

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Abstract. Let A be an abelian variety over a number field $\mathrm{E} \subset \mathbb{C}$ and let $\mathbf{G}$ denote the Mumford–Tate group of A. After replacing E by a finite extension, the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E})$ on the ℓ-adic cohomology $\mathrm{H}^1_{\text{ét}}(A_{\overline{\mathrm{E}}}, \mathbb{Q}_\ell)$ factors through $\mathbf{G}(\mathbb{Q}_\ell)$. We show that for v an odd prime of E where A has good reduction, the conjugacy class of Frobenius Frob_v in $\mathbf{G}(\mathbb{Q}_\ell)$ is independent of ℓ. Along the way, we prove that under certain hypotheses, every point in the μ-ordinary locus of the special fiber of Shimura varieties has a special point lifting it.

Contents

1. Introduction

Let A be an abelian variety over a number field $\mathrm{E} \subset \mathbb{C}$ and $\overline{\mathrm{E}}$ the algebraic closure of E in $\mathbb{C}$. We fix a prime p and $v|p$ a place of E where A has good reduction. Then for any prime $\ell \neq p$, the action of $\mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E})$ on the ℓ-adic cohomology $\mathrm{H}^1_{\acute{e}t}(A_{\overline{\mathrm{E}}}, \mathbb{Q}_\ell)$ is unramified at v, and the characteristic polynomial $P_{v,\ell}(t)$ of a geometric Frobenius $\mathrm{Frob}_v \in \mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E})$ has coefficients in $\mathbb{Z}$, and is independent of ℓ. The aim of this paper is to prove a refinement of this statement for the image of Frob_v in the *Mumford–Tate* group of A.

Recall that the Mumford–Tate group $\mathbf{G}$ of A is a reductive group over $\mathbb{Q}$, defined as the Tannakian group of the $\mathbb{Q}$-Hodge structure given by the Betti cohomology $V_B := \mathrm{H}^1_B(A(\mathbb{C}), \mathbb{Q})$. It may also be defined as the stabilizer in $\mathbf{GL}(V_B)$ of all Hodge cycles of type (0,0) on the tensor spaces $V_B^{\otimes r} \otimes (V_B^\vee)^{\otimes r}$ for $r \in \mathbb{Z}_{\geq 0}$. A fundamental result of Deligne [Del82a] asserts that there exists a finite extension E'/E in $\overline{\mathrm{E}}$ such that for any prime ℓ, the action of $\mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E}')$ on $\mathrm{H}^1_{\acute{e}t}(A_{\overline{\mathrm{E}}}, \mathbb{Q}_\ell)$ is induced by a representation

$$\rho_\ell^{\mathbf{G}} : \mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E}') \to \mathbf{G}(\mathbb{Q}_\ell).$$

It is not hard to see that for any finite extension E'/E, if $\rho_\ell^{\mathbf{G}}$ exists for one ℓ, then it exists for all ℓ. Moreover there is a minimal such extension E'. The existence of $\rho_\ell^{\mathbf{G}}$ is in fact predicted by the (in general still unproved) Hodge conjecture for A; Deligne's result on absolute Hodge cycles [Del82a] provides a reasonable substitute in this case so that the existence is unconditional. Upon replacing E by E', we assume there is a map $\rho_\ell^{\mathbf{G}} : \mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E}) \to \mathbf{G}(\mathbb{Q}_\ell)$.

For any reductive group $\mathbf{H}$ over $\mathbb{Q}$ we write $\mathrm{Conj}_{\mathbf{H}}$ for the variety of semisimple conjugacy classes of $\mathbf{H}$ (cf. §6.1.3) and $\chi_{\mathbf{H}} : \mathbf{H} \to \mathrm{Conj}_{\mathbf{H}}$ for the natural projection map which sends an element of $\mathbf{H}$ to the associated conjugacy class of its semisimple part. We thus obtain a well-defined element

$$\gamma_\ell = \gamma_\ell(v) := \chi_{\mathbf{G}}(\rho_\ell^{\mathbf{G}}(\mathrm{Frob}_v)) \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell),$$

the conjugacy class of ℓ-adic Frobenius at v. Our main theorem is the following.

Theorem 1.1. *Let $p > 2$ and $v|p$ a prime of* E *where A has good reduction. Then there exists $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that*

$$\gamma = \gamma_\ell \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell), \ \forall \ell \neq p.$$

Explicitly, γ is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathbf{G}(\overline{\mathbb{Q}})$-conjugacy class whose associated $\mathbf{G}(\overline{\mathbb{Q}}_\ell)$-conjugacy class contains $\rho_\ell^{\mathbf{G}}(\mathrm{Frob}_v)$ for all $\ell \neq p$. Since $P_{v,\ell}(t)$ is independent of ℓ, the image of γ_ℓ in $\mathrm{Conj}_{\mathbf{GL}(V_B)}(\mathbb{Q}_\ell)$ is defined over $\mathbb{Q}$ and independent of ℓ. However, in general the map $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}) \to \mathrm{Conj}_{\mathbf{GL}(V_B)}(\mathbb{Q})$ is not injective, so the theorem gives more information than the ℓ-independence of $P_{v,\ell}(t)$. We remark that $\mathbf{G}$ depends on the chosen embedding $\mathrm{E} \subset \mathbb{C}$; in general changing the embedding has the effect of replacing $\mathbf{G}$ by an inner form. However, the variety $\mathrm{Conj}_{\mathbf{G}}$ and the elements γ_ℓ do not depend on the choice of this embedding, and so neither does the statement of the theorem. When $\mathbf{G}$ is quasi-split at p it is expected that γ can be lifted to an element $\gamma_0 \in \mathbf{G}(\mathbb{Q})$, which is $\mathbf{G}(\mathbb{Q}_\ell)$-conjugate to $\rho_\ell^{\mathbf{G}}(\mathrm{Frob}_v)$. We prove a slightly weaker version of this result, when $\mathbf{G}^{\mathrm{der}}$ is simply connected; see §7.3 below for this and other potential refinements of the theorem.

An analogue of the above theorem for any algebraic variety (or more generally motive) over a number field was conjectured by Serre in [Ser94, 12.6], but in general one does not even know the analogue of Deligne's theorem on the existence of $\rho_\ell^{\mathbf{G}}$.

Previously proved cases of our theorem include a result of Noot who showed a version of this theorem where $\mathrm{Conj}_{\mathbf{G}}$ is replaced by a certain quotient $\mathrm{Conj}'_{\mathbf{G}_A}$ and under the additional assumption that the Frobenius element γ_ℓ is weakly neat [Noo09]. More recently, one of us [Kis17, Corollary 2.3.1] proved the Theorem when $\mathbf{G}$ is unramified at p. In fact, [Kis17] proves the stronger result that γ lifts to $\gamma_0 \in \mathbf{G}(\mathbb{Q})$ and is $\mathbf{G}(\mathbb{Q}_\ell)$-conjugate to $\rho_\ell^{\mathbf{G}}(\mathrm{Frob}_v)$. Noot's argument uses the independence of ℓ of $P_{v,\ell}(t)$, together with group theoretic arguments to analyze the map $\mathrm{Conj}_{\mathbf{G}} \to \mathrm{Conj}_{\mathbf{GL}(V_B)}$. The result of [Kis17] is proved by showing that, on the Shimura variety associated to $\mathbf{G}$, the isogeny class corresponding to A contains a point which admits a CM lift. It does not seem possible to extend either method to prove Theorem 1.1.

For the rest of the introduction we assume $p > 2$. Our proof of Theorem 1.1 makes use of families of abelian varieties with Mumford–Tate group contained in $\mathbf{G}$, and especially the structure of their mod p reductions. These families are parameterized by a Shimura variety $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ associated to $\mathbf{G}$, and defined over a number field (its reflex field) $\mathbf{E} \subset \mathbb{C}$ which is contained in E. We take $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$ with $\mathrm{K}_p \subset \mathbf{G}(\mathbb{Q}_p)$ a parahoric subgroup and $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ a compact open subgroup. Let w be the restriction of v to $\mathbf{E}$. Write $\mathbf{E}_w$ for the completion of $\mathbf{E}$ at w, $\mathcal{O}_{\mathbf{E}_w}$ for the ring of integers of $\mathbf{E}_w$ and $\kappa(w)$ for its residue field. Under some mild conditions we show that $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$ has an integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ over $\mathcal{O}_{\mathbf{E}_w}$, which is smoothly equivalent to a "local model", defined as the closure of an orbit of $\mathbf{G}$ acting on a certain Grassmannian. This extends the results of the first author and Pappas [KP18], which were restricted to the case when $\mathbf{G}_{\mathbb{Q}_p}$ was a tamely ramified group.

For each prime $\ell \neq p$, $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ is equipped with a $\mathbf{G}(\mathbb{Q}_\ell)$-torsor $\mathbb{L}_\ell$. In particular, for any finite extension $\kappa/\kappa(w)$ and $x \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)(\kappa)$, the $q = |\kappa|$-Frobenius acting on the geometric fiber of $\mathbb{L}_\ell$ at x, gives rise to an element $\gamma_{x,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$. We say x has the property (ℓ-ind), or the *ℓ-independence property*, if there exists an element $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that

$$\gamma = \gamma_{x,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell), \forall \ell \neq p.$$

Now suppose that $(\mathbf{G}, X)$ satisfies the conditions needed to guarantee the existence of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ (cf. Theorem 5.2.12); the general case of Theorem 1.1 is eventually reduced to this one. Then for a suitable choice of K, our abelian variety A corresponds to a point $\tilde{x}_A \in \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)(\mathrm{E})$ and its mod v reduction is a point x_A of the special fiber $\mathcal{S}_{\mathrm{K}} := \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X) \otimes_{\mathcal{O}_{\mathbf{E}_w}} \kappa(w)$. Moreover there is an equality $\gamma_\ell(v) = \gamma_{x_A,\ell}$ as elements of $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$. Thus in order to show Theorem 1.1, it suffices to prove

(†) If $\kappa/\kappa(w)$ is finite and $x \in \mathcal{S}_{\mathrm{K}}(\kappa)$, then x satisfies (ℓ-ind).

By considering A as a point on a larger Shimura variety related to a group of the form $\mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbf{G}$ where F is a suitably chosen totally real field, one can show that Theorem 1.1 follows from the following special case of (†).

Theorem 1.2. *Let $(\mathbf{G}, X)$ be a Shimura datum of Hodge type and assume $\mathbf{G}_{\mathbb{Q}_p}$ quasi-split, K_p is a very special parahoric and the triple $(\mathbf{G}, X, \mathrm{K}_p)$ is strongly acceptable. Then for any $\kappa/\kappa(w)$ finite and $x \in \mathcal{S}_{\mathrm{K}}(\kappa)$, x satisfies (ℓ-ind).*

The condition of strong acceptability of the triple $(\mathbf{G}, X, \mathrm{K}_p)$ is a technical one, and we refer the reader to §5.2.7 for the definition (cf. also Definition 3.3.2). We only mention here that this condition is always satisfied if $p > 3$ and the associated adjoint datum $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$.

As a first step towards Theorem 1.2, we prove the following analogue of Serre–Tate theory, which allows us to show that under the assumptions of Theorem 1.2, (ℓ-ind) holds on a dense, Zariski open subset of $\mathcal{S}_{\mathrm{K}}$.

Theorem 1.3. *Assume* $(\mathbf{G}, X)$ *is of Hodge type and the triple* $(\mathbf{G}, X, \mathrm{K}_p)$ *is strongly acceptable. Then*

1. *Any closed point* x *lying in the* μ*-ordinary locus* $\mathcal{S}_{\mathrm{K},[b]_\mu} \subset \mathcal{S}_{\mathrm{K}}$ *admits a lifting to a special point* $\widetilde{x} \in \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)$.
2. *If in addition* $\mathbf{G}_{\mathbb{Q}_p}$ *is quasi-split and* K_p *is very special. Then* $\mathcal{S}_{\mathrm{K},[b]_\mu}$ *is Zariski open and dense in* $\mathcal{S}_{\mathrm{K}}$.

The lifting constructed in (1) is the analogue in our setting of the canonical lift for ordinary abelian varieties and had been considered for Shimura varieties with good reduction in previous work of Moonen [Moo04] and Shankar and the second author [SZ21]. For these points, the Frobenius lifts to an automorphism of the associated CM abelian variety, and we obtain the desired element $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ by considering the induced action on Betti cohomology. Part (2) of the Theorem follows from our results about the local structure of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ and [KPS22, Corollary 1.3.16].

To prove Theorem 1.2, one considers a smooth curve $\mathcal{C}$ with a map $\pi : \mathcal{C} \to \mathcal{S}_{\mathrm{K}}$. Using a theorem of L. Lafforgue [Laf02, Théorème VII.6] on the existence of compatible local systems on smooth curves, we show that if the property (ℓ-ind) holds for a dense open subset of points on $\mathcal{C}$ then it holds for all points of $\mathcal{C}$. Our results on the structure of the integral models $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ imply that $\mathcal{S}_{\mathrm{K}}$ is equipped with a certain combinatorially described stratification, the Kottwitz–Rapoport stratification. The stratum of maximal dimension is the smooth locus of $\mathcal{S}_{\mathrm{K}}$. A theorem of Poonen [Poo04] shows that π can be chosen so that its image intersects $\mathcal{S}_{\mathrm{K},[b]_\mu}$ and any point x of the maximal stratum. The μ-ordinary case explained above then implies that any such x satisfies (ℓ-ind). We now argue by induction on the codimension of the strata; for a closed point x in some stratum of $\mathcal{S}_{\mathrm{K}}$, we show that π can be chosen so that its image contains x, and also meets some higher dimensional stratum.

In fact, using general arguments with ampleness, it is not hard to construct a map π from a smooth curve whose image contains any closed point $x \in \mathcal{S}_{\mathrm{K}}$, and meets the μ-ordinary locus. This would appear to avoid the induction on strata above. However, this argument would only allow us to prove the ℓ-independence result for some power of the Frobenius. To prove Theorem 1.2 in full, one needs the existence of a $y \in \mathcal{C}$, with $\pi(y) = x$, such that π induces an isomorphism of residue fields $\kappa(x) \simeq \kappa(y)$. To construct such curves, we first construct a sequence of smooth curves which are *subschemes* of the local model associated to $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$, using the explicit group theoretic description of this local model. These are then pulled back to $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ via the local model diagram. The assumption that K_p is very special is key to our argument, as this not only guarantees the density of $\mathcal{S}_{\mathrm{K},[b]_\mu}$, but also that the Kottwitz–Rapoport stratification on the local model has a particularly simple description (cf. §6.2.2) which is used in the construction of π.

The induction argument would also be unnecessary if one could show a conjecture of Deligne [Del80, Conjecture 1.2.10] on the existence of compatible local systems on a normal variety. For smooth schemes Deligne's conjecture has been proved by

Drinfeld [Dri12], but the special fiber $\mathcal{S}_{\mathrm{K}}$ is not smooth, so Drinfeld's theorem does not suffice for our purposes.

We now give some details about the construction of the integral models $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ as this may be of independent interest. The two main results we need about these models are the existence of a local model diagram as predicted in [Rap05], which relates the models to an orbit closure on a Grassmannian, and the analogue of Serre–Tate theory at μ-ordinary points, already mentioned in Theorem 1.3 above.

Under a very mild assumption, which we call *strong acceptability*, we prove (see Theorem 5.2.12) that that there exists a version of the local model diagram but where the torsor over the Shimura variety is for the parahoric of the adjoint group. The strategy for the construction follows that of [KP18], but there are new technical difficulties that need to be overcome. We first prove the existence of the local model diagram in some special Hodge-type cases; the general abelian type case is reduced to this via Deligne's formalism of connected Shimura varieties. Both the reduction step and the construction in the Hodge-type case make use of the notion of R-smoothness for tori introduced in §2.4. This is related to the failure of a closed immersion of tori to extend to a closed immersion of lft Néron models, a phenomenon which does not occur in the tamely ramified case.

In the Hodge-type case, we use the construction of [KP18, §3] to obtain a description of the formal neighborhood of the Shimura variety in terms of the deformation theory of a p-divisible group equipped with a collection of tensors in its crystalline cohomology. To apply the construction in *loc. cit.* we show that a suitable Hodge embedding induces a closed immersion of local models (cf. Proposition 3.2.6), which generalizes [KP18, Proposition 2.3.6], and a result of Anschutz [Ans, Proposition 10.3] on extending torsors, which generalizes [KP18, Proposition 1.4.3]. As a consequence of our results, we show that the local models we consider, which are constructed in [Lev16], satisfy the Scholze–Weinstein conjecture [SW20, Conjecture 21.4.1].

The special Hodge-type cases that we consider are actually not enough for applications to proving ℓ-independence for abelian varieties. This is due to pathologies in the local models when $p||\pi_1(\mathbf{G}^{\mathrm{der}})|$. In order to construct the integral models in these cases, we consider the Hodge-type Shimura datum as a datum of abelian type, and construct the integral model using a different, auxiliary, Hodge-type datum which does satisfy the required properties. The price of this indirect approach is that we have to do some work to prove that the integral model constructed in this way maps to an appropriate moduli space of abelian varieties. This is needed in order to define the μ-ordinary locus (which is however independent of auxiliary choices), and prove Theorem 1.3.

We now explain the organization of the paper. In §2-5 we construct the integral models of the Shimura varieties we need. These are then used to prove Theorem 1.1 in §6,7. The properties of the local models we use are established in §3. In §4 we review the deformation theoretic results of [KP18, §3], and show the existence of canonical deformations for μ-ordinary p-divisible groups. The latter uses a generalization to general parahorics of a result of Wortmann on μ-ordinary σ-conjugacy classes, which is proved in §2.3. We combine the previous results to construct the required integral models in §5, first in some special Hodge-type cases, then in general following [KP18, §4.4-6]. The notion of R-smoothness is used here to extend the twisting construction of [KP18, §4.4] beyond the tamely ramified case.

In §6, we prove Theorem 1.2 following the strategy outlined above and in §7 we prove Theorem 1.1 using Theorem 1.2. Finally we remark that for technical reasons related to the level structure on A, we actually work with Shimura stacks (i.e. Shimura varieties where the level structure is not neat) in §5-7.

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2. Group theoretic results

In this section, we prove some group theoretic results which will be used in §5. §2.1–§2.3 contains the results needed for the construction of canonical liftings in §4.3 and §5.4. In §2.4 we study some properties of Néron models of tori which will be used in §5.2 to construct integral models for Shimura varieties of abelian type.

2.1. σ-straight elements.

2.1.1. Let F be a non-archimedean local field with ring of integers $\mathcal{O}_F$. We fix a uniformizer $\varpi_F \in \mathcal{O}_F$ and we let k_F denote the residue field of $\mathcal{O}_F$. We let $\breve{F}$ denote the completion of the maximal unramified extension of F and $\mathcal{O}_{\breve{F}}$ its ring of integers, and we fix $\overline{F}$ an algebraic closure of F. We let k be the residue field of $\mathcal{O}_{\breve{F}}$ which is an algebraic closure of k_F. We write Γ for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ of F and I for the inertia subgroup, which is identified with $\mathrm{Gal}(\overline{\breve{F}}/\breve{F})$. We let σ denote the Frobenius element of $\mathrm{Aut}(\breve{F}/F)$.

Let S be a scheme. If X is a scheme over S and $S' \to S$ is a morphism of schemes, we write $X_{S'}$ for the base change of X along $S' \to S$.

2.1.2. Let G be a reductive group[1] over F. Let S be a maximal $\breve{F}$-split torus of G defined over F and T its centralizer (cf. [Tit79, 1.10] for the existence of S). By Steinberg's Theorem, G is quasi-split over $\breve{F}$ and T is a maximal torus of G. We let $\mathcal{B}(G,F)$ (resp. $\mathcal{B}(G,\breve{F})$) denote the (extended) Bruhat–Tits building of G over F (resp. $\breve{F}$). Let $\mathfrak{a}$ denote a σ-invariant alcove in the apartment $V := \mathcal{A}(G,S,\breve{F})$ over $\breve{F}$ associated to S; we write $\mathcal{I}$ for the corresponding Iwahori group scheme over $\mathcal{O}_F$. The relative Weyl group W_0 and the Iwahori Weyl group are defined as

$$W_0 = N(\breve{F})/T(\breve{F}), \quad W = N(\breve{F})/\mathcal{T}_0(\mathcal{O}_{\breve{F}}), \tag{2.1.2.1}$$

where N is the normalizer of T and $\mathcal{T}_0$ is the connected Néron model for T. These are related by an exact sequence

$$0 \longrightarrow X_*(T)_I \longrightarrow W \longrightarrow W_0 \longrightarrow 0.$$

For an element $\lambda \in X_*(T)_I$ we write t_λ for the corresponding element in W; such elements will be called translation elements. We will sometimes write W_G or $W_{G_{\breve{F}}}$ for W if we want to specify the group that we are working with.

[1]Our convention is that all reductive groups are connected.

2.1.3. We also fix a special vertex $\mathfrak{s}$ lying in the closure of $\mathfrak{a}$. Such a vertex induces a splitting of the exact sequence (2.1.2.1) and gives an identification

$$V \cong X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.1.3.1}$$

Let $\mathrm{Aff}(V)$ denote the group of affine transformations of V. Then we have an identification $\mathrm{Aff}(V) \cong V \rtimes \mathrm{GL}(V)$. The Frobenius σ acts on V via affine transformations and we write $\varsigma \in \mathrm{GL}(V)$ for the linear part of this action. The identification (2.1.3.1) also determines a dominant chamber $C_+ \subset X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$; namely by taking the one containing $\mathfrak{a}$, and we write B for the corresponding Borel subgroup defined over $\breve{F}$. We write σ_0 for the automorphism of $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ defined by $\sigma_0 := w_0 \circ \varsigma$ where $w_0 \in W_0$ is the unique element such that $w_0 \circ \varsigma(C_+) = C_+$. We call this the L-action on $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$; by definition it preserves C_+.

2.1.4. Let $\mathbb{S}$ denote the set of simple reflections about the walls of $\mathfrak{a}$. We let W_a denote the affine Weyl group; it is the subgroup of W generated by the reflections in $\mathbb{S}$. Then $(W_a, \mathbb{S})$ has the structure of a Coxeter group, and hence we have a notion of length and Bruhat order. The Iwahori Weyl group and affine Weyl group are related via the following exact sequence

$$0 \longrightarrow W_a \longrightarrow W \longrightarrow \pi_1(G)_I \longrightarrow 0. \tag{2.1.4.1}$$

The choice of $\mathfrak{a}$ induces a splitting of this exact sequence and $\pi_1(G)_I$ can be identified with the subgroup $\Omega \subset W$ consisting of elements which preserve $\mathfrak{a}$. The length function ℓ and Bruhat order $\leq$ extend to W via this choice of splitting and Ω is identified with the set of length 0 elements.

We let $\widetilde{\kappa}_G(w)$ denote the image of $w \in W$ in $\pi_1(G)_I$ and $\kappa_G(w)$ its projection to $\pi_1(G)_\Gamma$. For $w \in W$, there is an integer n such that σ^n acts trivially on W and $w\sigma(w)\dots\sigma^{n-1}(w) = t_\lambda$ for some $\lambda \in X_*(T)_I$. We define the (non-dominant) Newton cocharacter $\nu_w \in X_*(T)_{I,\mathbb{Q}} \cong X_*(T)_{\mathbb{Q}}^I$ to be $\frac{1}{n}\lambda$, which is easily seen to be independent of n. We let $\overline{\nu}_w \in X_*(T)_{\mathbb{Q}}^{I,+}$ be the dominant representative of ν_w.

2.1.5. Let T^{sc}, denote the preimage of T in the simply connected covering G^{sc} of the derived group of G. Then W_a is the Iwahori Weyl group for G^{sc} and we have the following exact sequence

$$0 \longrightarrow X_*(T^{\mathrm{sc}})_I \longrightarrow W_a \longrightarrow W_0 \longrightarrow 0.$$

Since the action of I permutes the set of absolute coroots, $X_*(T^{\mathrm{sc}})_I$ is torsion free and there is an inclusion $X_*(T^{\mathrm{sc}})_I \hookrightarrow X_*(T)_I$. By [HR08], there exists a reduced root system Σ such that

$$W_a \simeq Q^\vee(\Sigma) \rtimes W(\Sigma)$$

where $Q^\vee(\Sigma)$ and $W(\Sigma)$ denotes the coroot lattice and Weyl group of Σ, respectively, and there is a canonical isomorphism $W(\Sigma) \cong W_0$. The roots of Σ are proportional to the roots of the relative root system for $G_{\breve{F}}$; however the root systems themselves may not be proportional.

As explained in [HR08, p7], we may consider elements of Σ as functions on $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$, and we write $\langle\ ,\ \rangle$ for the induced pairing between $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ and the root lattice associated to Σ. We let ρ denote the half sum of all positive roots in Σ. Then for any $\lambda \in X_*(T)_I$ we have the equality

$$\ell(t_\lambda) = \langle \overline{\lambda}, 2\rho \rangle, \tag{2.1.5.1}$$

where $\overline{\lambda} \in W_0 \cdot \lambda$ is the dominant representative of λ, i.e. the image of $\overline{\lambda}$ in $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ lies in C_+.

2.1.6. We say that an element $w \in W$ is *σ-straight* if for any $n \in \mathbb{N}$, we have

$$\ell(w\sigma(w)\dots\sigma^{n-1}(w)) = n\ell(w).$$

It is straightforward to check that this is equivalent to the condition $\ell(w) = \langle \overline{\nu}_w, 2\rho\rangle$.

In this paper, we are particularly interested in translation elements $t_{\mu'}$ which are also σ-straight; the key property of these elements that we will need is that they are central for some Levi subgroup of G defined over F.

For any $v \in X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$, we let $\Phi_{v,0}$ be the set of relative roots α for $G_{\breve{F}}$ such that $\langle v, \alpha\rangle = 0$. We may then associate to v the semi-standard Levi subgroup $M_v \subset G_{\breve{F}}$ generated by T and the root subgroups U_α corresponding to $\alpha \in \Phi_{v,0}$. If in addition v is fixed by ς, then M_v is defined over F. We say $\lambda \in X_*(T)_I$ is central in G if it pairs with any relative root (equivalently any root in Σ) to give 0.

Lemma 2.1.7. *Let $\mu' \in X_*(T)_I$ such that $t_{\mu'}$ is a σ-straight element and let $M := M_{\nu_{t_{\mu'}}}$ be the semi-standard Levi subgroup of G associated to the Newton cocharacter $\nu_{t_{\mu'}}$. Then M is defined over F and μ' is central in M.*

Proof. For any $\lambda \in X_*(T)_I$, and for sufficiently divisible n we have

$$n\nu_{\sigma(t_\lambda)} = \sigma(t_\lambda)\dots\sigma^n(t_\lambda) = t_\lambda^{-1} n\nu_{t_\lambda} t_\lambda = n\nu_{t_\lambda}.$$

Note that $\sigma(t_\lambda) = t_{\varsigma(\lambda)}$; it follows that $\nu_{\sigma(t_\lambda)} = \varsigma(\nu_{t_\lambda})$ and hence ν_{t_λ} is fixed by ς. Therefore M is defined over F.

We let $u \in W_0$ be such that $u(\nu_{t_{\mu'}}) = \overline{\nu}_{t_{\mu'}}$. For a sufficiently divisible n, we have

$$\ell(t_{\mu'}) = \langle \overline{\nu}_{t_{\mu'}}, 2\rho\rangle = \frac{1}{n}\sum_{i=0}^{n-1}\langle u\varsigma^i(\mu'), 2\rho\rangle$$

where the first equality follows from the σ-straightness of $t_{\mu'}$. Now $\langle u\varsigma^i(\mu'), 2\rho\rangle \leq \ell(t_{\mu'})$ with equality if and only if $u\varsigma^i(\mu')$ is dominant. Therefore $u\varsigma^i(\mu')$ is dominant for all i and hence $\varsigma^i(\mu')$ is contained in the translate C' of the dominant chamber C_+ by u^{-1} for all i.

Now M corresponds to a sub-root system Σ_M of Σ consisting of the roots $\alpha \in \Sigma$ such that $\langle \nu_{t_{\mu'}}, \alpha\rangle = 0$. Then Σ_M is also the reduced root system associated to the affine Weyl group for M as in §2.1.5. We must show for all $\alpha \in \Sigma_M$, we have $\langle \mu', \alpha\rangle = 0$. Let $\alpha \in \Sigma_M$ be a root, then since $\varsigma^i(\mu')$ is contained in a single Weyl chamber for all i, it follows that $\langle \varsigma^i(\mu'), \alpha\rangle$ have the same sign for all i.

Without loss of generality, assume $\langle \varsigma^i(\mu'), \alpha\rangle \geq 0, \forall i$. Then we have

$$0 = \langle \nu_{t_{\mu'}}, \alpha\rangle = \frac{1}{n}\sum_{i=0}^{n-1}\langle \varsigma^i(\mu'), \alpha\rangle. \tag{2.1.7.1}$$

Since all terms in the sum are non-negative, they must be 0. Hence μ' is central in M. □

2.1.8. Now let $\{\mu\}$ be a geometric conjugacy class of cocharacters of G. Let $\mu \in X_*(T)_I$ denote the image of a dominant (with respect to the choice of Borel B defined above) representative $\widetilde{\mu} \in X_*(T)$ of $\{\mu\}$.

Lemma 2.1.9. *Let $w \in W_0$ such that for $\mu' := w(\mu)$, $t_{\mu'}$ is a σ-straight element. Let $\widetilde{\lambda} := w(\widetilde{\mu}) \in X_*(T)$. Then $\widetilde{\lambda}$ is central in $M := M_{\nu_{t_{\mu'}}}$. Here, we consider W_0 as a subgroup of the absolute Weyl group for G.*

Proof. Let $w(C_+) \subset X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ be the translate of the dominant chamber by w. Then $w(C_+)$ determines a chamber C_M for M (it is the unique chamber for M such that $w(C_+) \subset C_M$) and we have $\mu' \in C_M$. The chamber C_M determines an ordering of the root system Σ_M. Let α be a positive root for Σ_M and $\widetilde{\alpha} \in X^*(T)$ an (absolute) root lifting α; such a lift exists by the construction of Σ, see eg. [Bou68, VI, 2.1]. We let $(\ ,\ ) : X_*(T) \times X^*(T) \to \mathbb{Z}$ denote the natural pairing.

Let $K/\breve{F}$ be a finite Galois extension over which T splits. We have by definition of Σ_M

$$0 = \langle \mu', \alpha \rangle = c \sum_{\tau \in \mathrm{Gal}(K/\breve{F})} (\widetilde{\lambda}, \tau(\widetilde{\alpha}))$$

for some positive $c \in \mathbb{R}$, where the first equality follows since μ' is central in M. For any $\tau \in \mathrm{Gal}(K/\breve{F})$, C_M is preserved by τ and hence $\tau(\widetilde{\alpha})$ is a positive root for M. Therefore $(\widetilde{\lambda}, \tau(\widetilde{\alpha})) \geq 0$, and hence $(\widetilde{\lambda}, \tau(\widetilde{\alpha})) = 0$ for all τ. Applying this to every relative root α for M, we see that $\widetilde{\lambda}$ is central in M. □

2.2. μ-ordinary σ-conjugacy classes.

2.2.1. Let $\{\mu\}$ be a geometric conjugacy class of cocharacters of G; we let $\widetilde{\mu} \in X_*(T)$ and $\mu \in X_*(T)_I$ as above. The μ-admissible set is defined to be

$$\mathrm{Adm}(\{\mu\}) = \{w \in W | w \leq t_{x(\mu)} \text{ for some } x \in W_0\}.$$

It has a unique minimal element denoted $\tau_{\{\mu\}}$, which is also the unique element of $\mathrm{Adm}(\{\mu\}) \cap \Omega$.

For $b \in G(\breve{F})$, we let $[b]$ denote the set $\{g^{-1} b \sigma(g) | g \in G(\breve{F})\}$, the σ-conjugacy class of b. The set of σ-conjugacy classes $B(G)$ has been classified by Kottwitz in [Kot92] and [Kot97]. For $b \in G(\breve{F})$, we let $\nu_b : \mathbb{D} \to G_{\breve{F}}$ denote its Newton cocharacter and

$$\overline{\nu}_b \in X_*(T)^+_{I,\mathbb{Q}} \cong X_*(T)^{I,+}_{\mathbb{Q}}$$

the dominant representative for ν_b; it is known that $\overline{\nu}_b$ is invariant under the action of σ_0. We let $\widetilde{\kappa}_G : G(\breve{F}) \to \pi_1(G)_I$ denote the Kottwitz homomorphism and we write

$$\kappa_G : G(\breve{F}) \to \pi_1(G)_\Gamma$$

for the composition of $\widetilde{\kappa}_G$ and the projection map $\pi_1(G)_I \to \pi_1(G)_\Gamma$. This induces a well-defined map $B(G) \to \pi_1(G)_\Gamma$, also denoted κ_G. Then there is an injective map

$$B(G) \xrightarrow{(\kappa_G, b \mapsto \overline{\nu}_b)} \pi_1(G)_\Gamma \times (X_*(T)^{I,+}_{\mathbb{Q}})^{\sigma_0}. \tag{2.2.1.1}$$

2.2.2. There is a more explicit description of this map using the Iwahori Weyl group W. For $w \in W$, its σ-conjugacy class is the set $\{u^{-1} w \sigma(w) | u \in W\}$. We let $B(W, \sigma)$ denote the set of σ-conjugacy classes in W. For $w \in W$, we let $\dot{w} \in N(\breve{F})$ be a lift of w. Then to $w \in W$, we associate the σ-conjugacy class of $\dot{w}$; by Lang's theorem this does not depend on the choice of representative $\dot{w}$. We write

$$\Psi : B(W, \sigma) \to B(G)$$

for the map induced by $w \mapsto [\dot{w}]$.

By [He14, Theorem 3.7], Ψ is surjective and we have a commutative diagram

(2.2.2.1)
$$\begin{array}{ccc} B(W,\sigma) & \xrightarrow{\quad\Psi\quad} & B(G) \\ & {\scriptstyle(\overline{\nu},\kappa_G)}\searrow \quad \swarrow{\scriptstyle(\overline{\nu},\kappa_G)} & \\ & (X_*(T)_{\mathbb{Q}}^{I,+})\times\pi_1(G)_\Gamma & \end{array}.$$

The map Ψ is not injective in general, however it is proved in [He14, Theorem 3.7] that its restriction to the set of σ-straight σ-conjugacy classes is a bijection. Here, a σ-conjugacy class in W is said to be σ-straight if it contains a σ-straight element.

2.2.3. Note that there is a partial order on the set $X_*(T)_{\mathbb{Q}}^+$; for $\lambda,\lambda'\in X_*(T)_{\mathbb{Q}}^+$, we write $\lambda\leq\lambda'$ if $\lambda'-\lambda$ is a non-negative rational linear combination of positive roots. For $\{\mu\}$ as above, we write $\mu^\natural$ for the common image of an element of $\{\mu\}$ in $\pi_1(G)_\Gamma$ and we define

$$\mu^\diamond=\frac{1}{N}\sum_{i=1}^N\sigma_0^i(\mu)\in X_*(T)_{I,\mathbb{Q}}^+\cong X_*(T)_{\mathbb{Q}}^{I,+}.$$

where N is the order of the element σ_0 giving rise to the L-action on $X_*(T)_I\otimes_{\mathbb{Z}}\mathbb{Q}$. We set

$$B(G,\{\mu\})=\{[b]\in B(G):\kappa_G(b)=\mu^\natural,\overline{\nu}_b\leq\mu^\diamond\}.$$

Note that for $[b_1],[b_2]\in B(G,\{\mu\})$, we have $[b_1]=[b_2]$ if and only if $\overline{\nu}_{[b_1]}=\overline{\nu}_{[b_2]}$, since $[b_1]$ and $[b_2]$ have common image $\mu^\natural$ under κ_G.

Definition 2.2.4. Suppose there exists a class $[b]\in B(G,\{\mu\})$ such that $\overline{\nu}_{[b]}=\mu^\diamond$ (such a class is necessarily unique if it exists by the above remark). We write $[b]_\mu$ for this class; it is called the μ-ordinary σ-conjugacy class.

Remark 2.2.5. [HN18, Theorem 1.1] have shown that $B(G,\{\mu\})$ always contains a unique maximal element with respect to the partial order $\leq$. When G is quasi-split, this class is just $[b]_\mu$. However if G is not quasi-split, there may be no $[b]\in B(G,\{\mu\})$ such that $\overline{\nu}_{[b]}=\mu^\diamond$.

Lemma 2.2.6. *Assume there exists $[b]_\mu\in B(G,\{\mu\})$ with $\overline{\nu}_{[b]_\mu}=\mu^\diamond$. There exists $\mu'\in W_0\cdot\mu$ such that $t_{\mu'}$ is σ-straight and $\dot{t}_{\mu'}\in[b]_\mu$.*

Proof. Since $[b]_\mu\in B(G,\{\mu\})$, there exists a σ-straight element $w\in\mathrm{Adm}(\{\mu\})$ such that $\dot{w}\in[b]_\mu$ by [He16, Theorem 4.1]. The commutativity of diagram (2.2.2.1) implies that $\overline{\nu}_w=\mu^\diamond$. Since w is σ-straight, we have

$$\ell(w)=\langle\overline{\nu}_w,2\rho\rangle=\langle\mu^\diamond,2\rho\rangle=\langle\mu,2\rho\rangle=\ell(t_\mu),$$

where the third equality follows from the fact ρ is invariant under σ_0, and the final equality uses (2.1.5.1) and the fact that μ is dominant. Since $w\in\mathrm{Adm}(\{\mu\})$, $\ell(w)\leq\ell(t_\mu)$ with equality if and only if $w=t_{\mu'}$ for some $\mu'\in W_0\cdot\mu$. □

2.2.7. Now let G' be another reductive group over F and $f:G\to G'$ a group scheme morphism which induces an isogeny $G^{\mathrm{der}}\to G'^{\mathrm{der}}$. We write $\{\mu'\}$ for the G'-conjugacy class of cocharacters induced by $\{\mu\}$. We have the following relationship between μ-ordinary σ-conjugacy classes for G and G'.

Lemma 2.2.8. *(1) There exists $[b]_\mu\in B(G,\{\mu\})$ with $\overline{\nu}_{[b]_\mu}=\mu^\diamond$ if and only if there exists $[b']_{\mu'}\in B(G',\{\mu'\})$ with $\overline{\nu}_{[b']_{\mu'}}=\mu'^\diamond$.*

(2) Let $[b] \in B(G,\mu)$ and $[b'] := [f(b)] \in B(G', \{\mu'\})$. Then $[b] = [b]_\mu$ if and only if $[b'] = [b']_{\mu'}$.

Proof. (1) Note that we have a commutative diagram

$$\begin{array}{ccc} B(G) & \longrightarrow & (X_*(T)^{I,+}_{\mathbb{Q}}) \times \pi_1(G)_\Gamma \\ \downarrow & & \downarrow \\ B(G') & \longrightarrow & (X_*(T')^{I,+}_{\mathbb{Q}}) \times \pi_1(G')_\Gamma \end{array}$$

where T' is the centralizer of a maximal $\breve{F}$-split torus of G' containing $f(T)$. Thus one direction of (1) is clear.

For the converse, suppose there exists $[b']_{\mu'} \in B(G', \{\mu'\})$. Note that by assumption, there is an identification of relative Weyl groups for G and G'. Then by Lemma 2.2.6, there exists $w_0 \in W_0$ such that $t_{w_0(\mu')}$ is a σ-straight element of the Iwahori Weyl group for G' and $\dot{t}_{w_0(\mu')} \in [b']_\mu$. Then it is easy to check that $t_{w_0(\mu)}$ is a σ-straight element of the Iwahori Weyl group for G and that $\overline{\nu}_{t_{w_0(\mu)}} = \mu^\diamond$. It follows that $[\dot{t}_{w_0(\mu)}] = [b]_\mu \in B(G, \{\mu\})$.

(2) One direction is clear. Suppose then that $[b'] = [b']_{\mu'}$. It follows that $\overline{\nu}_{[b]} = \mu^\diamond + \alpha$ for some $\alpha \in X_*(\ker(G \to G'))^I$. But $[b] \in B(G, \{\mu\})$ and hence $\mu^\diamond - \overline{\nu}_{[b]}$ is a rational linear combination of positive coroots. Thus $\alpha = 0$ and $[b] = [b]_\mu$. □

2.3. **Parahoric group schemes.**

2.3.1. Recall that $\mathcal{B}(G,F)$ and $\mathcal{B}(G,\breve{F})$ denote the extended Bruhat–Tits buildings associated to G. For a non-empty bounded subset $\Xi \subset \mathcal{B}(G,F)$ which is contained in an apartment, we let $G(F)_\Xi$ (resp. $G(\breve{F})_\Xi$) denote the subgroup of $G(F)$ (resp. $G(\breve{F})$) which fixes Ξ pointwise. By the main result of [BT84], there exists a smooth affine group scheme $\widetilde{\mathcal{G}}_\Xi$ over $\mathcal{O}_F$ with generic fiber G which is uniquely characterized by the property $\widetilde{\mathcal{G}}_\Xi(\mathcal{O}_{\breve{F}}) = G(\breve{F})_\Xi$. As in [KP18, §1.1.2], we will call such a group scheme the Bruhat–Tits stabilizer scheme associated to Ξ. If $\Xi = \{x\}$ is a point we write $G(F)_x$ (resp. $\widetilde{\mathcal{G}}_x$) for $G(F)_{\{x\}}$ (resp. $\widetilde{\mathcal{G}}_{\{x\}}$).

For $\Xi \subset \mathcal{B}(G,F)$, we write $\mathcal{G}_\Xi$ for the "connected stabilizer" Ξ (cf. [BT84, §4]). We caution the reader that our convention differs from [KP18], where $\mathcal{G}_\Xi$ is used for the Bruhat–Tits stabilizer scheme and $\mathcal{G}_\Xi^\circ$ for the connected stabilizer. We are mainly interested in the cases where Ξ is a point x or an open facet $\mathfrak{f}$. In this case, $\mathcal{G}_x$ (resp. $\mathcal{G}_\mathfrak{f}$) is the parahoric group scheme associated to x (resp. $\mathfrak{f}$). By [HR08], $\mathcal{G}_\Xi(\mathcal{O}_{\breve{F}}) = \widetilde{\mathcal{G}}_\Xi(\mathcal{O}_{\breve{F}}) \cap \ker \widetilde{\kappa}_G$. It follows that $\mathcal{G}_\Xi(\mathcal{O}_F) = \widetilde{\mathcal{G}}_\Xi(\mathcal{O}_F) \cap \ker \widetilde{\kappa}_G$.

We may also consider the corresponding objects over $\breve{F}$ and we use the same notation in this case. When it is understood which point of $\mathcal{B}(G,F)$ or $\mathcal{B}(G,\breve{F})$ we are referring to, we simply write $\widetilde{\mathcal{G}}$ and $\mathcal{G}$ for the corresponding group schemes.

An important case that is needed for applications is when $\mathcal{G}_x = \widetilde{\mathcal{G}}_x$, i.e. the parahoric is equal to the Bruhat–Tits stabilizer. When this happens, we necessarily have that $\widetilde{\mathcal{G}}_x = \widetilde{\mathcal{G}}_\mathfrak{f}$, where $\mathfrak{f}$ is the facet which contains x, and $x \in \mathfrak{f}$ is a point which is "in general position." We say that $\mathcal{G}$ is a *connected parahoric* if there exists a point $x \in \mathcal{B}(G,F)$ such that $\widetilde{\mathcal{G}}_x = \mathcal{G}$.

Let G' be another reductive group and assume there is an identification $G^{\mathrm{ad}} \cong G'^{\mathrm{ad}}$ between their respective adjoint groups. Then there are surjective maps of buildings $\mathcal{B}(G,F)\to\mathcal{B}(G^{\mathrm{ad}},F)$ and $\mathcal{B}(G',F)\to\mathcal{B}(G'^{\mathrm{ad}},F)$ which are equivariant for $G(F)$ and $G'(F)$ respectively. Let $\mathcal{G}=\mathcal{G}_x$ be a parahoric group scheme of G corresponding to $x\in\mathcal{B}(G,F)$, and let $x^{\mathrm{ad}}\in\mathcal{B}(G^{\mathrm{ad}},F)$ denote the image of x. Then for any $x'\in\mathcal{B}(G',F)$ lifting x^{ad}, the parahoric $\mathcal{G}'=\mathcal{G}'_{x'}$ of G' is independent of the choice of x' lifting x^{ad}. Thus $\mathcal{G}$ determines a parahoric $\mathcal{G}'$ of G'; in this case we say that $\mathcal{G}$ and $\mathcal{G}'$ are *associated*.

2.3.2. Now let $J\subset\mathbb{S}$ be a subset and we write W_J for the subgroup of W generated by J. If W_J is finite, J corresponds to a parahoric group scheme $\mathcal{G}$ over $\mathcal{O}_{\breve{F}}$; such parahorics are called *standard* (with respect to $\mathfrak{a}$). We let W^J (resp. ${}^JW$) denote the set of minimal length representatives of the cosets W/W_J (resp $W_J\backslash W$).

We recall the Iwahori decomposition. For $w\in W$, the map $w\mapsto\dot{w}$ induces a bijection

$$W_J\backslash W/W_J\cong\mathcal{G}(\mathcal{O}_{\breve{F}})\backslash G(\breve{F})/\mathcal{G}(\mathcal{O}_{\breve{F}}).$$

We now assume J is σ-stable; in this case the parahoric group scheme $\mathcal{G}$ is defined over $\mathcal{O}_F$. For the rest of §2.3, we fix a geometric conjugacy class of cocharacters $\{\mu\}$ of G and assume the existence of $[b]_\mu\in B(G,\{\mu\})$. We define $\mathrm{Adm}(\{\mu\})_J$ to be the image of $\mathrm{Adm}(\{\mu\})$ in $W_J\backslash W/W_J$. We sometimes write $\mathrm{Adm}_G(\{\mu\})_J$ if we want to specify the group G we are working with. The following is the key group theoretic result needed to prove the existence of canonical liftings in §5.4.

Proposition 2.3.3. *Let* $b\in\Big(\bigcup_{w\in\mathrm{Adm}(\{\mu\})_J}\mathcal{G}(\mathcal{O}_{\breve{F}})\dot{w}\mathcal{G}(\mathcal{O}_{\breve{F}})\Big)\cap[b]_\mu$. *Then*

(1) $b\in\mathcal{G}(\mathcal{O}_{\breve{F}})\dot{t}_{\mu'}\mathcal{G}(\mathcal{O}_{\breve{F}})$ *for some σ-straight element* $t_{\mu'}$.

(2) *There exists $g\in\mathcal{G}(\mathcal{O}_{\breve{F}})$ such that $g^{-1}b\sigma(g)=\dot{t}_{\mu'}$, for $t_{\mu'}$ as in (1).*

Proof. By [HR17, Theorem 6.1 (b)], there exists $h\in\mathcal{G}(\mathcal{O}_{\breve{F}})$ such that $h^{-1}b\sigma(h)\in\mathcal{I}(\mathcal{O}_{\breve{F}})\dot{w}\mathcal{I}(\mathcal{O}_{\breve{F}})$ for some $w\in{}^JW$. Thus $w\in{}^JW\cap W_J\mathrm{Adm}(\{\mu\})W_J$ and hence lies in ${}^JW\cap\mathrm{Adm}(\{\mu\})$ by [He16, Theorem 6.1]. Thus upon replacing b by $h^{-1}b\sigma(h)$, we may assume $b\in\mathcal{I}(\mathcal{O}_{\breve{F}})\dot{w}\mathcal{I}(\mathcal{O}_{\breve{F}})$. By [HZ20, Theorem 4.1], there exists a σ-straight element $x\leq w$ such that $[b]_\mu\cap\mathcal{I}(\mathcal{O}_{\breve{F}})\dot{x}\mathcal{I}(\mathcal{O}_{\breve{F}})\neq\emptyset$ (the Theorem in *loc. cit.* proves the non-emptiness of the affine Deligne–Lusztig variety $X_x(b)$, which is equivalent to this statement). By the proof of [He14, Theorem 3.5], we have $\dot{x}\in[b]_\mu$ and by the same argument as in Lemma 2.2.6 we have $x=t_{\mu'}$ for some $\mu'\in W_0\cdot\mu$. Since $x\leq w$ and $w\in\mathrm{Adm}(\{\mu\})$, we have $w=t_{\mu'}$. This proves (1).

For (2), the above argument shows that we may assume $b\in\mathcal{I}(\mathcal{O}_{\breve{F}})\dot{t}_{\mu'}\mathcal{I}(\mathcal{O}_{\breve{F}})$ for $t_{\mu'}$ a σ-straight element. By [He14, Proposition 4.5], there exists $i\in\mathcal{I}(\mathcal{O}_{\breve{F}})$ such that $i^{-1}b\sigma(i)=\dot{t}_{\mu'}$; the result follows. □

Remark 2.3.4. This result is a generalization to general parahorics of [SZ21, Proposition 2.5] which is due to Wortmann. In the case when $\mathcal{G}$ is a hyperspecial parahoric, this result is the group theoretic analogue of the fact that there is exactly one isomorphism class of ordinary F-crystal over $\mathcal{O}_{\breve{F}}$.

2.4. **Néron models of tori.**

2.4.1. In this subsection, we introduce the notion of R-smooth tori and discuss some consequences for Bruhat–Tits group schemes.

Let T be a torus over a non-archimedean local field F; recall we have defined $\mathcal{T}_0$ to be the connected Néron model of T. We let $\mathcal{T}$ (resp. $\mathcal{T}_{\mathrm{ft}}$) denote the lft Néron model (resp. finite type Néron model) for T. Then we have $\mathcal{T}(\mathcal{O}_{\breve{F}})=T(\breve{F})$ and $\mathcal{T}_{\mathrm{ft}}$

is characterized by the condition $\mathcal{T}_{\mathrm{ft}}(\mathcal{O}_{\breve{F}}) = \{t \in T(\breve{F}) | \widetilde{\kappa}_T(t) \in X_*(T)_{I,\mathrm{tors}}\}$ where $X_*(T)_{I,\mathrm{tors}}$ is the torsion subgroup of $X_*(T)_I$. Alternatively, by [Rap05, n°1] the connected components of the special fiber of $\mathcal{T}$ are parameterized by $X_*(T)_I$ and $\mathcal{T}_{\mathrm{ft}}$ is the unique smooth subgroup scheme of $\mathcal{T}$ whose special fiber is given by the set of connected components corresponding to the subgroup $X_*(T)_{I,\mathrm{tors}}$ of $X_*(T)_I$.

2.4.2. Let $\widetilde{F}/F$ be a finite Galois extension over which T splits and $\mathcal{T}_{\mathcal{O}_{\widetilde{F}}}$ denote the lft Néron model of $T_{\widetilde{F}}$. [2] By [BLR90, §7.6, Proposition 6], $\mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{\mathcal{O}_{\widetilde{F}}}$ is the lft Néron model over $\mathcal{O}_F$ for $\mathrm{Res}_{\widetilde{F}/F}T_{\widetilde{F}}$. There is a natural map $T \to \mathrm{Res}_{\widetilde{F}/F}T_{\widetilde{F}}$ and we define $\mathcal{T}^c$ to be the Zariski closure of T inside $\mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{\mathcal{O}_{\widetilde{F}}}$. As in [BT84, §4.4.8], $\mathcal{T}^c$ does not depend on the choice of splitting field $\widetilde{F}$.

Definition 2.4.3. We say a torus T is *R-smooth* if $\mathcal{T}^c$ is smooth.

Since $\mathcal{T}^c$ satisfies the Néron mapping property (see [Edi92, Proof of Theorem 4.2] for the proof in the case of abelian varieties which also works for tori), we have $\mathcal{T} \cong \mathcal{T}^c$ if T is R-smooth.

We can similarly define a notion of R-smoothness for tori over $\breve{F}$. It is easy to see using compatibility of Néron models with base change along $\mathcal{O}_F \to \mathcal{O}_{\breve{F}}$ that a torus over F is R-smooth if and only if $T_{\breve{F}}$ is R-smooth.

The main property concerning R-smooth tori that we need is the following.

Lemma 2.4.4. *Suppose we have a closed immersion $f : T_1 \to T_2$ between tori where T_1 is R-smooth. Then*

(1) f extends to a closed immersion $\mathcal{T}_1 \to \mathcal{T}_2$ of lft Néron models.
(2) f extends to a closed immersion $\mathcal{T}_{1,\mathrm{ft}} \to \mathcal{T}_{2,\mathrm{ft}}$ of finite type Néron models.

Proof. (1) Let $\widetilde{F}/F$ be a finite Galois splitting field for both T_1 and T_2. Then since $T_{1,\widetilde{F}}$ and $T_{2,\widetilde{F}}$ are products of multiplicative group schemes, the map $T_{1,\widetilde{F}} \to T_{2,\widetilde{F}}$ extends to a closed immersion of lft Néron models $\mathcal{T}_{\mathcal{O}_{\widetilde{F}}} \to \mathcal{T}_{2,\mathcal{O}_{\widetilde{F}}}$ over $\mathcal{O}_{\widetilde{F}}$. We obtain a diagram

$$\begin{array}{ccc} \mathcal{T}_1 & \xrightarrow{f} & \mathcal{T}_2 \\ \downarrow g & & \downarrow h \\ \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{1,\mathcal{O}_{\widetilde{F}}} & \xrightarrow{i} & \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{2,\mathcal{O}_{\widetilde{F}}} \end{array}$$

where i is a closed immersion since it is obtained via restriction of scalars of a closed immersion, and g is a closed immersion since T_1 is R-smooth. It follows that $h \circ f = i \circ g$ is a closed immersion, and hence f is a closed immersion.

(2) By (1), we have a closed immersion $\mathcal{T}_1 \to \mathcal{T}_2$ of lft Néron models. We let $\phi : X_*(T_1)_I \to X_*(T_2)_I$ denote the morphism on the targets of the Kottwitz homomorphism. Using that $\ker(\phi)$ is torsion, one sees that

$$\phi^{-1}(X_*(T_2)_{I,\mathrm{tors}}) = X_*(T_1)_{I,\mathrm{tors}}.$$

As the finite type Néron models $\mathcal{T}_{1,\mathrm{ft}}$ and $\mathcal{T}_{2,\mathrm{ft}}$ correspond to the subschemes of $\mathcal{T}_1$ and $\mathcal{T}_2$ whose special fibers are given by the connected components parameterized by $X_*(T_1)_{I,\mathrm{tors}}$ and $X_*(T_2)_{I,\mathrm{tors}}$ respectively, it follows that $\mathcal{T}_1 \to \mathcal{T}_2$ induces a closed immersion $\mathcal{T}_{1,\mathrm{ft}} \to \mathcal{T}_{2,\mathrm{ft}}$ as desired. □

[2] We are abusing notation here since $\mathcal{T}_{\mathcal{O}_{\widetilde{F}}}$ is not necessarily the base change to $\mathcal{O}_{\widetilde{F}}$ of the Néron model $\mathcal{T}$ of T over $\mathcal{O}_F$.

2.4.5. The proof of [Edi92, Theorem 4.2] shows that if T splits over a tamely ramified extension of F, then T is R-smooth. In addition, the main examples of R-smooth tori that we will consider are given by the following proposition.

Proposition 2.4.6. *(1) Let $T = \prod_{i=1}^{s} \mathrm{Res}_{K_i/F} S_i$, where K_i are finite separable extensions of F and S_i are K_i-tori which split over a tamely ramified extension of K_i. Then T is R-smooth.*

(2) Let T be a torus which is an extension of an R-smooth torus by an R-smooth torus. Then T is R-smooth.

Proof. (1) We will make use of the following result which follows from [BLR90, §7.6 Proposition 6]: If S is a torus over a finite separable extension K of F with lft Néron model $\mathcal{S}$ over $\mathcal{O}_K$, then $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{S}$ is the lft Néron model for $\mathrm{Res}_{K/F}S$.

We may reduce to the case $s = 1$, in which case we write $T = \mathrm{Res}_{K/F}S$ for S a tamely ramified torus over K. Let $\widetilde{F}/F$ be a finite Galois splitting field of T which necessarily contains K. For any F-morphism $\tau : K \to \widetilde{F}$, the base change of S along τ is split. Since S is R-smooth, it follows that we have a closed immersion of $\mathcal{O}_K$-group schemes

$$\mathcal{S} \to \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_K}\mathcal{S}_{\mathcal{O}_{\widetilde{F}}},$$

where $\mathcal{S}$ (resp. $\mathcal{S}_{\mathcal{O}_{\widetilde{F}}}$) is the lft Néron model for S (resp. $S_{\widetilde{F}}$).

Applying $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}$ we obtain a closed immersion

$$\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{S} \to \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{S}_{\mathcal{O}_{\widetilde{F}}}.$$

Taking the product over all $\tau : K \to \widetilde{F}$ we obtain a closed immersion

$$\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{S} \to \prod_{\tau: K\to\widetilde{F}} \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{S}_{\mathcal{O}_{\widetilde{F}}} \cong \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{\mathcal{O}_{\widetilde{F}}}.$$

Since $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{S}$ is the lft Néron model $\mathcal{T}$ for T, it follows that $\mathcal{T}$ is the closure of its generic fiber inside $\mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{\mathcal{O}_{\widetilde{F}}}$ and hence T is R-smooth.

(2) We may assume $F = \breve{F}$. Assume we have an exact sequence

$$1 \longrightarrow T_1 \xrightarrow{f} T \xrightarrow{g} T_2 \longrightarrow 1$$

where T_1 and T_2 are R-smooth. Since T_1 is R-smooth, f extends to a closed immersion of lft Néron models $\mathcal{T}_1 \to \mathcal{T}$, by Lemma 2.4.4. The quotient $\mathcal{T}/\mathcal{T}_1$ is a smooth group scheme with generic fiber T_2, and by Steinberg's theorem it has the same $\mathcal{O}_{\breve{F}}$-points as the lft Néron model $\mathcal{T}_2$ for T_2. Thus by [BT84, Proposition 1.7.6], $\mathcal{T}/\mathcal{T}_1 \cong \mathcal{T}_2$ and we have an exact sequence of group schemes

$$1 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}_2 \longrightarrow 1\ .$$

Let $\widetilde{F}/F$ be a finite Galois extension over which T_1, T_2 and T split. We obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{T}_1 & \longrightarrow & \mathcal{T} & \longrightarrow & \mathcal{T}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{1,\mathcal{O}_{\widetilde{F}}} & \longrightarrow & \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{\mathcal{O}_{\widetilde{F}}} & \longrightarrow & \mathrm{Res}_{\mathcal{O}_{\widetilde{F}}/\mathcal{O}_F}\mathcal{T}_{2,\mathcal{O}_{\widetilde{F}}} & \longrightarrow & 1 \end{array}$$

Let $\mathcal{T}^c$ be the Zariski closure of T in $\mathrm{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F}\mathcal{T}_{\mathcal{O}_{\tilde{F}}}$. By R-smoothness of T_2 and T_1, the two outer vertical maps in the diagram above are closed immersions. Hence, $\mathcal{T}_1$ is closed in $\mathcal{T}^c$, and the composite

$$\mathcal{T}_2 \simeq \mathcal{T}/\mathcal{T}_1 \to \mathcal{T}^c/\mathcal{T}_1 \to \mathrm{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F}\mathcal{T}_{2,\mathcal{O}_{\tilde{F}}}$$

is a closed immersion. Thus, $\mathcal{T}/\mathcal{T}_1$ is closed in $\mathcal{T}^c/\mathcal{T}_1$. As these are two $\mathcal{O}_F$-flat schemes with the same generic fiber, it follows that $\mathcal{T}/\mathcal{T}_1 \simeq \mathcal{T}^c/\mathcal{T}_1$, and hence $\mathcal{T} \simeq \mathcal{T}^c$. Hence j is a closed immersion and T is R-smooth. □

Corollary 2.4.7. *Let T_2 be an F-torus which splits over a tame extension of F and $f : T \to T_2$ a surjective morphism of F-tori such that the identity component of* $\ker(f)$ *is an R-smooth torus. Then T is R-smooth.*

In particular, if T is torus which is an extension of $\mathbb{G}_m$ by a group scheme whose identity component is of the form $\prod_{i=1}^{s}\mathrm{Res}_{K_i/F}S_i$, where K_i/F are finite separable and S_i is a K_i-torus which splits over a tame extension of K_i, then T is R-smooth.

Proof. Let $Z := \ker f$ and let Z° be its identity component which is an R-smooth torus. We set $T_2' := T/Z^\circ$ which is equipped with an isogeny $T_2' \to T_2$. It follows that T_2' splits over a tame extension of F and hence is R-smooth. Then since T is an extension T_2' by Z, it is R-smooth by Proposition 2.4.6 (2).

The final statement then follows by the above and Proposition 2.4.6 (1). □

2.4.8. The previous results have the following consequences for Bruhat–Tits group schemes. Let G be a reductive group over F and $\widetilde{\mathcal{G}}$ a Bruhat–Tits stabilizer scheme corresponding to $x \in \mathcal{B}(G, F)$. Let $\beta : G \hookrightarrow G'$ be a closed immersion of reductive groups over F, which induces an isomorphism on derived groups. As in [KP18, §1.1.3], x determines a point $x' \in \mathcal{B}(G', F)$ and we write $\widetilde{\mathcal{G}}'$ for the corresponding r scheme of G'; then β extends to a group scheme morphism $\beta : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$.

Proposition 2.4.9. *Assume that there exists a maximal $\breve{F}$-split torus in G whose centralizer is an R-smooth torus. Then $\beta : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ is a closed immersion.*

Proof. As all maximal $\breve{F}$-split tori are $\breve{F}$-conjugate, the centralizer of any maximal $\breve{F}$-split torus is R-smooth if there exists one such centralizer which is R-smooth. Therefore we may assume that all such centralizers are R-smooth. Moreover, since the construction of Bruhat–Tits stabilizer schemes is compatible with unramified base extensions, it is enough to prove the result in the case $F = \breve{F}$.

Let S be a maximal $\breve{F}$-split torus in G such that x lies in $\mathcal{A}(G, S, \breve{F})$. Let T be the centralizer of S which by assumption is an R-smooth torus. Let S' be a maximal $\breve{F}$-split torus of G' such that $S' \cap G = S$ and T' the centralizer of S'.

By the construction of Bruhat–Tits stabilizer schemes in [BT84, §4.6], the Zariski closure of T (resp. T') inside $\widetilde{\mathcal{G}}$ (resp. $\widetilde{\mathcal{G}}'$) can be identified with the finite type Néron model $\mathcal{T}_{\mathrm{ft}}$ (resp. $\mathcal{T}'_{\mathrm{ft}}$). By Lemma 2.4.4, the natural map $T \to T'$ extends to a closed immersion $\mathcal{T}_{\mathrm{ft}} \to \mathcal{T}'_{\mathrm{ft}}$ of finite type Néron models.

For any relative root α, the map $G \to G'$ induces an isomorphism between the root subgroups U_α and U'_α. If we let $\mathcal{U}_\alpha$ and $\mathcal{U}'_\alpha$ denote the corresponding schematic closures, then by the construction of $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}'$ in [BT84, §4.6], the map $G \to G'$ also induces an isomorphism $\mathcal{U}_\alpha \to \mathcal{U}'_\alpha$. Thus by [BT84, Theorem 2.2.3] the schematic closure $\widehat{\mathcal{G}}$ of G in $\widetilde{\mathcal{G}}'$ contains the smooth big open cell

$$\prod_\alpha \mathcal{U}_{-\alpha} \times \mathcal{T}_{\mathrm{ft}} \times \prod_\alpha \mathcal{U}_\alpha;$$

hence by [BT84, Corollary 2.2.5], $\widehat{\mathcal{G}}$ is smooth. Since $\widehat{\mathcal{G}}(\mathcal{O}_{\breve{F}}) = G(\breve{F}) \cap \widetilde{\mathcal{G}}'(\mathcal{O}_{\breve{F}})$, it follows that $\widehat{\mathcal{G}} \cong \widetilde{\mathcal{G}}$, and hence we obtain a closed immersion $\widetilde{\mathcal{G}} \hookrightarrow \widetilde{\mathcal{G}}'$ as desired. □

2.4.10. Now let K/F be a finite separable extension. There is a natural embedding of buildings $\mathcal{B}(G,F) \to \mathcal{B}(G,K)$ and the image of x in $\mathcal{B}(G,K)$ determines a Bruhat–Tits stabilizer scheme $\widetilde{\mathcal{G}}_0$ over $\mathcal{O}_K$. Then by [Pra01, p. 172], there is an identification of buildings $\mathcal{B}(G,K) \cong \mathcal{B}(\mathrm{Res}_{K/F}G_K, F)$ and the stabilizer scheme for $\mathrm{Res}_{K/F}G_K$ corresponding to x can be identified with $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0$ (see eg. [HR20, §4.2]). By [BT84, §1.7.6], we obtain a natural morphism $i : \widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0$ of $\mathcal{O}_F$-group schemes. A similar argument to Proposition 2.4.9 gives the following proposition which generalizes [KP18, Prop. 1.3.9] (cf. [FHLR, Cor. 5.26]).

Proposition 2.4.11. *Assume $p > 2$ and that the centralizer of a maximal $\breve{F}$-split torus in G is R-smooth. Then $i : \widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0$ is a closed immersion.*

Proof. We may assume $F = \breve{F}$. It suffices to prove the result for K a field over which G splits. Indeed, if K'/K is an extension over which G splits and $\widetilde{\mathcal{G}}_0'$ is the Bruhat–Tits stabilizer scheme over $\mathcal{O}_{K'}$ corresponding to x, then $\widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0$ is a closed immersion if the composition $\widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0 \to \mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_F}\widetilde{\mathcal{G}}_0'$ is a closed immersion.

The same argument as in Proposition 2.4.9 shows that we can reduce to proving the following two statements:

(1) $i|_{\mathcal{T}_{\mathrm{ft}}}$ is a closed immersion, where T is the centralizer of a maximal $\breve{F}$-split torus S whose apartment contains x.
(2) $i|_{\mathcal{U}_\alpha}$ is a closed immersion, where α is a relative root for G and $\mathcal{U}_\alpha$ is the schematic closure of the root subgroup U_α inside $\widetilde{\mathcal{G}}$.

The first follows from Lemma 2.4.4 (2) applied to the map $T \to T'$, where T' is the centralizer in $\mathrm{Res}_{K/F}G_K$ of a maximal $\breve{F}$-split torus containing S.

For the second statement, let α be a relative root and let G_α denote the simply-connected cover of the subgroup of G generated by the root subgroups corresponding to relative roots which are proportional to α. Then G_α is isomorphic to either

(1) $\mathrm{Res}_{L/F}\mathrm{SL}_2$ for L/F a finite separable extension.
(2) $\mathrm{Res}_{L/F}\mathrm{SU}_3$, where SU_3 is the special unitary group over L associated to a hermitian space over a separable quadratic extension L'/L.

Let G_α' denote the subgroup of G generated by the image of G_α and T; then G_α' contains the maximal $\breve{F}$-split torus S of G. By the main result of [Lan00], the inclusion $G_\alpha' \to G$ induces a $G_\alpha'(\breve{F})$-equivariant embedding of buildings, which restricts to an identification of apartments $\mathcal{A}(G_\alpha', S, \breve{F}) \cong \mathcal{A}(G, S, \breve{F})$. The point $x \in \mathcal{A}(G, S, \breve{F})$ corresponding to $\widetilde{\mathcal{G}}$ determines a Bruhat–Tits stabilizer scheme of G_α', and since G_α and G_α' have the same adjoint group, we obtain a stabilizer scheme $\widetilde{\mathcal{G}}_\alpha$ of G_α via the choice of a lift $x_\alpha \in \mathcal{B}(G_\alpha, \breve{F})$ of the image of x in $\mathcal{B}(G_\alpha'^{\mathrm{ad}}, \breve{F})$.

We have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{G}}_\alpha & \longrightarrow & \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_{\alpha,0} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{G}} & \longrightarrow & \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0, \end{array}$$

where $\widetilde{\mathcal{G}}_{\alpha,0}$ denotes the parahoric for $G_{\alpha,K}$ corresponding to $x_\alpha \in \mathcal{B}(G_\alpha, K)$. The natural morphism $\widetilde{\mathcal{G}}_\alpha \to \widetilde{\mathcal{G}}$ induces an isomorphism on the integral root subgroups $\mathcal{U}_\alpha$ and similarly for the morphism $\widetilde{\mathcal{G}}_{\alpha,0} \to \widetilde{\mathcal{G}}_0$. It therefore suffices to prove the result for $G = G_\alpha$. Note that since we have assumed $p > 2$, G_α is the Weil-restriction of a tamely ramified group. Thus it suffices to prove the proposition in this case, which we now do.

We first consider the case that G itself splits over a tamely ramified extension K^t/F. We may assume K contains K^t. Let $\widetilde{\mathcal{G}}_0^t$ denote the Bruhat–Tits stabilizer scheme of G_{K^t} corresponding to $\widetilde{\mathcal{G}}$. Then i factors as

$$\widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_{K^t}/\mathcal{O}_F}\widetilde{\mathcal{G}}_0^t \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0.$$

The first morphism is a closed immersion by [KP18, Proposition 1.3.9]. The second morphism is obtained from $\widetilde{\mathcal{G}}_0^t \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_{K^t}}\widetilde{\mathcal{G}}_0$ by applying Weil-restriction. Since G_{K^t} is split, we can reduce to the case $G_{K^t} = \mathrm{SL}_2$, as above, and this follows from Lemma 2.4.12 below.

Now assume $G = \mathrm{Res}_{L/F}H$ where H is a group which splits over a tame extension of L and that K contains L. Then $G \to \mathrm{Res}_{K/F}G_K$ arises from Weil-restriction of a morphism $H \to \mathrm{Res}_{K/L}G_K$, which is given by a product of the diagonal morphisms $H \to \mathrm{Res}_{K/L}H_K$. Hence the result in this case follows from the tame case proved in the previous paragraph. The proposition follows. □

Lemma 2.4.12. *Let $G = \mathrm{SL}_2$. Then the morphism $i : \widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0$ is a closed immersion.*

Proof. We may assume $\widetilde{\mathcal{G}}$ corresponds to a point in the apartment for the diagonal torus T; let U be a root subgroup for T. Since T is split, hence R-smooth, it suffices as above to show $U \to \mathrm{Res}_{K/F}U_K$ extends to a closed immersion $\mathcal{U} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{U}_0$, where $\mathcal{U}$ (resp. $\mathcal{U}_0$) is the Zariski closure of U in $\widetilde{\mathcal{G}}$ (resp. U_K in $\widetilde{\mathcal{G}}_0$). The morphism $U \to \mathrm{Res}_{K/F}U_K$ can be identified with the diagonal morphism $\mathbb{G}_a \to \mathrm{Res}_{K/F}\mathbb{G}_a$.

Let ϖ_F (resp. ϖ_K) be a uniformizer for F (resp. K), and let e denote the ramification index of K/F. By the construction of the stabilizer schemes in [BT84], $\mathcal{U}_0$ is the $\mathcal{O}_K$-group scheme cordeoresponding to the $\mathcal{O}_K$-submodule $\varpi_K^{ne-k}\mathcal{O}_K$ of $K = \mathbb{G}_a(K)$, for some $n \in \mathbb{Z}$ and $k \in \{0, \ldots, e-1\}$, which depend on the choice of $x \in \mathcal{B}(G, F)$. Then $\mathcal{U}$ corresponds to the $\mathcal{O}_F$-submodule $\varpi_F^n\mathcal{O}_F$ of $F = \mathbb{G}_a(F)$. We can extend ϖ_F^n to an $\mathcal{O}_F$-basis for $\varpi_K^{ne-k}\mathcal{O}_K$ considered as an $\mathcal{O}_F$-module, and this induces an identification $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{U}_0 \cong \mathbb{A}^m$ where $m = [K : F]$. The map $\mathcal{U} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{U}_0$ is then identified with the closed immersion $\mathbb{A}^1 \to \mathbb{A}^m$ taking a to $(a, 0, \ldots, 0)$. □

2.4.13. Now let $\beta : G \to G'$ be a central extension between reductive groups with kernel Z and $\mathcal{G}$ the parahoric group scheme associated to some $x \in \mathcal{B}(G, F)$. We let $\mathcal{G}'$ denote the parahoric of G' corresponding to $\mathcal{G}$. As above, β extends to a group scheme homomorphism $\mathcal{G} \to \mathcal{G}'$.

Proposition 2.4.14. *Assume Z is an R-smooth torus. Then the Zariski closure $\widetilde{\mathcal{Z}}$ of Z inside $\mathcal{G}$ is smooth and there is an (fppf) exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{Z}} \longrightarrow \mathcal{G} \xrightarrow{\beta} \mathcal{G}' \longrightarrow 0 \tag{2.4.14.1}$$

of group schemes over $\mathcal{O}_F$.

Proof. As before, it suffices to prove the proposition when $F = \breve{F}$. Let S be a maximal $\breve{F}$-split torus of G such that x lies in $\mathcal{A}(G, S, \breve{F})$. Let T be the centralizer of S and we let T' be the corresponding maximal torus of G'.

Assume there exists an fppf exact sequence

$$1 \longrightarrow \widetilde{\mathcal{Z}} \longrightarrow \mathcal{T}_0 \longrightarrow \mathcal{T}_0' \longrightarrow 1 \tag{2.4.14.2}$$

where $\mathcal{T}_0$ and $\mathcal{T}_0'$ are the connected Néron models of T and T' respectively. Then we may argue as in [KP18, Proposition 1.1.4] to obtain the desired exact sequence (2.4.14.1).

It remains to exhibit the exact sequence (2.4.14.2); we follow the argument of [PR08, Lemma 6.7]. By Lemma 2.4.4 we obtain a closed immersion between lft Néron models $\mathcal{Z} \to \mathcal{T}$. We let $\widetilde{\mathcal{Z}}'$ denote the subgroup scheme of $\mathcal{Z}$ with generic fiber Z, and special fiber corresponding to the connected components of the special fiber of $\mathcal{Z}$ parameterized by $\ker(X_*(Z)_I \to X_*(T)_I)$. Then $\widetilde{\mathcal{Z}}'$ is smooth and we have a closed immersion $\widetilde{\mathcal{Z}}' \to \mathcal{T}_0$. It follows that $\widetilde{\mathcal{Z}}'$ coincides with $\widetilde{\mathcal{Z}}$ and we obtain a closed immersion $\widetilde{\mathcal{Z}} \to \mathcal{T}_0$. As in [PR08, Lemma 6.7] we have an exact sequence:

$$1 \longrightarrow \widetilde{\mathcal{Z}}(\mathcal{O}_{\breve{F}}) \longrightarrow \mathcal{T}_0(\mathcal{O}_{\breve{F}}) \longrightarrow \mathcal{T}_0'(\mathcal{O}_{\breve{F}}) \longrightarrow 1$$

The quotient $\mathcal{T}_0/\widetilde{\mathcal{Z}}$ is a smooth affine commutative group scheme with the same generic fiber as $\mathcal{T}_0'$ and with the same $\mathcal{O}_{\breve{F}}$-points; hence by [BT84, Proposition 1.7.6] we have $\mathcal{T}_0' \cong \mathcal{T}_0/\widetilde{\mathcal{Z}}$. The result follows. □

3. Local models of Shimura varieties

In this section we assume F is a finite extension of $\mathbb{Q}_p$ with residue field k_F. We recall the construction of local models for Weil-restricted groups following [Lev16] and prove properties of these models which will be needed in §5. We also show that the models we consider satisfy the Scholze–Weinstein conjecture [SW20, Conjecture 21.4.1].

3.1. **Local models for Weil-restricted groups.**

3.1.1. Let K_0/F be a finite unramified extension. Let $P(u) \in \mathcal{O}_{K_0}[u]$ be a monic polynomial and $\underline{\mathcal{G}}$ a smooth affine group scheme over $\mathcal{O}_{K_0}[u]$. We consider the functor $\mathrm{Fl}^{P(u)}_{\underline{\mathcal{G}},0}$ on $\mathcal{O}_{K_0}$-algebras R given by

$$\mathrm{Fl}^{P(u)}_{\underline{\mathcal{G}},0}(R) = \{\text{iso. classes of pairs } (\mathcal{E}, \beta)\},$$

where $\mathcal{E}$ is a $\underline{\mathcal{G}}$-torsor over $R[u]$ and $\beta : \mathcal{E}|_{R[u][1/P(u)]} \xrightarrow{\sim} \mathcal{E}^0$ is an isomorphism of $\underline{\mathcal{G}}$-torsors, where $\mathcal{E}^0$ denotes the trivial $\underline{\mathcal{G}}$-torsor. We then define the mixed characteristic affine Grassmannian

$$\mathrm{Fl}^{P(u)}_{\underline{\mathcal{G}}} := \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F} \mathrm{Fl}^{P(u)}_{\underline{\mathcal{G}},0}.$$

By embedding $\underline{\mathcal{G}}$ into a general linear group, one deduces as in [Lev16, Proposition 4.1.4], that $\mathrm{Fl}^{P(u)}_{\underline{\mathcal{G}}}$ is representable by an ind-scheme over $\mathcal{O}_F$.

3.1.2. Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple over F as in [HPR20, §2.1]. Thus

- G is a reductive group scheme over F.
- $\{\mu\}$ is a geometric conjugacy class of minuscule cocharacters of G.
- $\mathcal{G} = \mathcal{G}_x$ for some $x \in \mathcal{B}(G, F)$.

A morphism of local model triples $(G, \{\mu\}, \mathcal{G}) \to (G', \{\mu'\}, \mathcal{G}')$ is a morphism $\mathcal{G} \to \mathcal{G}'$ taking $\{\mu\}$ to $\{\mu'\}$.

We will sometimes make the following assumption.

$(*)$ G is isomorphic to $\prod_{i=1}^{r} \mathrm{Res}_{K_i/F} H_i$ where K_i/F is a finite extension and H_i is a reductive group over K_i which splits over a tamely ramified extension of K_i.

When $r = 1$, we simply write $G = \mathrm{Res}_{K/F} H$.

3.1.3. Let $(G, \{\mu\}, \mathcal{G})$ be a triple with $G \cong \mathrm{Res}_{K/F} H$ as above. As in §2.4.10, there is an identification of buildings $\mathcal{B}(G, F) \cong \mathcal{B}(H, K)$ and the image of x in $\mathcal{B}(H, K)$ determines a parahoric group scheme $\mathcal{H}$ of H over $\mathcal{O}_K$. By [HR20, Proposition 4.7], we have $\mathcal{G} \cong \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathcal{H}$.

Let K_0 denote the maximal unramified extension of F contained in K and write $\mathcal{O}_{K_0}$ (resp. k_0) for its ring of integers (resp. residue field). We let $\mathcal{O}_{K_0}[u^{\pm}]$ denote the ring $\mathcal{O}_{K_0}[u, u^{-1}]$. We fix a uniformizer ϖ of K and we write $Q(u) \in \mathcal{O}_{K_0}[u]$ for the Eisenstein polynomial which is the minimal polynomial for ϖ over K_0. Then [Lev16, §3.3] constructs a smooth affine group scheme $\underline{\mathcal{H}}$ over $\mathcal{O}_{K_0}[u]$ which specializes to $\mathcal{H}$ under the map $\mathcal{O}_{K_0}[u] \to \mathcal{O}_K$, $u \mapsto \varpi$ and such that

$$\underline{H} := \underline{\mathcal{H}}|_{\mathcal{O}_{K_0}[u^{\pm}]}$$

is a reductive group scheme. Applying the construction of §3.1.1 we obtain the ind-scheme $\mathrm{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ over $\mathcal{O}_F$ which is ind-projective by [Lev16, Theorem 4.2.11].

3.1.4. For a K_0-algebra R, the completion $\widehat{R[u]}$ of $R[u]$ along $Q(u)$, contains the completion of $K_0[u]$ along $Q(u)$. The latter ring may be identified with $K[[t]]$, by a map sending t to $Q(u)$ and inducing the identity on residue fields. Then $\widehat{R[u]}$ may be identified with $(K \otimes_{K_0} R)[[t]]$ by sending t to $Q(u)$. This induces an isomorphism from the generic fiber of $\mathrm{Fl}_{\underline{\mathcal{H}},0}^{Q(u)}$ with the affine Grassmannian $\mathrm{Gr}_{\mathrm{Res}_{K/K_0} H}$ (cf. [HR20, Corollary 3.5]), and hence an isomorphism between the generic fiber of $\mathrm{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ with $\mathrm{Gr}_{\mathrm{Res}_{K/F} H} \cong \mathrm{Gr}_G$ (recall that this is the fpqc sheaf associated to the functor on F-algebras R given by $R \mapsto G(R((t)))/G(R[[t]])$).

The special fiber of $\mathrm{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ can be identified with the partial affine flag variety $\mathrm{Res}_{k_0/k_F} \mathcal{FL}_{\underline{\mathcal{H}}_{k_0[[t]]}}$; here $\mathcal{FL}_{\underline{\mathcal{H}}_{k_0[[t]]}}$ is the fpqc sheaf associated to the functor

$$R \mapsto \underline{\mathcal{H}}_{k_0[[t]]}(R((t)))/\underline{\mathcal{H}}_{k_0[[t]]}(R[[t]])$$

on k_0-algebras. A representative μ of $\{\mu\}$ over $\overline{F}$ determines an element of $G(\overline{F}((t)))$ and hence a point $e_\mu := \mu(t) \in \mathrm{Gr}_G(\overline{F})$. The (affine) Schubert variety S_μ is then defined to be the closure of the $G(\overline{F}[[t]])$-orbit of e_μ in Gr_G. The conjugacy class $\{\mu\}$ has a minimal field of definition E known as the (local) reflex field, and the Schubert variety $S_\mu \subset \mathrm{Gr}_G$ is defined over E. The local model $\mathrm{M}_{\mathcal{G},\{\mu\}}^{\mathrm{loc}}$ is defined to be the Zariski closure of S_μ in $\mathrm{Fl}_{\underline{\mathcal{H}}}^{Q(u)} \otimes_{\mathcal{O}_F} \mathcal{O}_E$.

3.1.5. In general, if $G \cong \prod_{i=1}^r \mathrm{Res}_{K_i/F} H_i$ as in (*), we define $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ to be the product $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} := \prod_{i=1}^r \mathrm{M}^{\mathrm{loc}}_{\mathcal{G}_i,\{\mu_i\}} \otimes_{\mathcal{O}_{E_i}} \mathcal{O}_E$. Here the parahoric $\mathcal{G}_i$ of $\mathrm{Res}_{K_i/F} H_i$ is determined by $\mathcal{G} \cong \prod_{i=1}^r \mathcal{G}_i$, $\{\mu_i\}$ is the $\mathrm{Res}_{K_i/F} H_i$ factor of the G-conjugacy class $\{\mu\}$, and E_i (resp. E) is the field of definition of $\{\mu_i\}$ (resp. $\{\mu\}$). The following theorem follows immediately from [Lev16, Theorem 4.2.7].

Theorem 3.1.6. *Suppose G satisfies $(*)$ and that p does not divide the order of the algebraic fundamental group $\pi_1(G^{\mathrm{der}})$ of the derived group G^{der} of G. Then the scheme $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is normal with reduced special fiber. Moreover each geometric irreducible component of $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \otimes_{\mathcal{O}_E} k$ is normal and Cohen–Macaulay.* □

Remark 3.1.7.

(1) Note that the input for the constructions in this subsection is a parahoric group scheme $\mathcal{H}$ over $\mathcal{O}_K$ and a finite extension K/F. When $K = F$, the group scheme $\underline{\mathcal{H}}$ and the mixed characteristic affine Grassmannian $\mathrm{Fl}^{u-\varpi}_{\underline{\mathcal{H}}}$ agrees with those constructed by Pappas–Zhu [PZ13]. In this case, it follows from [HPR20, Theorem 2.7] that the local model $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ only depends on the local model triple $(G, \{\mu\}, \mathcal{G})$ and not on the choice of uniformizer ϖ.
(2) More generally, for an arbitrary K and under some additional assumptions, we show that the $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ satisfy Conjecture 21.4.1 of [SW20], and hence are independent of the choice of K, and uniformizer ϖ (cf. Theorem 3.3.11).

3.1.8. We may identify the geometric special fiber of $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ with a certain union of Schubert varieties corresponding to the μ-admissible set $\mathrm{Adm}(\{\mu\})_J$ defined in 2.2.1; we explain this in the remainder of §3.1. We first explain the relationship between the Iwahori Weyl group of G and a certain reductive group over $k_F[[u]]$.

Let S denote a maximal $\breve{K}$-split torus of H defined over K such that x lies in a σ_K-invariant facet in the apartment $\mathcal{A}(H, S, \breve{K})$ corresponding to S (here σ_K denotes the Frobenius of $\mathrm{Aut}(\breve{K}/K)$). Then the construction in [Lev16, Proposition 3.1.2] provides us with a maximal $\mathcal{O}_{\breve{K}_0}[u^{\pm}]$-split torus $\underline{S}$ of $\underline{H}$ defined over $\mathcal{O}_{K_0}[u^{\pm}]$ which extends S. The choice of $\underline{S}$ gives us an identification of apartments

$$\mathcal{A}(\underline{H}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u))) \cong \mathcal{A}(H, S, \breve{K}) \tag{3.1.8.1}$$

for $\kappa \in \{\breve{K}_0, k\}$. Moreover there is an identification of Iwahori Weyl groups

$$W_{\underline{H}_{\kappa((u))}} \cong W_{H_{\breve{K}}} \tag{3.1.8.2}$$

for $\underline{H}_{\kappa((u))}$ and $H_{\breve{K}}$ such that the identification (3.1.8.1) is equivariant for the actions of these groups on the respective apartments. We let

$$x_{\kappa((u))} \in \mathcal{A}(\underline{H}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u)))$$

be the point corresponding to x under the identification (3.1.8.1). Then the group scheme $\underline{\mathcal{H}}/\mathcal{O}_{K_0}[u]$ has the property that its specialization to $\kappa[[u]]$ is isomorphic to the parahoric group scheme corresponding to $x_{\kappa((u))}$.

3.1.9. Let $\underline{\mathcal{G}}_{k_F[[u]]}$ denote the group scheme $\underline{\mathcal{G}}_{k_F[[u]]} := \mathrm{Res}_{k_0[[u]]/k_F[[u]]} \underline{\mathcal{H}}_{k_0[[u]]}$ and we write $\underline{G}_{k_F((u))}$ for its generic fiber. We let $\underline{\mathcal{G}}_{k[[u]]}$ (resp. $\underline{G}_{k((u))}$) denote the base change of $\underline{\mathcal{G}}_{k_F[[u]]}$ (resp. $\underline{G}_{k_F((u))}$) to $k[[u]]$ (resp. $k((u))$). Then by construction, the special fiber of $\mathrm{Fl}^{Q(u)}_{\underline{\mathcal{H}}}$ is identified with the usual partial affine flag variety associated to $\underline{\mathcal{G}}_{k_F[[u]]}$; here we use [HR20, Corollary 3.6 and Lemma 3.7] for the identification $\mathrm{Res}_{k_0/k_F}\mathcal{FL}_{\underline{\mathcal{H}}_{k_0[[u]]}} \cong \mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}}$. The isomorphism (3.1.8.2) induces an isomorphism of Iwahori Weyl groups

$$W_G \cong W_{\underline{G}_{k_F((u))}}. \tag{3.1.9.1}$$

Indeed we have identifications

$$W_G \cong \prod_{\psi:K_0\to\breve{F}} W_H, \quad W_{\underline{G}_{k_F((u))}} \cong \prod_{\psi:k_0\to k} W_{\underline{H}_{k_F((u))}},$$

where ψ runs over F (resp. k_F)-embeddings. Identifying $k_0 \to k$ with the unique lift $K_0 \to \breve{F}$ and using (3.1.8.2), we obtain the identification (3.1.9.1).

Similarly, we obtain an identification of apartments

$$\mathcal{A}(\underline{G}_{k_F((u))}, \underline{S}'_{k((u))}, k((u))) \cong \mathcal{A}(G, S', \breve{F}). \tag{3.1.9.2}$$

Here S' is the maximal $\breve{F}$-split torus of G determined by the maximal $\breve{K}$-split torus of H as in [HR20, §4.2], and $\underline{S}'_{k((u))}$ is the maximal $k((u))$-split torus of $\underline{G}_{k_F((u))}$ obtained from the maximal $\mathcal{O}_{\breve{K}_0}[u^{\pm}]$-split torus $\underline{S}$ of $\underline{H}$. Moreover the identification (3.1.9.2) is compatible with the action of Iwahori Weyl groups under the identification (3.1.9.1).

3.1.10. We fix a σ-invariant alcove $\mathfrak{a} \subset \mathcal{A}(G, S', \breve{F})$ whose closure contains x. This determines a set of simple reflections $\mathbb{S}$ for W_G and the parahoric $\mathcal{G}$ is a standard parahoric for this choice of alcove; hence it corresponds to a σ-stable subset $J \subset \mathbb{S}$. We let $\underline{\mathfrak{a}}$ denote the alcove in $\mathcal{A}(\underline{G}_{k_F((u))}, \underline{S}'_{k((u))}, k((u)))$ corresponding to $\mathfrak{a}$ and $\underline{\mathbb{S}}$ the set of simple reflections in the walls of $\underline{\mathfrak{a}}$. There is an identification $\mathbb{S} \cong \underline{\mathbb{S}}$ and we let $\underline{J} \subset \underline{\mathbb{S}}$ be the subset corresponding to $J \subset \mathbb{S}$. Then $\underline{\mathcal{G}}_{k_F[[u]]}$ is the standard parahoric group scheme of $\underline{G}_{k_F((u))}$ associated to $\underline{J}$. Writing W_J (resp. $W_{\underline{J}}$) for the finite group generated by the reflections in J (resp. $\underline{J}$), we obtain an identification $W_J \cong W_{\underline{J}}$, and an identification

$$W_J\backslash W_G/W_J \cong W_{\underline{J}}\backslash W_{\underline{G}_{k_F((u))}}/W_{\underline{J}}. \tag{3.1.10.1}$$

In particular we may consider $\mathrm{Adm}(\{\mu\})_J$ as a subset of $W_{\underline{J}}\backslash W_{\underline{G}_{k_F((u))}}/W_{\underline{J}}$.

For an element $w \in W_G$, we write $\underline{w} \in W_{\underline{G}_{k_F((u))}}$ for the corresponding element and $\underline{\dot{w}} \in \underline{G}_{k_F((u))}(k((u)))$ a lift of w. We let $\underline{S}_w$ denote the closure of the $\underline{\mathcal{G}}_{k_F[[u]]}(k[[u]])$-orbit of $\underline{\dot{w}}$ considered as a point of the partial affine flag variety $\mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}} \otimes_{k_F} k$ for $\underline{\mathcal{G}}_{k_F[[u]]}$.

3.1.11. If $G \cong \prod_{i=1}^r \mathrm{Res}_{K_i/F}H_i$, we may define $\underline{\mathcal{G}}_{k_F[[u]]} := \prod_{i=1}^r \underline{\mathcal{G}}_{i,k_F[[u]]}$, where the $\underline{\mathcal{G}}_{i,k_F[[u]]}$ are the $k_F[[u]]$-group schemes constructed in the previous paragraphs using the groups $\mathrm{Res}_{K_i/F}H_i$. We let $\underline{G}_{i,k_F((u))}$ denote the generic fiber of $\underline{\mathcal{G}}_{i,k_F[[u]]}$ and we define $\underline{G}_{k_F((u))} := \prod_{i=1}^r \underline{G}_{i,k_F((u))}$. Since the construction of Iwahori Weyl groups and apartments are compatible with products, the above discussion extends to this case. In particular, we have an identification of double cosets for the Iwahori Weyl group (3.1.10.1), and for $w \in W_J\backslash W_G/W_J$ we have the associated Schubert variety $\underline{S}_w$ in $\mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}} \otimes_{k_F} k$. Applying [Lev16, Proposition 4.3.2] to each of the factors $\mathrm{Res}_{K_i/F}H_i$, we obtain the following theorem.

Theorem 3.1.12. *Let $G \cong \prod_{i=1}^r \mathrm{Res}_{K_i/F} H_i$ with K_i and H_i as in assumption $(*)$ of §3.1.2, and assume that $p \nmid |\pi_1(G^{\mathrm{der}})|$. We have an identification*

$$\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \otimes_{\mathcal{O}_E} k \cong \bigcup_{w \in \mathrm{Adm}(\{\mu\})_J} \underline{S}_w$$

as closed subschemes of $\mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}} \otimes_{k_F} k$. □

3.2. **Embedding local models.**

3.2.1. We now show that a suitable embedding of local model triples induces a closed immersion of local models (Proposition 3.2.6). This generalizes [KP18, Proposition 2.3.6] and is proved using a similar method. This result and its consequences in §3.3 are the main technical results needed to establish the local model diagram in §5.1.

We begin by recalling the construction of certain lattice chains of $\mathcal{O}_{K_0}[u]$-modules from [PZ13, §5.2.1]. Let $\underline{W} = \mathcal{O}_{K_0}[u]^n$ and $W = \underline{W} \otimes_{\mathcal{O}_{K_0}[u], u \mapsto 0} \mathcal{O}_{K_0} \cong \mathcal{O}_{K_0}^n$. Write $W = \oplus_{i=0}^r V_i$ for some r and direct summands V_i of W, and let $U_i = \oplus_{j \geq i} V_j$ which forms a flag of subspaces of W; we write $P \subset \mathrm{GL}(W)$ for the corresponding parabolic. For $i = 0, \dots, r-1$ we let $\underline{W}_i \subset \underline{W}$ denote the inverse image of U_i under $\underline{W} \to W$; the sequence W_i satisfies

$$u\underline{W} \subset \underline{W}_{r-1} \subset \dots \subset \underline{W}_0 = \underline{W}.$$

We extend the sequence to $\mathbb{Z}$ by letting $\underline{W}_{i+kr} = u^k \underline{W}_i$ and we write $\underline{W}_\bullet$ for the resulting chain indexed by $\mathbb{Z}$. As in [PZ13, §5.2.1], the dilatation $\mathrm{GL}(\underline{W}_\bullet)$ of $\mathrm{GL}(\underline{W})$ along P can be identified with the closed subscheme of $\prod_{i=0}^{r-1} \mathrm{GL}(\underline{W}_i)$ which respect the maps $\underline{W}_{i+1} \to \underline{W}_i$. Let $\mathcal{GL}$ be the parahoric group scheme over $\mathcal{O}_K$ of $\mathrm{GL}_n(K)$ corresponding to the stabilizer of the lattice chain $\underline{W}_i \otimes_{\mathcal{O}_{K_0}[u], u \mapsto \varpi} \mathcal{O}_K$ in K^n. Then $\mathrm{GL}(\underline{W}_\bullet)$ is isomorphic to the $\mathcal{O}_{K_0}[u]$-group scheme $\underline{\mathcal{GL}}$ associated to $\mathcal{GL}$ and the extension K/F in §3.1.3. Since every parahoric of $\mathrm{GL}_n(K)$ arises in this way, this gives an explicit description of the associated $\mathcal{O}_{K_0}[u]$-group scheme $\underline{\mathcal{GL}}$ attached to any parahoric of $\mathrm{GL}_n(K)$.

3.2.2. Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple as in §3.1.2 with $G \cong \mathrm{Res}_{K/F} H$. Let $\rho : G \to \mathrm{GL}(W)$ be a faithful minuscule representation, where W is a finite dimensional vector space over F, such that $\rho \circ \mu$ is conjugate to a standard (i.e. having weights $0, -1$) minuscule coweight and $\rho(G)$ contains the scalars.

Base changing ρ to K, we obtain a map $H \to \mathrm{GL}(W_K)$ given by composing

$$\rho_K : G_K \to \mathrm{GL}(W_K)$$

with the diagonal map $H \to G_K$. Let W' denote the underlying F-vector space of W_K. We consider the composition

$$\rho' : G = \mathrm{Res}_{K/F} H \xrightarrow{\rho_1} \mathrm{Res}_{K/F} \mathrm{GL}(W_K) \xrightarrow{\rho_2} \mathrm{GL}(W') \tag{3.2.2.1}$$

where ρ_1 is obtained by applying restriction of scalars to the map $H \to \mathrm{GL}(W_K)$, and ρ_2 is induced by the restriction of structure functor from K-vector spaces to F-vector spaces. Then ρ' is also a faithful minuscule representation, and $\rho' \circ \mu$ is conjugate to a standard minuscule coweight and $\rho'(G)$ contains the scalars.

Thus upon replacing ρ by ρ', we may assume that ρ factors as a composition

$$G = \mathrm{Res}_{K/F} H \xrightarrow{\rho_1} \mathrm{Res}_{K/F} \mathrm{GL}(V) \xrightarrow{\rho_2} \mathrm{GL}(W) \tag{3.2.2.2}$$

where V is a K-vector space whose underlying F-vector space is identified with W, ρ_1 is obtained via Weil-restriction from a minuscule representation $H \to \mathrm{GL}(V)$, and ρ_2 arises from the restriction of structure functor from K-vector spaces to F-vector spaces.

We will show that under these assumptions, ρ induces a closed immersion of local models

$$\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho\circ\mu\}} \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

where $\mathcal{GL}_W$ is a certain parahoric group scheme of $\mathrm{GL}(W)$.

3.2.3. Since H splits over a tame extension of K and $H \to \mathrm{GL}(V)$ is a minuscule representation, it follows from [KP18, §1.2] that there exists an $H(K)$-equivariant toral embedding of buildings

$$\mathcal{B}(H,K) \to \mathcal{B}(\mathrm{GL}(V),K). \tag{3.2.3.1}$$

There are canonical identifications of $\mathcal{B}(G,F)$ (resp. $\mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V),F)$) with $\mathcal{B}(H,K)$ (resp. $\mathcal{B}(\mathrm{GL}(V),K)$); we thus obtain a $G(F)$-equivariant toral embedding of buildings

$$\mathcal{B}(G,F) \to \mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V),F). \tag{3.2.3.2}$$

Similarly, restriction of structure induces a $\mathrm{GL}(V)$-equivariant map of buildings

$$\mathcal{B}(\mathrm{GL}(V),K) \cong \mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V),F) \to \mathcal{B}(\mathrm{GL}(W),F).$$

Let y (resp. z) denote the image of x in $\mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V),F)$ (resp. $\mathcal{B}(\mathrm{GL}(W),F)$). We write $\mathcal{GL}_{K/F}$ (resp. $\mathcal{GL}_W$) for the parahoric group schemes over $\mathcal{O}_F$ for $\mathrm{Res}_{K/F}\mathrm{GL}(V)$ (resp. $\mathrm{GL}(W)$) corresponding to y (resp. z), and we write $\{\mu_{K/F}\}$ and $\{\mu_W\}$ for the respective conjugacy class of cocharacters of $\mathrm{Res}_{K/F}\mathrm{GL}(V)$ and $\mathrm{GL}(W)$ induced by $\{\mu\}$. If we write $\mathcal{GL}$ for the parahoric $\mathcal{O}_K$-group scheme of $\mathrm{GL}(V)$ associated to y, then we have $\mathcal{GL}_{K/F} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{GL}$. Similarly we have $\widetilde{\mathcal{G}} = \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{G}}$ (resp. $\widetilde{\mathcal{H}}$) is the Bruhat–Tits stabilizer scheme of G (resp. H) associated to x.

The natural map of group schemes $\widetilde{\mathcal{G}} \to \mathcal{GL}_{K/F}$ is a closed immersion since this map is obtained by Weil-restriction of a closed immersion $\widetilde{\mathcal{H}} \to \mathcal{GL}$ between $\mathcal{O}_K$-group schemes as in [KP18, Proposition 1.3.3]. We need the following lemma.

Lemma 3.2.4. *Let K be a non-archimedean local field (in possibly equal characteristic) and K'/K a finite (not necessarily separable) extension. Let V be a vector space over K' and let W denote V considered as a vector space over K. Let $\mathcal{GL}$ be a parahoric group scheme of $\mathrm{GL}(V)$ corresponding to the stabilizer of an $\mathcal{O}_{K'}$-lattice chain $\{\Lambda_i\}_{i=1,\dots,r}$ in V. We write $\{\Lambda_{W,i}\}_{i=1,\dots,r}$ for the associated $\mathcal{O}_K$-lattice chain of W and we let $\mathcal{GL}_W$ denote the parahoric group scheme of $\mathrm{GL}(W)$ stabilizing $\{\Lambda_{W,i}\}_{i=1,\dots,r}$. Then the natural closed immersion $\mathrm{Res}_{K'/K}\mathrm{GL}(V) \to \mathrm{GL}(W)$ extends to a closed immersion of $\mathcal{O}_K$-group schemes*

$$\mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_K}\mathcal{GL} \hookrightarrow \mathcal{GL}_W.$$

Proof. The group scheme $\mathcal{GL}$ is the schematic closure of $\mathrm{GL}(V) \to \prod_{i=1}^r \mathrm{GL}(V)$ (under the diagonal embedding) in $\prod_{i=1}^r \mathrm{GL}(\Lambda_i)$. Similarly $\mathcal{GL}_W$ is the schematic closure of $\mathrm{GL}(W) \to \prod_{i=1}^r \mathrm{GL}(W)$ in $\prod_{i=1}^r \mathrm{GL}(\Lambda_{W,i})$. Thus we have a commutative diagram of $\mathcal{O}_K$-schemes

$$\begin{array}{ccc} \mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_K}\mathcal{GL} & \longrightarrow & \mathcal{GL}_W \\ \downarrow & & \downarrow \\ \prod_{i=1}^r \mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_K}\mathrm{GL}(\Lambda_i) & \longrightarrow & \prod_{i=1}^r \mathrm{GL}(\Lambda_{W,i}) \end{array}$$

where the vertical maps are closed immersions. It therefore suffices to show the bottom map is a closed immersion, and hence we reduce to proving the lemma when $r = 1$, i.e. when $\mathcal{GL}$ is the stabilizer $\mathrm{GL}(\Lambda)$ of an $\mathcal{O}_K$-lattice $\Lambda \subset V$. This case can be proved by explicitly writing down the equations for the morphism. □

By Lemma 3.2.4, the map $\mathcal{GL}_{K/F} \to \mathcal{GL}_W$ is a closed immersion. Composing with $\widetilde{\mathcal{G}} \to \mathcal{GL}_{K/F}$ we obtain a closed immersion of $\mathcal{O}_F$-group schemes $\widetilde{\mathcal{G}} \to \mathcal{GL}_W$ extending ρ.

3.2.5. By assumption, $\rho \circ \mu = \mu_W$ is conjugate to a standard minuscule coweight

$$a \mapsto \mathrm{diag}(1^{(n-d)}, (a^{-1})^{(d)})$$

of $\mathrm{GL}(W)$, where $n = \dim_F W$. The generic fiber of $\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\mu_W\}}$ is the Grassmannian $\mathrm{Gr}(d, n)$ of d-dimensional subspaces of W. We let X_μ denote the generic fiber of $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$; it can be identified with the E-variety G/P_μ, where P_μ is the parabolic subgroup of G corresponding to μ (we use the convention that μ acts on the Lie algebra of P_μ by weights less than or equal to 0). Then the representation $\rho : G \to \mathrm{GL}(W)$ induces a closed immersion

$$X_\mu \to \mathrm{Gr}(d, n) \otimes_{\mathcal{O}_F} E. \tag{3.2.5.1}$$

Proposition 3.2.6. *The map (3.2.5.1) extends to a closed immersion of local models*

$$\rho^{\mathrm{loc}} : \mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \to \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\mu_W\}} \otimes_{\mathcal{O}_F} \mathcal{O}_E. \tag{3.2.6.1}$$

Proof. By assumption, ρ factors as $\rho_2 \circ \rho_1$; it suffices to show there are closed immersions

$$\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F},\{\mu_{K/F}\}} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\mu_W\}} \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

where the first map is induced by ρ_1 and the second map is induced by ρ_2. Here, $\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F},\{\mu_{K/F}\}}$ is the local model attached to the $\mathcal{O}_K$-group scheme $\mathcal{GL}$ and the extension K/F as in §3.1.3, and E' is the local reflex field for the $\mathrm{Res}_{K/F}\mathrm{GL}(V)$-conjugacy class of cocharacters $\{\mu_{K/F}\}$.

Step (1): $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F},\{\mu_{K/F}\}} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E$.

As in [KP18, Proposition 2.3.7], it follows from descent that it suffices to show that such a closed immersion exists upon base change to $\breve{E}$. Thus we need to show that there exists a closed immersion

$$\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}_{\mathcal{O}_{\breve{F}}},\{\mu\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F,\mathcal{O}_{\breve{F}}},\{\mu_{K/F}\}} \otimes_{\mathcal{O}_{\breve{E}'}} \mathcal{O}_{\breve{E}}$$

where $\mathcal{G}_{\mathcal{O}_{\breve{F}}}$ (resp. $\mathcal{GL}_{K/F,\mathcal{O}_{\breve{F}}}$) denotes the corresponding parahoric group schemes for $G_{\breve{F}}$ (resp. $\mathrm{Res}_{K/F}\mathrm{GL}(V) \otimes_F \breve{F}$) and these are the analogues of the local models defined over $\breve{F}$.

We have isomorphisms

$$G_{\breve{F}} \cong \prod_{\tau: K_0 \to \breve{F}} \mathrm{Res}_{\breve{K}/\breve{K}_0} H_{\breve{K}}, \quad \mathrm{Res}_{K/F}\mathrm{GL}(V) \otimes_F \breve{F} \cong \prod_{\tau: K_0 \to \breve{F}} \mathrm{Res}_{\breve{K}/\breve{K}_0} \mathrm{GL}(V_{\breve{K}})$$

and the embedding $\rho_{1,\breve{F}}$ is given by the product embedding; it suffices to consider each factor separately. Thus upon relabeling we may assume $G_{\breve{F}} \cong \mathrm{Res}_{\breve{K}/\breve{K}_0} H_{\breve{K}}$ and that ρ_1 is induced by restriction of scalars from an embedding

$$\phi : H_{\breve{K}} \to \mathrm{GL}(V_{\breve{K}}).$$

For notational simplicity, we write $\underline{\mathcal{H}}$ for the $\mathcal{O}_{\breve{K}_0}[u]$-group scheme associated to $\mathcal{H}_{\breve{K}}$ as explained in §3.1.3.

By the same proof as [PZ13, Proposition 8.1], it suffices to show that there exists a lattice chain $\underline{V}_\bullet$ in $\mathcal{O}_{\breve{K}_0}[u]^n$ such that ϕ extends to a homomorphism of $\mathcal{O}_{\breve{K}_0}[u]$-group schemes

$$\phi_{\mathcal{O}_{\breve{K}_0}[u]} : \underline{\mathcal{H}} \to \mathrm{GL}(\underline{V}_\bullet)$$

satisfying the following two conditions

- ρ extends to a group scheme morphism $\underline{\mathcal{H}} \to \mathrm{GL}(\underline{V}_\bullet)$ over $\mathcal{O}_{\breve{K}_0}[u]$.
- The homomorphism

$$\underline{\mathcal{H}}_{k[[u]]} := \underline{\mathcal{H}} \otimes_{\mathcal{O}_{\breve{K}_0}[u]} k[[u]] \to \mathrm{GL}(\underline{V}_\bullet \otimes_{\mathcal{O}_{\breve{K}_0}[u]} k[[u]])$$

is a locally closed immersion, and the Zariski closure of $\underline{H}_{k((u))} := \underline{H} \otimes_{\mathcal{O}_{\breve{K}_0}[u]} k((u))$ in $\mathrm{GL}(\underline{V}_\bullet \otimes_{\mathcal{O}_{\breve{K}_0}[u]} k[[u]])$ is a smooth group scheme $\mathcal{P}'$ whose connected component may be identified with $\underline{\mathcal{H}}_{k[[u]]}$.

Indeed, under these assumptions, the proof in [PZ13, Proposition 8.1] shows that pushing out torsors along $\phi_{\mathcal{O}_{\breve{K}_0}[u]}$ gives a morphism $\mathrm{Fl}^{Q(u)}_{\underline{\mathcal{H}}} \to \mathrm{Fl}^{Q(u)}_{\mathrm{GL}(\underline{V}_\bullet)}$ which restricts to a closed immersion $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}_{\mathcal{O}_{\breve{F}}},\{\mu\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F,\mathcal{O}_{\breve{F}}},\{\mu_{K/F}\}} \otimes_{\mathcal{O}_{\breve{E}'}} \mathcal{O}_{\breve{E}}$.

The construction of the map $\phi_{\mathcal{O}_{\breve{K}_0}[u]}$ follows, with some minor modifications, from the same argument as [KP18, Proposition 2.3.7]; as in the construction of the group scheme $\underline{H}$ in [Lev16], the key point is to realize the tame descent over $\mathcal{O}_{\breve{K}_0}[u^\pm]$ as opposed to $\mathcal{O}_{\breve{K}}[u^\pm]$ in [KP18]. We briefly sketch their argument, pointing out what modifications are needed in our situation.

Let $\breve{K}'/\breve{K}$ be a splitting field for $H_{\breve{K}}$ which we may assume is finite, tamely ramified and Galois. We let $\widetilde{e} := [\breve{K}' : \breve{K}]$ and fix a uniformizer $\widetilde{\varpi}$ of $\breve{K}'$. The action of $\mathrm{Gal}(\breve{K}'/\breve{K})$ extends to an action on $\mathcal{O}_{\breve{K}_0}[w^\pm]/\mathcal{O}_{\breve{K}_0}[u^\pm]$, where $w^{\widetilde{e}} = u$. Using the argument in [KP18, Proposition 2.3.7, Step 1], we obtain a representation

$$\phi_{\mathcal{O}_{\breve{K}_0}[u^\pm]} : \underline{\mathcal{H}}_{\mathcal{O}_{\breve{K}_0}[u^\pm]} \to \mathrm{GL}_n(\mathcal{O}_{\breve{K}_0[u^\pm]})$$

which extends ϕ under the map $u \mapsto \varpi$; this is constructed by descending along the cover $\mathcal{O}_{\breve{K}_0}[w^\pm]/\mathcal{O}_{\breve{K}_0}[u^\pm]$. (In *loc. cit.*, they apply the argument to the cover $\mathcal{O}_{\breve{K}}[w^\pm]/\mathcal{O}_{\breve{K}}[u^\pm]$ to obtain a representation over $\mathcal{O}_{\breve{K}}[u^\pm]$). Here, the specialization of $\mathrm{GL}_n(\mathcal{O}_{\breve{K}_0}[u^\pm])$ along $u \mapsto \varpi$ is identified with $\mathrm{GL}(V_{\breve{K}})$ via a suitable choice of basis for $V_{\breve{K}}$.

The construction of $\underline{V}_\bullet$ then proceeds in the same way as [KP18, Proposition 2.3.7, Step 1]. We write T for the diagonal torus of GL_n; then the basis of $V_{\breve{K}}$ is chosen so that $y \in \mathcal{A}(\mathrm{GL}_n, T, \breve{K})$. Using the identification of apartments

$$\mathcal{A}(\mathrm{GL}_n, T, \breve{K}) \cong \mathcal{A}(\underline{GL}_{n,\breve{K}_0((u))}, \underline{T}_{\breve{K}_0((u))}, \breve{K}_0((u))). \tag{3.2.6.2}$$

we obtain a lattice chain $\underline{N}_\bullet$ of $\breve{K}_0[[u]]$-modules in $\breve{K}_0((u))^n$ corresponding to the image of y in $\mathcal{A}(\underline{\mathrm{GL}}_{n,\breve{K}_0((u))}, \underline{T}_{\breve{K}_0((u))}, \breve{K}_0((u)))$. Then if we define $\underline{V}_\bullet := \underline{N}_\bullet \cap \mathcal{O}_{\breve{K}_0}[u^\pm]^n$, $\phi_{\mathcal{O}_{\breve{K}_0}[u^\pm]}$ extends to a map $\phi_{\mathcal{O}_{\breve{K}_0}[u]} : \underline{\mathcal{H}} \to \mathrm{GL}(\underline{V}_\bullet)$ satisfying the required conditions.

Step (2): $\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F},\{\mu_{K/F}\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\mu_W\}} \otimes_{\mathcal{O}_F} \mathcal{O}_{E'}$.

Since $\mathrm{GL}(W)$ is a split F-group, the local model $\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\mu_W\}}$ is naturally a subscheme of $\mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_W}$. Here, $\underline{\mathcal{GL}}_W$ is an $\mathcal{O}_F[v]$-group scheme and $\mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_W}$ is defined by applying §3.1.3 with $K = F$. We first show there exists a map $\mathrm{Fl}^{Q(u)}_{\underline{\mathcal{GL}}} \to \mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_W}$; here $\underline{\mathcal{GL}}$ is the $\mathcal{O}_{K_0}[u]$-group scheme associated to the $\mathcal{O}_K$-group scheme $\mathcal{GL}$ and the extension K/F as in §3.1.3.

Let W_0 denote the underlying K_0-vector space of V. Denote by $\mathcal{GL}_{W_0}$ the parahoric group scheme over $\mathcal{O}_{K_0}$ corresponding to the image of y under the map of buildings

$$\mathcal{B}(\mathrm{GL}(V), K) = \mathcal{B}(\mathrm{Res}_{K/K_0}\mathrm{GL}(V), K_0) \to \mathcal{B}(\mathrm{GL}(W_0), K_0)$$

We first define a map $\mathrm{Fl}^{Q(u)}_{\underline{\mathcal{GL}},0} \to \mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{W_0},0}$; see §3.1.1 for the definition of these ind-schemes. (This amounts to constructing the map in the special case $F = K_0$).

Define the map

$$r : \mathcal{O}_{K_0}[v] \to \mathcal{O}_{K_0}[u], \quad v \mapsto Q(u) + \varpi_F,$$

which lifts the inclusion $\mathcal{O}_{K_0} \to \mathcal{O}_K$, via $v \mapsto \varpi_F$, and $u \mapsto \varpi$. Then r is a finite flat morphism of rings. Let $\underline{\mathcal{GL}}_{K/K_0}$ denote the group scheme given by Weil-restriction of $\underline{\mathcal{GL}}$ along r; then the base change of $\underline{\mathcal{GL}}_{K/K_0}$ along $\mathcal{O}_{K_0}[v] \to \mathcal{O}_{K_0},\ v \mapsto \varpi_F$ is identified with $\mathcal{GL}_{K/K_0} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_{K_0}}\mathcal{GL}$. We begin by constructing a map

$$i : \underline{\mathcal{GL}}_{K/K_0} \to \underline{\mathcal{GL}}_{W_0}$$

extending the map of $\mathcal{O}_{K_0}$-schemes $\mathcal{GL}_{K/K_0} \to \mathcal{GL}_{W_0}$ under the specialization $v \mapsto \varpi_F$, such that the base change to $k[[v]]$

$$i_{k[[v]]} : \underline{\mathcal{GL}}_{K/K_0,k[[v]]} \to \underline{\mathcal{GL}}_{W_0,k[[v]]}$$

is a closed immersion.

To construct i, let $\underline{W}_\bullet$ denote the lattice chain of $\mathcal{O}_{K_0}[u]$-modules associated to $\mathcal{GL}$ via the construction in §3.2.1; then $\underline{\mathcal{GL}}$ may be identified with the automorphism group of $\underline{W}_\bullet$. We may view $\underline{W}_\bullet$, via r, as a lattice chain of $\mathcal{O}_{K_0}[v]$-modules $\underline{W}_{0,\bullet}$. Then we may identify $\underline{\mathcal{GL}}_{W_0}$ with the automorphism group of $\underline{W}_{0,\bullet}$. Since any $\mathcal{O}_{K_0}[u]$-automorphism of $\underline{W}_\bullet$ gives an $\mathcal{O}_{K_0}[v]$-automorphism of $\underline{W}_{0,\bullet}$, we obtain a natural map of $\mathcal{O}_{K_0}[v]$-group schemes $i : \underline{\mathcal{GL}}_{K/K_0} \to \underline{\mathcal{GL}}_{W_0}$ as desired. The base change $i_{k[[v]]} : \underline{\mathcal{GL}}_{K/K_0,k[[v]]} \to \underline{\mathcal{GL}}_{W_0,k[[v]]}$ is induced by restriction of structure from $k[[u]]$-lattices to $k[[v]]$-lattices under the map $v \mapsto u^e$, where $e = [K : K_0]$. Therefore it is a closed immersion by Lemma 3.2.4.

By [HR20, Corollary 3.6], the Weil-restriction of torsors along r induces an isomorphism

$$\mathrm{Fl}^{Q(u)}_{\underline{\mathcal{GL}},0} \xrightarrow{\sim} \mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{K/K_0},0}.$$

Combining this isomorphism with the map given by taking push-outs of torsors along i, we obtain the required map

$$\iota_0 : \mathrm{Fl}^{Q(u)}_{\underline{\mathcal{GL}},0} \cong \mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{K/K_0},0} \to \mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{W_0},0}.$$

Now applying $\mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F}$ we obtain a map

$$\iota : \mathrm{FL}^{Q(u)}_{\underline{\mathcal{GL}}} \to \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F}\mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{W_0}}.$$

A standard argument (cf. [PR08, Theorem 1.4]) shows that $\iota \otimes_{\mathcal{O}_F} k$ is a locally closed immersion. Since the domain of this map is ind-projective it follows that $\iota \otimes_{\mathcal{O}_F} k$ is a closed immersion.

We compose ι with the map

$$\iota' : \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F}\mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{W_0}} \to \mathrm{Fl}^{v-\varpi_F}_{\underline{\mathcal{GL}}_{W}}$$

induced by the embedding $\mathrm{Res}_{K_0/F}\mathrm{GL}(W_0) \to \mathrm{GL}(W)$. As in [PZ13, Proof of Proposition 8.1], $\iota' \otimes_{\mathcal{O}_F} k$ is a closed immersion, since $\mathrm{Res}_{K_0/F}\mathrm{GL}(W_0)$ is an unramified group and the embedding $\mathrm{Res}_{K_0/F}\mathrm{GL}(W_0) \to \mathrm{GL}(W)$ is minuscule. It follows that the composite map $\iota' \circ \iota$ is a closed immersion on special fibers.

Restricting to the local models we obtain a map

$$\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{K/F},\{\mu_{K/F}\}} \to \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\mu_W\}} \otimes_{\mathcal{O}_F} \mathcal{O}_{E'} \tag{3.2.6.3}$$

which is a closed immersion on special fibers. An argument involving Nakayama's Lemma as in [PZ13, Proposition 8.1] shows that (3.2.6.3) itself is a closed immersion.

It remains to check that (3.2.6.3) extends the natural morphism on the generic fiber. This follows from the definition of local models in §3.1.4, and the fact that the map r takes $v - \varpi_F$ to $Q(u)$. □

3.2.7. More generally if $G \cong \prod_{i=1}^r \mathrm{Res}_{K_i/F}H_i$ as in $(*)$ and $\rho : G \to \mathrm{GL}(W)$ is a faithful minuscule representation such that $\rho \circ \mu$ is conjugate to a standard minuscule coweight and $\rho(G)$ contains the scalars, we let W_i denote the underlying F-vector space of $W \otimes_{\mathcal{O}_F} K_i$. Then as before we obtain a new faithful minuscule representation given by the composition

$$\rho' : G \cong \prod_{i=1}^r \mathrm{Res}_{K_i/F}H_i \to \prod_{i=1}^r \mathrm{GL}(W_i) \to \mathrm{GL}(W').$$

where the first map is induced from a product of maps $\rho_i' : \mathrm{Res}_{K_i/F}H_i \to \mathrm{GL}(W_i)$ and $W' := \prod_{i=1}^r W_i$. We let $\mathcal{GL}_{W_i}$ denote the parahoric for $\mathrm{GL}(W_i)$ as constructed in §3.2.3; this determines a parahoric $\mathcal{GL}_{W'}$ of $\mathrm{GL}(W')$ given by the stabilizer of the lattice chain in W' formed by all possible products of the lattice chains in W_i corresponding to $\mathcal{GL}_{W_i}$. We let μ_{W_i} denote the i^{th}-component of the $\prod_{i=1}^r \mathrm{GL}(W_i)$-conjugacy class of cocharacters induced by $\{\mu\}$. By [KP18, Proposition 2.3.7], there is a closed immersion

$$\prod_{i=1}^r \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{W_i},\{\mu_{W_i}\}} \hookrightarrow \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{W'},\{\rho'\circ\mu\}} \tag{3.2.7.1}$$

Applying Proposition 3.2.6 to each factor and composing with (3.2.7.1), we obtain:

Proposition 3.2.8. *ρ' extends to a closed immersion $\widetilde{\mathcal{G}} \to \mathcal{GL}_{W'}$, and there is a closed immersion*

$$\rho'^{\mathrm{loc}} : \mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \to \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_{W'},\{\rho'\circ\mu\}} \otimes_{\mathcal{O}_F} \mathcal{O}_E. \tag{3.2.8.1}$$

extending the natural map on generic fibers. □

3.3. **More general local models.**

3.3.1. In this subsection we extend the construction of local models to certain triples $(G, \{\mu\}, \mathcal{G})$ with the condition $(*)$ relaxed. This is necessary for the later applications to Shimura varieties because groups of the form $\mathrm{Res}_{K/F} H$ rarely arise as the group at p of a Shimura datum of Hodge type. The triples we consider are those given by the next definition, which amounts to condition $(*)$ for the adjoint group.

Definition 3.3.2. A reductive group G over F is said to be *acceptable* if $G^{\mathrm{ad}} \cong \prod_{i=1}^{r} \mathrm{Res}_{K_i/F} H_i^{\mathrm{ad}}$ where K_i/F is a finite extension and H_i^{ad} is an adjoint group over K_i which splits over a tame extension of K_i.

A local model triple $(G, \{\mu\}, \mathcal{G}\})$ is said to be acceptable if G is acceptable.

If $p > 3$, there are no automorphisms of a connected Dynkin diagram of order divisible by p, hence any such reductive group is acceptable. Moreover, for $p = 3$, any reductive group which has no factors of type D_4 is acceptable, as this is the only connected Dynkin diagram with an automorphism of order 3.

3.3.3. We will also consider the following condition on the pair $(G, \{\mu\})$.

Definition 3.3.4. A pair $(G, \{\mu\})$ consisting of a reductive group G over F and a geometric conjugacy class $\{\mu\}$ of minuscule cocharacters is said to be of *local abelian type*, if there exists a similar pair $(G', \{\mu'\})$ such that

(1) There is an isomorphism of the associated adjoint pairs
$$(G^{\mathrm{ad}}, \{\mu^{\mathrm{ad}}\}) \cong (G'^{\mathrm{ad}}, \{\mu'^{\mathrm{ad}}\}).$$
(2) G' admits a faithful minuscule representation $\rho : G' \to \mathrm{GL}(V)$ such that $\rho \circ \mu'$ is conjugate to a standard minuscule cocharacter, and such that $\rho(G')$ contains the scalars.

A local model triple $(G, \{\mu\}, \mathcal{G})$ is said to be of local abelian type if $(G, \{\mu\})$ is.

Remark 3.3.5. This definition is modeled on [HLR, Definition 9.6] (cf. also [PRa, Definition 2.1.3]). The difference is that we require the representation $\rho : G' \to \mathrm{GL}(V)$ to be minuscule and its image to contain the scalars.

Following [HPR20], we call the pair $(G', \{\mu'\})$ in Definition 3.3.4 a *realization* of $(G, \{\mu\})$. The next proposition, which concerns the property of a pair $(G, \{\mu\})$ of local abelian type, can be proved in a similar way to [Del79, Prop. 2.3.10] (cf. also [KP18, Lem. 4.6.22]). We refer to [PRa, Prop. 8.2.1] for a detailed proof.

Proposition 3.3.6. *Assume $p > 2$. Let G be an acceptable reductive group over F and $\{\mu\}$ a geometric conjugacy class of minuscule cocharacters such that the pair $(G, \{\mu\})$ is of local abelian type. We assume that $\{\mu^{\mathrm{ad}}\}$ is non-trivial in every $\mathbb{Q}_p$-simple factor of G^{ad}. Then there exists a realization $(G', \{\mu'\})$ of $(G, \{\mu\})$ satisfying the following properties.*

(1) $p \nmid |\pi_1(G'^{\mathrm{der}})|$.
(2) $G' = \prod_{i=1}^{r} \mathrm{Res}_{K_i/F} H_i$ where K_i/F are finite extensions and H_i is a reductive group over K_i which splits over a tame extension.
(3) $E' = E^{\mathrm{ad}}$, where E' (resp. E^{ad}) is the local reflex field for $\{\mu\}$ (resp. $\{\mu^{\mathrm{ad}}\}$).

3.3.7. For the rest of this section, we assume that $p>2$. Now let $(G,\{\mu\},\mathcal{G})$ be an acceptable local model triple of local abelian type. We construct a local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ for $(G,\{\mu\},\mathcal{G})$ as follows. We write $G^{\mathrm{ad}}=G_1^{\mathrm{ad}}\times G_2^{\mathrm{ad}}$, where G_1^{ad} (resp. G_2^{ad}) is the product of the $\mathbb{Q}_p$-simple factors of G^{ad} where $\{\mu^{\mathrm{ad}}\}$ is non-trivial (resp. trivial). Let G_1 denote the kernel of $G\to G_2^{\mathrm{ad}}$. Then $\{\mu\}$ factors through G_1 and and we write $\{\mu_1\}$ for the induced conjugacy class of cocharacters. The morphism $G_1\to G_1^{\mathrm{ad}}$ is a central extension and $(G_1,\{\mu_1\})$ is of local abelian type and satisfies the assumptions in Proposition 3.3.6.

The parahoric $\mathcal{G}^{\mathrm{ad}}$ breaks up into a product $\mathcal{G}_1^{\mathrm{ad}}\times\mathcal{G}_2^{\mathrm{ad}}$, and we let $\mathcal{G}_1$ denote the parahoric of G_1 determined by $\mathcal{G}_1^{\mathrm{ad}}$. We obtain a local model triple $(G_1,\{\mu_1\},\mathcal{G}_1)$. Choose a realization $(G',\{\mu'\})$ for $(G_1,\{\mu_1\})$ as in Proposition 3.3.6. Then $\mathcal{G}_1$ determines a parahoric $\mathcal{G}'$ of G' and we obtain the local model triple $(G',\{\mu'\},\mathcal{G}')$ with reflex field E'. Since G' satisfies assumption $(*)$ of §3.1.2, we have the local model $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}',\{\mu'\}}$ constructed in the previous subsection. We then define

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}:=\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}',\{\mu'\}}\otimes_{\mathcal{O}_{E'}}\mathcal{O}_E,$$

which is a flat projective $\mathcal{O}_E$-scheme with reduced special fiber, by Theorem 3.1.6.

Remark 3.3.8. If G itself satisfies assumption $(*)$ of §3.1.2, we have the associated local model $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ from §3.1.5. When p divides $|\pi_1(G^{\mathrm{der}})|$, the schemes $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ and $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ are not necessarily isomorphic. However it can be shown using the method of [KP18, Prop. 2.2.4], that $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is isomorphic to the normalization of $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$.

3.3.9. The definition of the local models $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is justified by the following theorem which shows that they satisfy the Scholze–Weinstein conjecture [SW20, Conjecture 21.4.1]. In our setting, this conjecture states that there exists a flat projective $\mathcal{O}_E$-scheme $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ with reduced special fiber such that there is an identification

$$\mathcal{M}^{\mathrm{loc},\diamond}_{\mathcal{G},\{\mu\}}\cong\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu}. \tag{3.3.9.1}$$

Here $\mathcal{M}^{\mathrm{loc},\diamond}_{\mathcal{G},\{\mu\}}$ is the associated diamond and $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu}$ is the closure of the minuscule Schubert variety corresponding to $\{\mu\}$ in the v-sheaf Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E}$ for $\mathcal{G}$ over $\mathrm{Spd}\mathcal{O}_E$, as defined in [SW20, Definition 20.3.1]. Moreover (3.3.9.1) is required to extend the natural identification on the generic fiber. It is shown in *loc. cit.* that the scheme $\mathcal{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is unique if it exists (cf. [SW20, Proposition 18.4.1]).

We will make use of the following proposition.

Proposition 3.3.10. *Let $(G,\{\mu\},\mathcal{G})$ be a local model triple and let $\rho:G\to\mathrm{GL}(W)$ be a faithful representation such that $\rho\circ\mu$ is conjugate to a standard minuscule cocharacter. We assume the following two conditions are satisfied.*

(1) There exists a parahoric group scheme $\mathcal{GL}_W$ of $\mathrm{GL}(W)$ such that ρ extends to a closed immersion $\widetilde{\mathcal{G}}\to\mathcal{GL}_W$ of $\mathcal{O}_F$-group schemes.

(2) The Zariski closure $\mathcal{M}$ of X_μ inside $\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho\circ\mu\}}\otimes_{\mathcal{O}_F}\mathcal{O}_E$ has reduced special fiber. Here the embedding of homogenous spaces $X_\mu\to\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho\circ\mu\}}\otimes_{\mathcal{O}_F}E$ is induced by ρ.

Then $\mathcal{M}$ satisfies the Scholze–Weinstein conjecture.

Proof. The argument of [KP18, §2.3.15] shows that upon replacing ρ by a direct sum $\rho^{\oplus m}$, for some $m\geq 1$, we may assume $\mathcal{GL}_W$ is the stabilizer $\mathrm{GL}(\Lambda)$ of a single lattice $\Lambda \subset W$ (cf. [KP18, §2.3.15]). In this case $\mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho\circ\mu\}}$ can be identified with $\mathrm{Gr}(\Lambda)$, the smooth Grassmannian of subspaces $\mathcal{F} \subset \Lambda$ of rank d, where d is the integer such that $\{\rho \circ \mu\}$ is the conjugacy class of the standard minuscule cocharacter $a \mapsto \mathrm{diag}(1^{(n-d)}, (a^{-1})^{(d)})$.

By [SW20, Corollary 21.6.1], we have an isomorphism

$$\mathrm{Gr}(\Lambda)^{\diamond}_{\mathcal{O}_E} \cong \mathrm{Gr}_{\mathcal{GL}_W,\mathrm{Spd}\mathcal{O}_E,\rho\circ\mu}.$$

Thus by (1) and [SW20, Proposition 20.3.7] and [SW20, Proposition 21.4.3], we obtain a closed immersion $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu} \to \mathrm{Gr}(\Lambda)^{\diamond}_{\mathcal{O}_E}$ extending the closed immersion $X^{\diamond}_{\mu} \to \mathrm{Gr}(\Lambda)^{\diamond}_{E}$. It follows that we have an isomorphism $\mathcal{M}^{\diamond} \cong \mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu}$, and the result follows since $\mathcal{M}$ is a flat projective $\mathcal{O}_E$-scheme with reduced special fiber. □

Theorem 3.3.11. *Assume $p > 2$ and let $(G, \{\mu\}, \mathcal{G})$ be an acceptable local model triple of local abelian type. Then the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ defined above using a choice of realization $(G', \{\mu'\})$, as in Proposition 3.3.6, satisfies the Scholze–Weinstein conjecture. In particular, $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is unique up to isomorphism and hence independent of all choices made in the construction.*

Proof. Since $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is flat and projective with reduced special fiber, it suffices to show that $\mathbb{M}^{\mathrm{loc},\diamond}_{\mathcal{G},\{\mu\}}$ can be identified with $\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu}$. We use the notation of §3.3.7, so that $G^{\mathrm{ad}} = G_1^{\mathrm{ad}} \times G_2^{\mathrm{ad}}$.

By [SW20, Prop. 21.4.3], [SW20, Prop. 21.5.1], there are natural isomorphisms

$$\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu} \cong \mathrm{Gr}_{\mathcal{G}^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu^{\mathrm{ad}}}$$

$$\mathrm{Gr}_{\mathcal{G}',\mathrm{Spd}\mathcal{O}_E,\mu'} \cong \mathrm{Gr}_{\mathcal{G}_1^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu_1^{\mathrm{ad}}}$$

induced by the surjective morphisms $G \to G^{\mathrm{ad}}$ and $G' \to G_1^{\mathrm{ad}}$. Since $\mathcal{G}^{\mathrm{ad}} = \mathcal{G}_1^{\mathrm{ad}} \times \mathcal{G}_2^{\mathrm{ad}}$, we have an isomorphism

$$\mathrm{Gr}_{\mathcal{G}^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu^{\mathrm{ad}}} \cong \mathrm{Gr}_{\mathcal{G}_1^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu_1^{\mathrm{ad}}} \times_{\mathrm{Spd}\mathcal{O}_E} \mathrm{Gr}_{\mathcal{G}_2^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu_2^{\mathrm{ad}}},$$

where for $i = 1, 2$, $\{\mu_i^{\mathrm{ad}}\}$ is factor of $\{\mu^{\mathrm{ad}}\}$ in G_i. By assumption, μ_2^{ad} is trivial and hence $\mathrm{Gr}_{\mathcal{G}_2^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu_2^{\mathrm{ad}}} \cong \mathrm{Spd}\mathcal{O}_E$. It follows that $\mathrm{Gr}_{\mathcal{G}^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu^{\mathrm{ad}}} \cong \mathrm{Gr}_{\mathcal{G}_1^{\mathrm{ad}},\mathrm{Spd}\mathcal{O}_E,\mu_1^{\mathrm{ad}}}$ and hence we obtain an isomorphism

$$\mathrm{Gr}_{\mathcal{G},\mathrm{Spd}\mathcal{O}_E,\mu} \cong \mathrm{Gr}_{\mathcal{G}',\mathrm{Spd}\mathcal{O}_E,\mu'}.$$

Since the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ is defined using G', it suffices to prove the result in the case $(G, \{\mu\}, \mathcal{G}) = (G', \{\mu'\}), \mathcal{G}')$.

Let $\rho : G' \to \mathrm{GL}(V)$ be a faithful minuscule representation as in Definition 3.3.4, and let $\rho' : G' \to \mathrm{GL}(W)$ be the representation obtained by the construction in §3.2.7. Then by Proposition 3.2.8, ρ' extends to a closed immersion $\widetilde{\mathcal{G}} \to \mathcal{GL}_W$ for some parahoric $\mathcal{GL}_W$ of GL_W and we have a closed immersion

$$\rho'^{\mathrm{loc}} : \mathrm{M}^{\mathrm{loc}}_{\mathcal{G}',\{\mu'\}} \to \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho'\circ\mu'\}} \otimes_{\mathcal{O}_F} \mathcal{O}_E.$$

Since $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}',\{\mu'\}}$ has reduced fiber (Theorem 3.1.6), the result follows from Proposition 3.3.10.

□

3.3.12. As an immediate corollary, we find that $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$ only depends on the associated adjoint triple $(G^{\mathrm{ad}}, \{\mu^{\mathrm{ad}}\}, \mathcal{G}^{\mathrm{ad}})$.

Corollary 3.3.13. *Let $(G_1, \{\mu_1\}, \mathcal{G}_1)$ and $(G_2, \{\mu_2\}, \mathcal{G}_2)$ be acceptable local model triples of local abelian type with reflex fields E_1, E_2 respectively, and let $E' = E_1E_2$. Assume $p > 2$ and that there is an isomorphism $(G_1^{\mathrm{ad}}, \{\mu_1^{\mathrm{ad}}\}, \mathcal{G}_1^{\mathrm{ad}}) \cong (G_2^{\mathrm{ad}}, \{\mu_2^{\mathrm{ad}}\}, \mathcal{G}_2^{\mathrm{ad}})$ of associated adjoint triples. Then there is an isomorphism*

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_1,\{\mu_1\}} \otimes_{\mathcal{O}_{E_1}} \mathcal{O}_{E'} \cong \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_2\}} \otimes_{\mathcal{O}_{E_2}} \mathcal{O}_{E'}$$

extending the natural isomorphism on the generic fiber.

Proof. This follows by noting that the same (G', μ') in Proposition 3.3.6 can be used for the construction of both $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_1,\{\mu_1\}}$ and $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_2\}}$. □

3.3.14. Now let $(G, \{\mu\}, \mathcal{G})$ be an acceptable local model triple of local abelian type. The following notion will be needed for applications in §4.

Definition 3.3.15. Let $\rho : G \to \mathrm{GL}(V)$ a faithful minuscule representation where V is an n-dimensional vector space and $\Lambda \subset V$ an $\mathcal{O}_F$-lattice. The representation $\rho : G \to \mathrm{GL}(V)$ is *good* with respect to Λ if the following conditions are satisfied.

(1) $\rho(G)$ contains the scalars.
(2) ρ extends to a closed immersion $\widetilde{\mathcal{G}} \hookrightarrow \mathcal{GL}_V := \mathrm{GL}(\Lambda)$.
(3) There is a closed immersion of local models

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \hookrightarrow \mathrm{Gr}(\Lambda) \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

extending the natural map on the generic fiber. Here $\mathrm{Gr}(\Lambda)$ is the smooth Grassmannian of subspaces $\mathcal{F} \subset \Lambda$ of rank d, where $d \in \mathbb{Z}_{\geq 0}$ is such that $\{\rho \circ \mu\}$ is the conjugacy class of $a \mapsto \mathrm{diag}(1^{(n-d)}, (a^{-1})^{(d)})$.

We say that the representation $\rho : G \to \mathrm{GL}(V)$ is good if there exists a lattice Λ such that ρ is good with respect to Λ.

Remark 3.3.16. If we assume in addition that $\widetilde{\mathcal{G}} = \mathcal{G}$ (i.e. $\mathcal{G}$ is a connected parahoric and x is chosen to be in general position in its facet), then we recover the definition of a strongly integral local Hodge embedding for $(\mathcal{G}, \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}})$ as in [Pap22, §3.1.4].

3.3.17. Now let $\rho : G \to \mathrm{GSp}(V)$ be a faithful symplectic representation where V is a $2n$-dimensional vector space over F equipped with a perfect alternating bilinear form Ψ. We assume that $\rho \circ \mu$ is conjugate to the standard minuscule coweight $a \mapsto \mathrm{diag}(1^{(n)}, (a^{-1})^{(n)})$ and that $\rho(G)$ contains the scalars. We call such an embedding a local Hodge embedding. We say ρ is *good* if the corresponding representation $G \to \mathrm{GL}(V)$ is good.

Proposition 3.3.18. *Let $p > 2$ and $(G, \{\mu\}, \mathcal{G})$ an acceptable local model triple of local abelian type. Assume $p \nmid |\pi_1(G^{\mathrm{der}})|$, the centralizer of a maximal $\breve{F}$-split torus in G is R-smooth and that G admits a local Hodge embedding $\rho : G \to \mathrm{GSp}(V)$.*

Then $(G, \{\mu\}, \mathcal{G})$ admits a good local Hodge embedding $\rho'' : G \to \mathrm{GSp}(W')$.

Proof. Let K/F be a finite extension over which G splits. Then we may replace ρ by a new Hodge embedding $\rho' : G \to \mathrm{GSp}(W)$ such that the induced map $G \to \mathrm{GL}(W)$ factors as

$$G \to \mathrm{Res}_{K/F} G_K \xrightarrow{\rho_K} \mathrm{Res}_{K/F}\mathrm{GL}(V_K) \to \mathrm{GL}(W) \quad (3.3.18.1)$$

(cf. §3.2.2). Here W is the underlying F-vector space of V_K, which we equip with the alternating bilinear form given by

$$\psi : W \times W \xrightarrow{\Psi\otimes_F K} K \xrightarrow{\mathrm{Tr}_{K/F}} F.$$

Then $\rho'(G)$ also contains the scalars.

Let $\widetilde{\mathcal{G}}_0$ denote the Bruhat–Tits stabilizer scheme over $\mathcal{O}_K$ corresponding to the image of x in $\mathcal{B}(G, K)$. By [KP18, Proposition 1.3.3] there is a parahoric $\mathcal{GL}_0$ of $\mathrm{GL}(V_K)$ such that there is a closed immersion $\widetilde{\mathcal{G}}_0 \to \mathcal{GL}_0$ of $\mathcal{O}_K$-group schemes. The lattice-chain corresponding to $\mathcal{GL}_0$ determines a parahoric group scheme $\mathcal{GL}_W$ of $\mathrm{GL}(W)$, and (3.3.18.1) extends to morphisms

$$\widetilde{\mathcal{G}} \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\widetilde{\mathcal{G}}_0 \to \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{GL}_0 \to \mathcal{GL}_W.$$

The first map is a closed immersion by Proposition 2.4.11, the second map is a closed immersion since it is obtained via Weil-restriction of a closed immersion, and the last map is a closed immersion by Lemma 3.2.4.

Let $\{\mu_{K/F}\}$ denote the conjugacy class of cocharacters of $\mathrm{Res}_{K/F}G_K$ induced by μ and set $\mathcal{G}_{K/F} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{G}_0$. We let E' denote the local reflex field of $\{\mu_{K/F}\}$. By Proposition 3.2.6, we have a closed immersion

$$\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}_{K/F},\{\mu_{K/F}\}} \to \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho'\circ\mu\}} \otimes_{\mathcal{O}_F} \mathcal{O}_{E'}$$

where $\mathrm{M}^{\mathrm{loc}}_{\mathcal{G}_{K/F},\{\mu_{K/F}\}}$ is the local model associated to $(\mathrm{Res}_{K/F}G_K, \{\mu_{K/F}\}, \mathcal{G}_{K/F})$ as in §3.1.5. Thus by Proposition 3.3.10, we have an isomorphism $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_{K/F},\{\mu_{K/F}\}} \cong \mathrm{M}^{\mathrm{loc}}_{\mathcal{G}_{K/F},\{\mu_{K/F}\}}$. By Lemma 3.3.19 below, we obtain a closed immersion

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_{K/F},\{\mu_{K/F}\}} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E \to \mathrm{M}^{\mathrm{loc}}_{\mathcal{GL}_W,\{\rho'\circ\mu\}} \otimes_{\mathcal{O}_F} \mathcal{O}_E.$$

Now arguing as in [KP18, §2.3.15], upon replacing ρ' by a direct sum $\rho'' := \rho'^{\oplus m}$ and W by $W' = W^m$ for some $m \geq 1$, we may assume that there exists an $\mathcal{O}_F$-lattice $\Lambda \subset W'$ such that ρ'' induces closed immersions $\widetilde{\mathcal{G}} \to \mathrm{GL}(\Lambda)$ and

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \to \mathrm{Gr}(\Lambda) \otimes_{\mathcal{O}_F} \mathcal{O}_E;$$

(cf. Proposition 3.3.10). Hence ρ'' is a good local Hodge embedding. □

Lemma 3.3.19. *With the notation and assumptions of the previous proposition, there is a closed immersion*

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_{K/F},\{\mu_{K/F}\}} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E. \tag{3.3.19.1}$$

Proof. This follows from [FHLR, Lemma 5.27] and [AGLR, Theorem 7.21]. More precisely, [AGLR, Theorem 7.21] shows that the models constructed in [FHLR, Lemma 5.27] agree with our $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}$. [FHLR, Lemma 5.27] then shows the existence of the closed immersion noting that Hypotheses 2.1 and 5.24 of *loc. cit.* are satisfied by our assumptions of acceptability and that $p > 2$. □

3.3.20. Now let $(G, \{\mu\}, \mathcal{G})$ be an acceptable local model triple of local abelian type and $\rho'' : G \to \mathrm{GL}(W')$ a representation which is good with respect to an $\mathcal{O}_F$-lattice Λ. We finish this section by giving a more explicit description of the embedding $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}} \to \mathrm{Gr}(\Lambda) \otimes_{\mathcal{O}_F} \mathcal{O}_E$ on the level of k-points which will be needed in §4.3. We assume $\mathcal{G}$ is a standard parahoric corresponding to a subset $J \subset \mathbb{S}$.

As explained in [Zho20, §3.6], we may identify the k-points of $\mathrm{Gr}(\Lambda)$ with a subset of $\mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}})$, where $\mathcal{GL}_{W'} := \mathrm{GL}(\Lambda)$. The convention in *loc.*

cit. is that $g \in \mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}}) \cap \mathrm{Gr}(\Lambda)(k)$ corresponds to the subspace of $\Lambda \otimes_{\mathcal{O}_F} k$ induced by the reduction mod ϖ_F of the lattice $\varpi_F g\Lambda$. We thus obtain an inclusion $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}(k) \subset \mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}})$.

Proposition 3.3.21. *Assume $p > 2$ and $\tilde{\mathcal{G}} = \mathcal{G}$. Let $g \in G(\breve{F})$ with*

$$g \in \mathcal{G}(\mathcal{O}_{\breve{F}})\dot{w}\mathcal{G}(\mathcal{O}_{\breve{F}})$$

for some $w \in W_J\backslash W/W_J$. Then the image of $\rho''(g)$ in $\mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}})$ lies in $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}(k)$ if and only if $w \in \mathrm{Adm}(\{\mu\})_J$.

Proof. By [AGLR, Theorem 7.23] and Proposition 3.3.11, the inclusion $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}(k) \subset \mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}})$ lifts to an inclusion $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}(k) \subset G(\breve{F})/\mathcal{G}(\mathcal{O}_{\breve{F}})$ which identifies $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu\}}(k)$ with the μ-admissible locus in $G(\breve{F})/\mathcal{G}(\mathcal{O}_{\breve{F}})$ (i.e. elements of the form $\mathcal{G}(\mathcal{O}_{\breve{F}})\dot{w}\mathcal{G}(\mathcal{O}_{\breve{F}})/\mathcal{G}(\mathcal{O}_{\breve{F}})$ for $w \in \mathrm{Adm}(\{\mu\})_J$). By our assumption that $\mathcal{G} = \widetilde{\mathcal{G}}$, the morphism $G(\breve{F})/\mathcal{G}(\mathcal{O}_{\breve{F}}) \to \mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}})$ induced by ρ'' is injective, and the result follows. □

Remark 3.3.22. The reason for the convention in [Zho20, §3.6] is as follows. Let μ be the standard minuscule cocharacter of GL_n given by $a \mapsto \mathrm{diag}(1^{(n-d)}, (a^{-1})^{(d)})$. Then on the generic fiber, μ corresponds to the subspace of W' where it acts by weight -1. The specialization of this point in $\mathrm{Gr}(\Lambda)(k)$ is the subspace of $\Lambda \otimes_{\mathcal{O}_F} k$ given by the reduction mod ϖ_F of $\varpi_F\mu(\varpi_F)\Lambda$. Thus with this convention, $\mathrm{Gr}(\Lambda)(k)$ is identified with the μ-admissible locus of $\mathrm{GL}_{W'}(\breve{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\breve{F}})$.

4. Deformation theory of p-divisible groups

In this section we prove the deformation theoretic results needed to study integral models of Shimura varieties in §5. §4.1-§4.2 contains the results needed to establish the local model diagram and in §4.3 we study the deformation theory of μ-ordinary p-divisible groups and construct an analogue of the Serre–Tate canonical lift.

4.1. **The versal deformation space with tensors.**

4.1.1. We recall the deformation theory of p-divisible groups equipped with a collection of crystalline tensors following [KP18, §3]. As most of the arguments of *loc. cit.* go through unchanged in our setting, we discuss in detail only those points which do not.

In this section, we assume $p > 2$ and we work over the base field $\mathbb{Q}_p$ so that $\breve{\mathbb{Q}}_p = W(k)[\frac{1}{p}]$, where $W(k)$ denotes the Witt vectors of k. For any ring R and an R-module M, we let $M^{\otimes}$ denote the direct sum of all R-modules obtained from M by taking duals, tensor products, symmetric and exterior products. If R is a complete local ring with residue field of positive characteristic and $\mathscr{G}$ is a p-divisible group over R, we write $\mathbb{D}(\mathscr{G})$ for its (contravariant) Dieudonné crystal.

4.1.2. Let $\mathscr{G}_0$ be a p-divisible group over k and set $\mathbb{D} := \mathbb{D}(\mathscr{G}_0)(\breve{\mathbb{Z}}_p)$. We write φ for the Frobenius on $\mathbb{D}$. Let $(s_{\alpha,0}) \subset \mathbb{D}^{\otimes}$ be a collection of φ-invariant tensors whose image in $\mathbb{D}(\mathscr{G}_0)(k)^{\otimes}$ lie in Fil^0. We assume that there exists a $\mathbb{Z}_p$-module U and an isomorphism

$$U \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong \mathbb{D} \tag{4.1.2.1}$$

such that $s_{\alpha,0} \in U^{\otimes}$. Write $\widetilde{\mathcal{G}} \subset \mathrm{GL}(U)$ for the pointwise stabilizer of $\{s_{\alpha,0}\}_\alpha$ so that $\widetilde{\mathcal{G}}_{\breve{\mathbb{Z}}_p}$ can be identified with the stabilizer of $s_{\alpha,0}$ in $\mathrm{GL}(\mathbb{D})$. We assume that the generic fiber $G := \widetilde{\mathcal{G}}_{\otimes_{\mathbb{Z}_p}}\mathbb{Q}_p$ is a reductive group and that $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_x$ for some $x \in \mathcal{B}(G, \mathbb{Q}_p)$. We write $\mathcal{G}$ for the associated parahoric group scheme.

Let $P \subset \mathrm{GL}(\mathbb{D})$ be a parabolic subgroup lifting the parabolic P_0 corresponding to the filtration on $\mathbb{D}(\mathscr{G}_0)(k)$. Write $\mathbb{M}^{\mathrm{loc}} = \mathrm{GL}(\mathbb{D})/P$ and $\mathrm{Spf}A = \widehat{\mathbb{M}}^{\mathrm{loc}}$ the completion of $\mathbb{M}^{\mathrm{loc}}$ at the identity; then A is isomorphic to a power series ring over $\breve{\mathbb{Z}}_p$. Let $K'/\breve{\mathbb{Q}}_p$ be a finite extensions and $y : A \to K'$ a continuous map such that $s_{\alpha,0} \in \mathrm{Fil}^0\mathbb{D}^{\otimes} \otimes_{\breve{\mathbb{Z}}_p} K'$ for the filtration induced by y on $\mathbb{D}^{\otimes} \otimes_{\breve{\mathbb{Z}}_p} K'$. By [Kis10, Lemma 1.4.5], the filtration corresponding to y is induced by a G-valued cocharacter μ_y (by convention μ_y has weights $(0, 1)$). Let $G.y$ be the orbit of y in $\mathbb{M}^{\mathrm{loc}} \otimes_{\breve{\mathbb{Z}}_p} K'$ which is defined over a finite extension $\breve{E}/\breve{\mathbb{Q}}_p$, and we write $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}}$ for the closure of this orbit in $\mathbb{M}^{\mathrm{loc}}$.

4.1.3. Let R be a complete local ring with maximal ideal $\mathfrak{m}$ and residue field k. We let $W(R)$ denote the Witt vectors of R. Recall [Zin01] we have a subring

$$\widehat{W}(R) = W(k) \oplus \mathbb{W}(\mathfrak{m}) \subset W(R),$$

where $\mathbb{W}(\mathfrak{m}) \subset W(R)$ consists of Witt vectors $(w_i)_{i\geq 1}$ with $w_i \in \mathfrak{m}$ and $w_i \to 0$ in the $\mathfrak{m}$-adic topology. The Frobenius of $W(R)$ induces a map $\varphi : \widehat{W}(R) \to \widehat{W}(R)$, and we write I_R for the kernel of the projection $\widehat{W}(R) \to R$. We recall the following definition, which is [Zho20, Definition 4.6] in the case that G splits over a tamely ramified extension of $\mathbb{Q}_p$.

Definition 4.1.4. Let $K/\breve{\mathbb{Q}}_p$ be a finite extension. Let $\mathscr{G}$ be a p-divisible group over $\mathcal{O}_K$ whose special fiber is isomorphic to $\mathscr{G}_0$. We say $\mathscr{G}$ is $(\widetilde{\mathcal{G}}, \mu_y)$-adapted if the tensors $s_{\alpha,0}$ extend to Frobenius invariant tensors $\widetilde{s}_\alpha \in \mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K))^{\otimes}$ such that the following two conditions hold:

(1) There is an isomorphism $\mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K)) \cong \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \widehat{W}(\mathcal{O}_K)$ taking $\widetilde{s}_\alpha$ to $s_{\alpha,0}$.

(2) Under the canonical identification

$$\mathbb{D}(\mathscr{G})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \cong \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} K$$

given by [KP18, Lemma 3.1.17], the filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} K$ is induced by a G-valued cocharacter conjugate to μ_y.

4.1.5. Consider the local model triple $(G, \{\mu_y^{-1}\}, \mathcal{G})$. We assume in addition that the following conditions are satisfied:

$$(G, \{\mu_y^{-1}\}, \mathcal{G}) \text{ is acceptable of local abelian type.} \tag{4.1.5.1}$$

$$\text{The embedding } G \subset \mathrm{GL}(U_{\mathbb{Q}_p}) \text{ is good with respect to } U. \tag{4.1.5.2}$$

Under these assumptions, property (3) of Definition 3.3.15 implies that the definition of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}}$ above agrees with the local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_y^{-1}\}} \otimes_{\mathcal{O}_E} \mathcal{O}_{\breve{E}}$. We write $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}} \cong \mathrm{Spf} A_{\widetilde{\mathcal{G}}}$ for the completion of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}}$ at the identity element. By Theorem 3.1.6, $A_{\widetilde{\mathcal{G}}}$ is normal and we have a natural surjective map $A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}} \to A_{\widetilde{\mathcal{G}}}$ corresponding to the closed immersion $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}} \subset \widehat{\mathbb{M}}^{\mathrm{loc}} \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$.

4.1.6. We now apply the construction in [KP18, 3.2]; the following is essentially [KP18, Proposition 3.2.17].

Proposition 4.1.7. *There exists a versal p-divisible group* $\mathscr{G}_A$ *over* $\mathrm{Spf} A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$ *deforming* $\mathscr{G}_0$ *such that for any* $K/\breve{\mathbb{Q}}_p$ *finite, a map* $\varpi : A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_E \to K$ *factors through* $A_{\widetilde{\mathcal{G}}}$ *if and only if the p-divisible group* $\mathscr{G}_\varpi$ *given by the base change of* $\mathscr{G}_A$ *along* ϖ *is* $(\widetilde{\mathcal{G}}, \mu_y)$*-adapted.*

Proof. Under our assumptions and using [Ans, Proposition 10.3] (see also [PRb, Proposition 5.3.2] for a different proof which applies in our setting) in place of [KP18, Proposition 1.4.3], we find that the conditions (3.2.2)-(3.2.4) of [KP18] are satisfied; we may thus apply the construction in [KP18, §3.2] to obtain $\mathscr{G}_A$. As we will make use of it later, we briefly recall the construction.

We set $M = \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \widehat{W}(A)$. Let $\overline{M}_1 \subset M/I_A M$ be the universal direct summand of $\widehat{\mathbb{M}}^{\mathrm{loc}}$ and $M_1 \subset M$ the preimage of $\overline{M}_1$. Let $\widetilde{M}_1$ denote the image of the map

$$\widehat{W}(A) \otimes_{\widehat{W}(A),\varphi} M_1 \to \widehat{W}(A) \otimes_{\widehat{W}(A),\varphi} M.$$

By [KP18, Corollary 4.2.1], we have $s_{\alpha,0} \in \widetilde{M}_1^{\otimes} \otimes_{\widehat{W}(A)} \widehat{W}(A_{\widetilde{\mathcal{G}}})$, and the scheme

$$\mathcal{T} = \underline{\mathrm{Isom}}_{s_{\alpha,0}}(\widetilde{M}_1 \otimes_{\widehat{W}(A)} \widehat{W}(A_{\widetilde{\mathcal{G}}}), M \otimes_{\widehat{W}(A)} \widehat{W}(A_{\widetilde{\mathcal{G}}})) \tag{4.1.7.1}$$

of isomorphisms which preserve the tensors $s_{\alpha,0}$ is a trivial $\widetilde{\mathcal{G}}$-torsor.

Let $\mathfrak{m}_{A_E}$ denote the maximal ideal in $A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$ and set $\mathfrak{a}_E := \mathfrak{m}_{A_E}^2 + \pi_E A_E$, where $\pi_E \in E$ is a unifomizer. We let

$$\Psi_{A_{\widetilde{\mathcal{G}}}} : \widetilde{M}_1 \otimes_{\widehat{W}(A)} \widehat{W}(A_{\widetilde{\mathcal{G}}}) \xrightarrow{\sim} M \otimes_{\widehat{W}(A)} \widehat{W}(A_{\widetilde{\mathcal{G}}})$$

be a section of $\mathcal{T}$ which is constant mod $\mathfrak{a}_E$ in the sense of [KP18, §3.1.11]; such a section exists as explained in [KP18, §3.2.12]. We will call such an isomorphism *rigid* (cf. [Pap22, Definition 4.5.8]). We then lift $\Psi_{A_{\widetilde{\mathcal{G}}}}$ to an isomorphism

$$\Psi : \widetilde{M}_1 \otimes_{\widehat{W}(A)} \widehat{W}(A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}) \xrightarrow{\sim} M \otimes_{\widehat{W}(A)} \widehat{W}(A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}})$$

which is constant mod $\mathfrak{a}_E$. By [KP18, Lemma 3.1.5], this gives rise to a Dieudonné display over $A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$, and hence to a p-divisible group $\mathscr{G}_A$ over $\mathrm{Spf} A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$ which is versal by [KP18, Lemma 3.1.12].

By construction, the base change $\mathscr{G}_{A_{\tilde{\mathcal{G}}}} := \mathscr{G}_A \otimes_{A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}} A_{\tilde{\mathcal{G}}}$ is equipped with Frobenius invariant tensors $s_{\alpha,0,A_{\widetilde{\mathcal{G}}}} \in \mathbb{D}(\mathscr{G}_{A_{\tilde{\mathcal{G}}}})(\widehat{W}(A_{\tilde{\mathcal{G}}}))^{\otimes}$. It is then clear that for $\varpi : A_{\widetilde{\mathcal{G}}} \to K$, the tensors $s_{\alpha,0}$ extend to

$$\widetilde{s}_\alpha \in \mathbb{D}(\mathscr{G}_\varpi)(\widehat{W}(\mathcal{O}_K))^{\otimes}$$

so that Definition 4.1.4 (1) is satisfied. Indeed the tensors $\widetilde{s}_\alpha$ are obtained from $s_{\alpha,0,A_{\widetilde{\mathcal{G}}}}$ via base change. The argument in [Zho20, Proposition 4.8] shows that condition (2) is also satisfied, so that $\mathscr{G}_\varpi$ is $(\widetilde{\mathcal{G}}, \mu_y)$-adapted.

The converse is [KP18, Proposition 3.2.17]. □

4.2. **Deformations with étale tensors.**

4.2.1. Let $K/\breve{\mathbb{Q}}_p$ be a finite extension and $\mathscr{G}$ a p-divisible group over $\mathcal{O}_K$ with special fiber $\mathscr{G}_0$. We write $T_p\mathscr{G}$ for the p-adic Tate-module of $\mathscr{G}$ and $T_p\mathscr{G}^\vee$ its linear dual. We let $s_{\alpha,\text{ét}} \in T_p\mathscr{G}^{\vee\otimes}$ be a collection of tensors whose stabilizer $\widetilde{\mathcal{G}}$ has reductive generic fiber G and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_x$ for some $x \in \mathcal{B}(G,\mathbb{Q}_p)$. We write $\mathbb{D} := \mathbb{D}(\mathscr{G}_0)(\breve{\mathbb{Z}}_p)$ and we let

$$s_{\alpha,0} \in D_{\text{cris}}(T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^\otimes \simeq \mathbb{D}^\otimes \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p$$

denote the φ-invariant tensors corresponding to the image of $s_{\alpha,\text{ét}}$ under the p-adic comparison isomorphism.

Proposition 4.2.2. *(1) We have $s_{\alpha,0} \in \mathbb{D}^\otimes \subset \mathbb{D}^\otimes \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p$. Moreover the $s_{\alpha,0}$ extend canonically to tensors $\widetilde{s}_\alpha \in \mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K))^\otimes$ and there exists an isomorphism*

$$T_p\mathscr{G}^\vee \otimes_{\breve{\mathbb{Z}}_p} \widehat{W}(\mathcal{O}_K) \cong \mathbb{D}(\mathscr{G})(\widehat{W}(\mathcal{O}_K)) \tag{4.2.2.1}$$

taking $s_{\alpha,0}$ to $\widetilde{s}_\alpha$.

(2) There exists a G-valued cocharacter μ_y such that

(i) Under the canonical isomorphism

$$\gamma : \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} K \cong \mathbb{D}(\mathscr{G})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K,$$

the filtration is induced by a G-valued cocharacter conjugate to μ_y.

(ii) The filtration on $\mathbb{D}\otimes_{\breve{\mathbb{Z}}_p} K$ induced by μ_y lifts the filtration on the module $\mathbb{D}(\mathscr{G}_0) \otimes_{\breve{\mathbb{Z}}_p} k$.

Here we consider $G_{\breve{\mathbb{Q}}_p} \subset \mathrm{GL}(\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p)$ via the base change of (4.2.2.1) to $\breve{\mathbb{Q}}_p$.

Proof. The argument is the same as [KP18, Proposition 3.3.8, Corollary 3.3.10], where again we are using [Ans, Proposition 10.3] in place of [KP18, Proposition 1.4.3]. □

4.2.3. The isomorphism (4.2.2.1) induces an isomorphism

$$T_p\mathscr{G}^\vee \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong \mathbb{D}$$

taking $s_{\alpha,\text{ét}}$ to $s_{\alpha,0}$ which we now fix. Taking $T_p\mathscr{G}^\vee$ to be U, we place ourselves in the setting of §4.1.2. Therefore we have a notion of $(\widetilde{\mathcal{G}},\mu_y)$-adapted lifting where μ_y is as in Proposition 4.2.2. Moreover it follows from the same proposition that $\mathscr{G}$ itself is a $(\widetilde{\mathcal{G}},\mu_y)$-adapted lifting. The next proposition then follows immediately from Proposition 4.2.2 and the definition of $(\widetilde{\mathcal{G}},\mu_y)$-adapted liftings (cf. [KP18, Proposition 3.3.13]).

Proposition 4.2.4. *Let $K'/\breve{\mathbb{Q}}_p$ be a finite extension and let $\mathscr{G}'$ be a deformation of $\mathscr{G}_0$ to $\mathcal{O}_{K'}$ such that*

(1) The filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} K'$ corresponding to $\mathscr{G}'$ is induced by a G-valued cocharacter conjugate to μ_y.

(2) The tensors $s_{\alpha,0} \in \mathbb{D}^\otimes$ correspond to tensors $s_{\alpha,\text{ét}} \in T_p\mathscr{G}'^{\vee\otimes}$ under the p-adic comparison isomorphism.

Then $\mathscr{G}'$ is $(\widetilde{\mathcal{G}},\mu_y)$-adapted lifting.

□

4.3. **Canonical liftings for μ-ordinary p-divisible groups.**

4.3.1. We now study the deformation theory of μ-ordinary p-divisible groups. The results in this subsection will be used in §5.4 to prove our main result on CM (special) liftings for Shimura varieties.

We return to the setting of §4.1. Thus $\mathscr{G}_0$ is a p-divisible group over k equipped with $s_{\alpha,0} \in \mathbb{D}^{\otimes}$. We fix a $\breve{\mathbb{Z}}_p$-linear isomorphism

$$U \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong \mathbb{D}(\mathscr{G}_0) \tag{4.3.1.1}$$

as in (4.1.2.1) so that $s_{\alpha,0} \in U^{\otimes}$. In §4.3, we will assume in addition to (4.1.5.1)–(4.1.5.2), that $\mathcal{G} = \widetilde{\mathcal{G}}$. Since the $s_{\alpha,0}$ are φ-invariant, the Frobenius is given by $b\sigma$ for an element $b \in G(\breve{\mathbb{Q}}_p)$, and modifying (4.3.1.1) by an element $h \in \mathcal{G}(\breve{\mathbb{Z}}_p)$ modifies b by $b \mapsto h^{-1}b\sigma(h)$. Therefore b is well-defined up to σ-conjugation by an element of $\mathcal{G}(\breve{\mathbb{Z}}_p)$ and in particular we obtain a well-defined class $[b] \in B(G)$.

We choose a maximal $\breve{\mathbb{Q}}_p$-split torus S of G defined over $\mathbb{Q}_p$ such that $x \in \mathcal{A}(G, S, \breve{\mathbb{Q}}_p)$ and we let T be its centralizer. We fix a σ-stable alcove $\mathfrak{a} \subset \mathcal{A}(G, S, \breve{\mathbb{Q}}_p)$ such that x lies in the closure of $\mathfrak{a}$; this determines a set of simple reflections $\mathbb{S}$ for W, and $\mathcal{G}$ corresponds to the subset $J \subset \mathbb{S}$ of reflections which fix x. We follow the notation of §2 and let $\widetilde{\mu} \in X_*(T)$ denote the dominant (with respect to a choice of Borel defined over $\breve{\mathbb{Q}}_p$) representative of the conjugacy class $\{\mu_y\}$; we write μ for its image in $X_*(T)_I$. We have a closed immersion of local models

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_y^{-1}\}} \hookrightarrow \mathrm{Gr}(U) \otimes_{\mathbb{Z}_p} \mathcal{O}_E,$$

where $\mathrm{Gr}(U)$ classifies submodules of U of rank $\dim_k \mathrm{Fil}^1\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} k$. By definition, the filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} k$ corresponds to an element of $\mathrm{Gr}(U)(k)$ which lies in $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_y^{-1}\}}(k)$. This filtration is by definition the kernel of φ. Thus its preimage in $\mathbb{D}$ is given by $\{v \in \mathbb{D} | b\sigma(v) \in p\mathbb{D}\}$, which is just the $\breve{\mathbb{Z}}_p$-lattice $\sigma^{-1}(b^{-1})p\mathbb{D}$. It follows from Proposition 3.3.21 that $\sigma^{-1}(b^{-1}) \in \mathcal{G}(\breve{\mathbb{Z}}_p)\dot{w}\mathcal{G}(\breve{\mathbb{Z}}_p)$ for some element $w \in \mathrm{Adm}(\{\mu_y^{-1}\})_J$, and hence that

$$b \in \mathcal{G}(\breve{\mathbb{Z}}_p)\sigma(\dot{u})\mathcal{G}(\breve{\mathbb{Z}}_p)$$

for some $u \in \mathrm{Adm}(\{\mu_y\})_J$. In particular we have $[\sigma^{-1}(b)] \in B(G, \{\mu_y\})$ by [He16, Theorem 1.1].

4.3.2. Now assume the existence of $[b]_\mu \in B(G, \{\mu_y\})$ as in Definition 2.2.4, and that $\sigma^{-1}(b) \in [b]_\mu$. We will construct a $(\mathcal{G}, \mu_y)$-adapted (recall $\widetilde{\mathcal{G}} = \mathcal{G}$) deformation of $\mathscr{G}_0$ which will be the analogue of the Serre–Tate canonical lift in this context.

By Proposition 2.3.3 applied to $\sigma^{-1}(b)$, there exists an element $h \in \mathcal{G}(\breve{\mathbb{Z}}_p)$ such that $h^{-1}b\sigma(h) = \sigma(\dot{t}_{\mu'})$ for some $\mu' \in W_0 \cdot \mu$ with $t_{\mu'}$ σ-straight. Upon modifying the isomorphism (4.3.1.1), we may assume $b = \sigma(\dot{t}_{\mu'})$; we fix this choice of (4.3.1.1) from now on. Let M be the semistandard Levi subgroup of G corresponding to $\nu_{t_{\mu'}} = \nu_{\sigma(t_{\mu'})}$; then $t_{\mu'}$ is central in W_M by Lemma 2.1.7. Let $w \in W_0$ such that $w \cdot \mu = \mu'$ and write $\widetilde{\lambda} := (w \cdot \widetilde{\mu})$; then by Lemma 2.1.9, $\widetilde{\lambda}$ is central in M.

Let

$$\mathcal{M}(\breve{\mathbb{Z}}_p) := M(\breve{\mathbb{Q}}_p) \cap \mathcal{G}(\breve{\mathbb{Z}}_p),$$

which is the $\breve{\mathbb{Z}}_p$-points of a parahoric group scheme $\mathcal{M}$ of M defined over $\mathbb{Z}_p$. Explicitly, we have an identification of apartments $\mathcal{A}(G, S, \breve{\mathbb{Q}}_p) \cong \mathcal{A}(M, S, \breve{\mathbb{Q}}_p)$ and hence we may consider x as an element of $\mathcal{A}(M,S,\breve{\mathbb{Q}}_p)$ which determines the parahoric $\mathcal{M}=\mathcal{M}_x$. Since $\mathcal{M}(\breve{\mathbb{Z}}_p)$ is stable under σ, $\mathcal{M}$ is defined over $\mathbb{Z}_p$.

The kernel of the map $\pi_1(M)\to\pi_1(G)$ is freely generated by a subset of the roots of G which are not roots of M, and which are stable under the action of Γ. Hence $\ker(\pi_1(M)\to\pi_1(G))$ is an induced module for the action of Γ and $\pi_1(M)_I\to\pi_1(G)_I$ has torsion-free kernel. Since $\widetilde{\mathcal{G}}=\mathcal{G}$, it follows from this fact that the image of $\widetilde{\mathcal{M}}(\breve{\mathbb{Z}}_p)$ in $\pi_1(M)_I$ is trivial, and hence $\widetilde{\mathcal{M}}=\mathcal{M}$.

Lemma 4.3.3. *Let K be the field of definition of $\widetilde{\lambda}$. The filtration induced by $\widetilde{\lambda}$ on $\mathbb{D}\otimes_{\breve{\mathbb{Z}}_p}K$ specializes to $\mathrm{Fil}^1\mathbb{D}\otimes_{\breve{\mathbb{Z}}_p}k$.*

Proof. The cocharacter $\widetilde{\lambda}^{-1}$ determines a K-point $s_{\widetilde{\lambda}^{-1}}$ of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_y^{-1}\}}$ and whose image in $\mathbb{M}^{\mathrm{loc}}=\mathrm{Gr}(U)\otimes_{\mathbb{Z}_p}\breve{\mathbb{Z}}_p$ corresponds to the filtration induced by $\widetilde{\lambda}$.

By [SW20, 21.3.1], applied to the torus T, the point $s_{\widetilde{\lambda}^{-1}}$ reduces to the point $\dot{t}_{\mu'}^{-1}\in\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_y^{-1}\}}(k)\subset G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$. By construction of the embedding

$$\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_y^{-1}\}}(k)\hookrightarrow \mathrm{GL}_U(\breve{\mathbb{Q}}_p)/\mathrm{GL}_U(\breve{\mathbb{Z}}_p)$$

in §3.3.20, the filtration on $\mathbb{D}\otimes_{\breve{\mathbb{Z}}_p}k$ corresponding to the image of this element is given by the mod p reduction of $\dot{t}_{\mu'}^{-1}p\mathbb{D}=\sigma^{-1}(b^{-1})p\mathbb{D}$. The proposition follows. □

4.3.4. We extend the tensors $s_{\alpha,0}\in U^{\otimes}$ to a set of tensors $t_{\beta,0}\in U^{\otimes}$ whose stabilizer is $\mathcal{M}$. Viewed in $\mathbb{D}\simeq U\otimes_{\mathbb{Z}_p}\breve{\mathbb{Z}}_p$, the $t_{\beta,0}$ are φ-invariant as $b=\sigma(\dot{t}_{\mu'})\in M(\breve{\mathbb{Q}}_p)$. Since $\widetilde{\lambda}$ is an M-valued cocharacter, we may apply the construction in §4.1 to M and the tensors $t_{\beta,0}$. In particular we have a notion of $(\mathcal{M},\widetilde{\lambda})$-adapted liftings of $\mathscr{G}_0$. It is clear from the definition that any $(\mathcal{M},\widetilde{\lambda})$-adapted lifting is also a $(\mathcal{G},\mu_y)$-adapted lifting.

Let J_b denote the σ-centralizer group for b. It is a reductive group over $\mathbb{Q}_p$ such that

$$J_b(R):=\{g\in G(\breve{\mathbb{Q}}_p\otimes_{\mathbb{Q}_p}R)|g^{-1}b\sigma(g)=b\}$$

for any $\mathbb{Q}_p$-algebra R. There is an action of $J_b(\mathbb{Q}_p)$ on $\mathscr{G}_0$ in the isogeny category. Since $\nu_{g^{-1}b\sigma(g)}=g^{-1}\nu_b g$ for any $g\in G(\breve{\mathbb{Q}}_p)$, it follows that for $b=\sigma(\dot{t}_{\mu'})$, we have $J_b(\mathbb{Q}_p)\subset M(\breve{\mathbb{Q}}_p)$.

Theorem 4.3.5. *Assume we are in the setting of §4.3.2 so that $b=\sigma(\dot{t}_{\mu'})$. Let $K/\breve{\mathbb{Q}}_p$ be an extension over which $\widetilde{\lambda}$ is defined, and suppose $\widetilde{\mathcal{G}}=\mathcal{G}$. There exists a $(\mathcal{G},\mu_y)$-adapted lifting $\mathscr{G}$ to $\mathcal{O}_K$ such that the action of $J_b(\mathbb{Q}_p)$ on $\mathscr{G}_0$ lifts to $\mathscr{G}$ in the isogeny category.*

Proof. Suppose there exists an $(\mathcal{M},\widetilde{\lambda})$-adapted lifting $\mathscr{G}$ of $\mathscr{G}_0$; from the above discussion, we have that $\mathscr{G}$ is also a $(\mathcal{G},\mu_y)$-adapted lifting. By Definition 4.1.4 (2), the filtration on the weakly admissible filtered φ-module associated to $T_p\mathscr{G}^\vee$ is induced by an M-valued cocharacter conjugate to $\widetilde{\lambda}$, hence by $\widetilde{\lambda}$ itself since it is central in M. Since $J_b(\mathbb{Q}_p)\subset M(\breve{\mathbb{Q}}_p)$, the action of $J_b(\mathbb{Q}_p)$ respects the filtration and hence lifts to an action on $\mathscr{G}$ in the isogeny category.

It suffices to show the existence of an $(\mathcal{M},\widetilde{\lambda})$-adapted lifting. This follows from the same argument as [Zho20, Proposition 4.9]; we briefly recall the construction for the convenience of the reader.

We set $\mathfrak{S} := \breve{\mathbb{Z}}_p[[u]]$ and we let $\sigma : \mathfrak{S} \to \mathfrak{S}$ be the map given by the usual Frobenius on $\breve{\mathbb{Z}}_p$ and $u \mapsto u^p$. We define $\mathfrak{M} := \mathbb{D} \otimes_{\sigma^{-1}, \breve{\mathbb{Z}}_p} \mathfrak{S}$, so that $\sigma^*(\mathfrak{M}) \cong \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \mathfrak{S}$, and we let $\mathcal{F} \subset \sigma^*(\mathfrak{M})$ denote the preimage of the filtration induced by $\widetilde{\lambda}$ on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_K$; here the map $\mathfrak{S} \to \mathcal{O}_K$ is induced by sending u to a uniformizer ϖ in $\mathcal{O}_K$. Then $\mathcal{F}$ is a free $\mathfrak{S}$-module and $t_{\beta,0} \in \mathcal{F}^{\otimes}$; this follows from the argument in [KP18, Lemma 3.2.6] using [Ans, Proposition 10.3] in place of [KP18, Proposition 1.4.3]. Moreover the scheme of $\mathfrak{S}$-linear isomorphisms $\underline{\mathrm{Isom}}_{t_{\beta,0}}(\mathcal{F}, \sigma^*(\mathfrak{M}))$ taking $t_{\beta,0}$ to $t_{\beta,0}$ is a trivial $\mathcal{M}$-torsor. Then arguing as in [Zho20, Proposition 4.9], we may construct a morphism $\varphi : \sigma^*(\mathfrak{M}) \to \mathfrak{M}$ satisfying the following properties:

- The map φ gives $\mathfrak{M}$ the structure of an element of BT^{φ} (see [Zho20, §4.1] for the definition of BT^{φ}).
- The canonical identification $\sigma^*(\mathfrak{M}/u\mathfrak{M}) \cong \mathbb{D}$ is an isomorphism of F-crystals.
- φ preserves the tensors $t_{\beta,0}$.

By [Kis10, Theorem 1.4.2], $\mathfrak{M}$ corresponds to a p-divisible group $\mathscr{G}$ over $\mathcal{O}_K$, and the argument of [Zho20, Proposition 4.9] shows that $\mathscr{G}$ is an $(\mathcal{M}, \widetilde{\lambda})$-adapted lifting. □

5. Integral models of Shimura varieties and canonical liftings

In this section we construct integral models for Shimura varieties with parahoric level structure and establish the main geometric properties needed for later applications. These include a version of the local model diagram, which is first proved in some Hodge-type cases in §5.1 and then under some mild assumptions in the abelian type case in §5.2. In §5.3, we prove some functoriality properties concerning the integral models that are needed in §5.4 to define the μ-ordinary locus. We then prove the existence of canonical liftings in §5.4.

5.1. **Integral models.**

5.1.1. For the rest of this paper we fix an algebraic closure $\overline{\mathbb{Q}}$, and for each place v of $\mathbb{Q}$ (including $v = \infty$) an algebraic closure $\overline{\mathbb{Q}}_v$ together with an embedding $i_v : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ (here $\overline{\mathbb{Q}}_\infty \cong \mathbb{C}$).

Let $\mathbf{G}$ be a reductive group over $\mathbb{Q}$ and X a $\mathbf{G}_{\mathbb{R}}$-conjugacy class of homomorphisms

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathbf{G}_{\mathbb{R}}$$

such that $(\mathbf{G}, X)$ is a Shimura datum in the sense of [Del71].

Let c be complex conjugation. Then $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \cong \mathbb{C}^\times \times c^*(\mathbb{C}^\times)$ and we write μ_h for the cocharacter given by

$$\mathbb{C}^\times \to \mathbb{C}^\times \times c^*(\mathbb{C}^\times) \xrightarrow{h} \mathbf{G}(\mathbb{C}).$$

We set $w_h := \mu_h^{-1} \mu_h^{c-1}$.

For the rest of this section, we fix a prime $p > 2$. Let $\mathbb{A}_f$ denote the ring of finite adeles and $\mathbb{A}_f^p$ the ring of prime-to-p adeles which we consider as the subgroup of $\mathbb{A}_f$ with trivial p-component. Let $\mathrm{K}_p \subset \mathbf{G}(\mathbb{Q}_p)$ and $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f)$ be compact open subgroups and write $\mathrm{K} := \mathrm{K}_p\mathrm{K}^p$. Then if K^p is sufficiently small (in fact if K^p is neat, see [Mil92, p. 34]), the set

$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)_{\mathbb{C}} = \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f)/\mathrm{K} \tag{5.1.1.1}$$

can be identified with the complex points of a smooth algebraic variety. The theory of canonical models implies that $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X)_{\mathbb{C}}$ has a model $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X)$ over the reflex field $\mathbf{E}\subset\mathbb{C}$, which is defined to be the field of definition of the conjugacy class $\{\mu_h\}$. We may consider $\mathbf{E}$ as a subfield of $\overline{\mathbb{Q}}$ via the embedding $i_\infty:\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$ and we write $\mathcal{O}_{\mathbf{E}}$ for the ring of integers of $\mathbf{E}$. For a general compact open subgroup K, we take a sufficiently small compact open subgroup K_1^p which is normal in K^p and define the Shimura stack $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X)$ to be the quotient $\mathrm{Sh}_{\mathrm{K}_p\mathrm{K}_1^p}(\mathbf{G},X)/(\mathrm{K}^p/\mathrm{K}_1^p)$; it is a smooth algebraic stack over $\mathbf{E}$.

We also define

$$\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G},X):=\lim_{\leftarrow \mathrm{K}^p}\mathrm{Sh}_{\mathrm{K}_p\mathrm{K}^p}(\mathbf{G},X)$$
$$\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X):=\lim_{\leftarrow \mathrm{K}}\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X);$$

these are pro-varieties equipped with actions of $\mathbf{G}(\mathbb{A}_f^p)$ and $\mathbf{G}(\mathbb{A}_f)$ respectively.

5.1.2. We now assume that $G:=\mathbf{G}_{\mathbb{Q}_p}$ is acceptable; in this case, we say that the Shimura datum $(\mathbf{G},X)$ is acceptable. We also assume that there is an embedding of Shimura data

$$\iota:(\mathbf{G},X)\to(\mathbf{GSp}(V),S^{\pm}).$$

Here $\mathbf{GSp}(V)$ is the group of symplectic similitudes of a $\mathbb{Q}$-vector space V equipped with a perfect alternating bilinear form Ψ, and $S^{\pm}$ is the Siegel double space. Such an ι is called a Hodge embedding.

Let $v|p$ be a prime of $\mathbf{E}$; upon modifying $i_p:\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$, we may assume v is induced by this embedding. We let $\mathcal{O}_{\mathbf{E}_{(v)}}$ denote the localization of $\mathcal{O}_{\mathbf{E}}$ at v, and we write E for the completion of $\mathbf{E}$ at v. We let k_E denote the residue field at v and we fix an algebraic closure k of k_E. We let $\widetilde{\mathcal{G}}:=\widetilde{\mathcal{G}}_x$ for some $x\in\mathcal{B}(G,\mathbb{Q}_p)$ and we write $\mathcal{G}$ for the associated parahoric. We obtain a local model triple $(G,\{\mu_h\},\mathcal{G})$ which is acceptable and of local abelian type. Then we have the attached local model $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$ from §3.3. The Hodge embedding ι is said to be *good* (with respect to $(\mathbf{G},X,\mathcal{G})$) if the corresponding local Hodge embedding $\iota_{\mathbb{Q}_p}:G\to\mathrm{GSp}(V_{\mathbb{Q}_p})$ is good (note that $\iota(\mathbf{G})$ contains the scalars since it contains the image of w_h).

Lemma 5.1.3. *Let $(\mathbf{G},X)$ be an acceptable Shimura datum of Hodge type and $\mathcal{G}$ a parahoric. Assume that $p\nmid|\pi_1(G^{\mathrm{der}})|$ and that the centralizer of a maximal $\breve{\mathbb{Q}}_p$-split torus in G is R-smooth. Then $(\mathbf{G},X,\mathcal{G})$ admits a good Hodge embedding.*

Proof. Our assumptions imply that $\iota_{\mathbb{Q}_p}$ satisfies the conditions in Proposition 3.3.18. The construction there provides us with a good local Hodge embedding ρ' which is easily seen to come from a global Hodge embedding; cf. [KP18, §4.1.5]. □

5.1.4. For the rest of §5.1, we make the following assumptions (recall $p>2$).

(5.1.4.1) $\quad p\nmid|\pi_1(G^{\mathrm{der}})|$ and $(\mathbf{G},X)$ is acceptable of Hodge type.

(5.1.4.2) $\quad$ The centralizer of a maximal $\breve{\mathbb{Q}}_p$-split torus in G is R-smooth.

We set $\widetilde{\mathrm{K}}_p:=\widetilde{\mathcal{G}}(\mathbb{Z}_p)$, $\mathrm{K}_p:=\mathcal{G}(\mathbb{Z}_p)$, and we let $\widetilde{\mathrm{K}}:=\widetilde{\mathrm{K}}_p\mathrm{K}^p$, $\mathrm{K}:=\mathrm{K}_p\mathrm{K}^p$. Let $\iota:(\mathbf{G},X)\to(\mathbf{GSp}(V),S^{\pm})$ be a good Hodge embedding which exists by Lemma 5.1.3 and let $V_{\mathbb{Z}_p}\subset V_{\mathbb{Q}_p}$ be a $\mathbb{Z}_p$-lattice such that $G\to\mathrm{GL}(V_{\mathbb{Q}_p})$ is good with respect to $V_{\mathbb{Z}_p}$. Upon scaling, we may assume $V_{\mathbb{Z}_p}$ is contained in the dual lattice $V_{\mathbb{Z}_p}^\vee$.

Let $V_{\mathbb{Z}_{(p)}}=V_{\mathbb{Z}_p}\cap V$. We write $G_{\mathbb{Z}_{(p)}}$ for the Zariski closure of $\mathbf{G}$ in $\mathrm{GL}(V_{\mathbb{Z}_{(p)}})$; then $G_{\mathbb{Z}_{(p)}}\otimes_{\mathbb{Z}_{(p)}}\mathbb{Z}_p\cong\widetilde{\mathcal{G}}$. Let $\mathrm{K}'=\mathrm{K}'_p\mathrm{K}'^p$ where K'_p is the stabilizer in $\mathrm{GSp}(V_{\mathbb{Q}_p})$

of the lattice $V_{\mathbb{Z}_p}$ and $\mathrm{K}'^p \subset \mathbf{GSp}(\mathbb{A}_f^p)$ is a compact open subgroup. The choice of $V_{\mathbb{Z}_{(p)}}$ gives rise to an interpretation of $\mathrm{Sh}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm})$ as a moduli stack of abelian varieties up to prime-to-p isogeny and hence an integral model $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm})$ over $\mathbb{Z}_{(p)}$, see [KP18, §4] and [Zho20, §6].

Assume that K^p is a neat compact open subgroup. By [Kis10, Lemma 2.1.2], we can choose K'^p such that ι induces a closed immersion

$$\mathrm{Sh}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X) \hookrightarrow \mathrm{Sh}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm}) \otimes_{\mathbb{Q}} \mathbf{E}.$$

Let $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)^-$ be the Zariski closure of $\mathrm{Sh}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ inside $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm}) \otimes_{\mathbb{Z}_{(p)}} \mathcal{O}_{\mathbf{E}_{(v)}}$, and $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ the normalization of $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)^-$. We also define the pro-scheme

$$\mathscr{S}_{\widetilde{\mathrm{K}}_p}(\mathbf{G}, X) := \lim_{\leftarrow \mathrm{K}^p} \mathscr{S}_{\widetilde{\mathrm{K}}_p \mathrm{K}^p}(\mathbf{G}, X).$$

The $\mathbf{G}(\mathbb{A}_f^p)$-action on $\mathrm{Sh}_{\widetilde{\mathrm{K}}_p}(\mathbf{G}, X)$ extends to $\mathscr{S}_{\widetilde{\mathrm{K}}_p}(\mathbf{G}, X)$. Hence we may define $\mathscr{S}_{\widetilde{\mathrm{K}}_p \mathrm{K}^p}(\mathbf{G}, X)$ for a general (not necessarily neat) compact open subgroup $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f)$ as the quotient stack $\mathscr{S}_{\widetilde{\mathrm{K}}_p}(\mathbf{G}, X)/\mathrm{K}^p$. Alternatively, we may take a compact open subgroup $\mathrm{K}_1^p \subset \mathrm{K}^p$ which is neat and normal in K^p, and define $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ as the quotient of $\mathscr{S}_{\widetilde{\mathrm{K}}_p \mathrm{K}_1^p}(\mathbf{G}, X)$ under the action of the finite group $\mathrm{K}^p/\mathrm{K}_1^p$.

5.1.5. In order to understand the local structure of $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$, we need to introduce Hodge cycles. By [Kis10, Proposition 1.3.2], the subgroup $G_{\mathbb{Z}_{(p)}}$ is the stabilizer of a collection of tensors $s_\alpha \in V_{\mathbb{Z}_{(p)}}^{\otimes}$. Let $h : \mathcal{A} \to \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ denote the pullback of the universal abelian scheme on $\mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}, S^{\pm})$ and let $V_{\mathrm{B}} := R^1 h_{\mathrm{an},*}\mathbb{Z}_{(p)}$, where h_{an} is the map of complex analytic spaces associated to h. Since the tensors s_α are $\mathbf{G}$-invariant, they give rise to sections $s_{\alpha,\mathrm{B}} \in V_B^{\otimes}$. We also let $\mathcal{V} = R^1 h_* \Omega^\bullet$ be the relative de Rham cohomology of $\mathcal{A}$. Using the de Rham isomorphism, the $s_{\alpha,B}$ give rise to a collection of Hodge cycles $s_{\alpha,\mathrm{dR}} \in \mathcal{V}_{\mathbb{C}}^{\otimes}$, where $\mathcal{V}_{\mathbb{C}}$ is the complex analytic vector bundle associated to $\mathcal{V}$. By [Kis10, Corollary 2.2.2], these tensors are defined over $\mathbf{E}$.

Similarly for a finite prime $\ell \neq p$, we let $\mathcal{V}_\ell = \mathcal{V}_\ell(\mathcal{A}) = R^1 h_{\text{ét}*}\mathbb{Q}_\ell$ and $\mathcal{V}_p = \mathcal{V}_p(\mathcal{A}) = R^1 h_{\eta,\text{ét}*}\mathbb{Z}_p$ where h_η is the generic fiber of h. Using the étale-Betti comparison isomorphism, we obtain tensors $s_{\alpha,\ell} \in \mathcal{V}_\ell^{\otimes}$ and $s_{\alpha,p} \in \mathcal{V}_p^{\otimes}$.

For T an $\mathcal{O}_{\mathbf{E}_{(v)}}$-scheme and $x \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)(T)$, we write $\mathcal{A}_x$ for the pullback of $\mathcal{A}$ to x, and for $* = \ell$ or dR, we write $s_{\alpha,*,x}$ for the pullback of $s_{\alpha,*}$ to x. Similarly, for T an $\mathbf{E}$-scheme (resp. $\mathbb{C}$-scheme) and $x \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)(T)$, we write $s_{\alpha,p,x}$ (resp. $s_{\alpha,B,x}$) for the pullback of $s_{\alpha,p}$ (resp. $s_{\alpha,B}$) to x.

For T an $\mathcal{O}_{\mathbf{E}_{(v)}}$-scheme, an element $x \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)(T)$ corresponds to a triple $(\mathcal{A}_x, \lambda, \epsilon_{\mathrm{K}'}^p)$, where λ is a weak polarization (cf. [Zho20, §6.3]) and $\epsilon_{\mathrm{K}'}^p$ is a section of the étale sheaf $\underline{\mathrm{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}_x), V_{\mathbb{A}_f^p})/\mathrm{K}'^p$; here

$$\widehat{V}(\mathcal{A}_x) = \varprojlim_{p \nmid n} \mathcal{A}_x[n]$$

is the adelic prime-to-p Tate module of $\mathcal{A}_x$. As in [Kis10, §3.4.2], $\epsilon_{\mathrm{K}'}^p$ can be promoted to a section

$$\epsilon_{\mathrm{K}}^p \in \Gamma(T, \underline{\mathrm{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}_x), V_{\mathbb{A}_f^p})/\mathrm{K}^p)$$

which takes $s_{\alpha,\ell,x}$ to s_α for $\ell \neq p$.

5.1.6. Recall that k is an algebraic closure of k_E and $\breve{\mathbb{Q}}_p = W(k)[1/p]$. Let $\overline{x} \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)(k)$ and $\widetilde{x} \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)(\mathcal{O}_K)$ a point lifting $\overline{x}$, where $K/\breve{\mathbb{Q}}_p$ is a finite extension.

Let $\mathscr{G}_{\widetilde{x}}$ denote the p-divisible group associated to $\mathcal{A}_{\widetilde{x}}$ and $\mathscr{G}_{\overline{x}}$ its special fiber; we let $\mathbb{D} := \mathbb{D}(\mathscr{G}_{\overline{x}})(\breve{\mathbb{Z}}_p)$. Then $T_p\mathscr{G}_{\widetilde{x}}^\vee$ is identified with $\mathrm{H}^1_{\text{ét}}(\mathcal{A}_{\widetilde{x},\overline{K}}, \mathbb{Z}_p)$ and we obtain $\mathrm{Gal}(\overline{K}/K)$-invariant tensors $s_{\alpha,p,\widetilde{x}} \in T_p\mathscr{G}_{\widetilde{x}}^{\vee\otimes}$ whose stabilizer can be identified with $\widetilde{\mathcal{G}}$. Let $s_{\alpha,0,\widetilde{x}} \in \mathbb{D}[\frac{1}{p}]^\otimes$ denote the tensors corresponding to $s_{\alpha,p,\widetilde{x}}$ via the p-adic comparison isomorphism. By [KPS, Proposition 1.3.7], $s_{\alpha,0,\widetilde{x}}$ are independent of the choice of lifting $\widetilde{x} \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)(\mathcal{O}_K)$. We may therefore denote them by $s_{\alpha,0,\overline{x}}$.

By Proposition 4.2.2, we have $s_{\alpha,0,\overline{x}} \in \mathbb{D}^\otimes$ and there is a $\breve{\mathbb{Z}}_p$-linear bijection

$$V^\vee_{\mathbb{Z}_p} \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong T_p\mathscr{G}_{\widetilde{x}}^\vee \otimes_{\mathbb{Z}_p} \breve{\mathbb{Z}}_p \cong \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Z}}_p \tag{5.1.6.1}$$

taking s_α to $s_{\alpha,0,\overline{x}}$. The filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} K$ corresponding to $\mathscr{G}_{\widetilde{x}}$ is induced by a G-valued cocharacter conjugate to μ_h^{-1}. By a result of Blasius and Wintenberger [Bla91], $s_{\alpha,\mathrm{dR},\widetilde{x}} \in \widetilde{x}^*(\mathcal{V})^\otimes \cong \mathbb{D}(\mathscr{G}_{\widetilde{x}})(\mathcal{O}_K)^\otimes$ corresponds to $s_{\alpha,p,\widetilde{x}}$ via the p-adic comparison isomorphism. Hence $s_{\alpha,\mathrm{dR},\widetilde{x}}$ may be identified with the image of the elements $\widetilde{s}_\alpha \in \mathbb{D}(\mathscr{G}_{\widetilde{x}})(\widehat{W}(\mathcal{O}_K))^\otimes$ of Proposition 4.2.2 inside $\mathbb{D}(\mathscr{G}_{\widetilde{x}})(\mathcal{O}_K)^\otimes$. The same Proposition implies that there is an $\mathcal{O}_K$-linear bijection

$$\mathbb{D}(\mathscr{G}_{\widetilde{x}})(\mathcal{O}_K) \cong \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_K$$

taking $s_{\alpha,\mathrm{dR},\widetilde{x}}$ to $s_{\alpha,0,\overline{x}}$ and which lifts the identity over k. Thus there is a G-valued cocharacter μ_y which is G-conjugate to μ_h^{-1} and which induces a filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_K$ lifting the filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} k$. We may therefore define the notion of $(\widetilde{\mathcal{G}}, \mu_y)$-adapted liftings as in §4 and it follows from Proposition 4.2.2 that $\mathscr{G}_{\widetilde{x}}$ is a $(\widetilde{\mathcal{G}}, \mu_y)$-adapted lifting.

5.1.7. Note that $G \subset \mathrm{GL}(V_{\mathbb{Q}_p})$ contains the scalars. It follows that under our assumptions, conditions (4.1.5.1)–(4.1.5.2) are satisfied. We let $P \subset \mathrm{GL}(\mathbb{D})$ be a parabolic lifting P_0 as in §4.1. We obtain formal local models $\widehat{\mathbb{M}}^{\mathrm{loc}} = \mathrm{Spf} A$ and $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\widetilde{\mathcal{G}}} = \mathrm{Spf} A_{\widetilde{\mathcal{G}}} \cong \widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$, and the filtration corresponding to μ_y is given by a point $y : A_{\widetilde{\mathcal{G}}} \to \mathcal{O}_K$.

Proposition 5.1.8. *Assume* K^p *is neat. Let* $\widehat{U}_{\overline{x}}$ *be the completion of* $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)^-$ *at the image of* $\overline{x}$.

1. *(1)* $\widehat{U}_{\overline{x}}$ *can be identified with a closed subspace of* $\mathrm{Spf} A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$ *containing* $\mathrm{Spf} A_{\widetilde{\mathcal{G}}}$.
2. *(2) A deformation* $\mathscr{G}$ *of* $\mathscr{G}_{\overline{x}}$ *corresponds to a point on the irreducible component of* $\widehat{U}_{\overline{x}}$ *containing* $\widetilde{x}$ *if and only if* $\mathscr{G}$ *is* $(\widetilde{\mathcal{G}}, \mu_y)$*-adapted.*
3. *(3) Let* $\overline{x}' \in \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)(k)$ *whose image in* $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)^-(k)$ *coincides with that of* $\overline{x}$. *Then* $s_{\alpha,0,\overline{x}'} = s_{\alpha,0,\overline{x}} \in \mathbb{D}^\otimes$ *if and only if* $\overline{x} = \overline{x}'$.

Proof. Since the conditions (4.1.5.1)–(4.1.5.2) are satisfied, we may apply the construction of Proposition 4.1.7; this allows us to view $\mathrm{Spf} A$ as a versal deformation space for $\mathscr{G}_{\overline{x}}$ and hence we obtain a map $\Theta : \widehat{U}_{\overline{x}} \to \mathrm{Spf} A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$ such that the universal p-divisible group over $\mathrm{Spf} A \otimes_{\breve{\mathbb{Z}}_p} \mathcal{O}_{\breve{E}}$ pulls back to the one over $\widehat{U}_{\overline{x}}$ arising from the universal abelian scheme over $\widehat{U}_{\overline{x}}$. The map Θ is a closed immersion by the Serre–Tate theorem.

Let $Z \subset \widehat{U}_{\overline{x}}$ denote the irreducible component of $\widehat{U}_{\overline{x}}$ containing $\widetilde{x}$. Let K' be a finite extension of $\breve{E}$ and let $\widetilde{x}' \in Z(K')$. Then the tensors $s_{\alpha,p,\widetilde{x}'}$ correspond to $s_{\alpha,0,\overline{x}}$ under the p-adic comparison isomorphism. Moreover the filtration on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} K'$ corresponding to $\mathscr{G}_{\widetilde{x}'}$ is induced by a G-valued cocharacter conjugate to μ_h^{-1}, and hence conjugate to μ_y. By Proposition 4.2.4, $\mathscr{G}_{\widetilde{x}'}$ is a $(\widetilde{\mathcal{G}}, \mu_y)$-adapted deformation of $\mathscr{G}_{\overline{x}}$ and hence $\widetilde{x}'$ corresponds to a point of $\mathrm{Spf} A_{\widetilde{\mathcal{G}}}$. Since this is true for any $\widetilde{x}'$, it follows that $\Theta|_Z$ factors through $\mathrm{Spf} A_{\widetilde{\mathcal{G}}}$. Since Z and $\mathrm{Spf} A_{\widetilde{\mathcal{G}}}$ have the same dimension, it follows that $Z \cong \mathrm{Spf} A_{\widetilde{\mathcal{G}}}$. We thus obtain (1) and (2).

One direction of (3) is clear. For the other direction, let $\widetilde{x}' \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)(\mathcal{O}_{K'})$ be a lift of $\overline{x}'$. Then by Proposition 4.2.2, $s_{\alpha,0,\overline{x}'}$ arises from the specialization of tensors $\widetilde{s}_\alpha \in \mathbb{D}(\mathscr{G}_{\widetilde{x}'})(\widehat{W}(\mathcal{O}_K))$. By assumption, we have $s_{\alpha,0,\overline{x}'} = s_{\alpha,0,\overline{x}}$. It follows that $\mathscr{G}_{\widetilde{x}'}$ corresponds to a $(\widetilde{\mathcal{G}}, \mu_y)$-adapted lifting and hence to a point of $\mathrm{Spf} A_{\widetilde{\mathcal{G}}}$. By what we have seen, $\widetilde{x}'$ corresponds to a point in the same irreducible component $Z \subset \widehat{U}_{\overline{x}}$ containing $\widetilde{x}$ and hence $\overline{x} = \overline{x}'$. □

5.1.9. The above description of the local structure of $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ may be globalized as follows.

Theorem 5.1.10. *(1) $\mathscr{S}_{\widetilde{\mathrm{K}}_p}(\mathbf{G}, X)$ is an $\mathcal{O}_{\mathbf{E}_{(v)}}$-flat, $\mathbf{G}(\mathbb{A}_f^p)$-equivariant extension of $\mathrm{Sh}_{\widetilde{\mathrm{K}}_p}(\mathbf{G}, X)$.*

(2) Assume K^p is neat. Let $\widehat{U}_{\overline{x}}$ be the completion of $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ at some k-point $\overline{x}$. Then there exists a point $\overline{z} \in \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}(k)$ such that $\widehat{U}_{\overline{x}}$ is isomorphic to the completion of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$ at $\overline{z}$.

(3) $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ fits in a local model diagram

$$\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)_{\mathcal{O}_E} \xleftarrow{\ \pi\ } \widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)_{\mathcal{O}_E} \xrightarrow{\ q\ } \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$$

where π is a $\widetilde{\mathcal{G}}$-torsor and q is smooth of relative dimension $\dim G$.

Proof. (1) is clear and (2) follows from Proposition 5.1.8.

For (3), we first assume K^p is neat. Recall we have the vector bundle $\mathcal{V}$ over $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$ corresponding to the de Rham cohomology of the universal abelian variety over $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$. Its generic fiber $\mathcal{V}_{\mathbf{E}}$ is equipped with tensors $s_{\alpha,\mathrm{dR}} \in \mathcal{V}_{\mathbf{E}}^{\otimes}$ and these extend to $\mathcal{V}$ by the same argument as [KP18, Proposition 4.2.6]. Moreover the argument of *loc. cit.* also shows that the scheme classifying isomorphisms $f : V^\vee_{\mathcal{O}_{\mathbf{E}_{(v)}}} \cong \mathcal{V}$ which take s_α to $s_{\alpha,\mathrm{dR}}$ is a $\widetilde{\mathcal{G}}$-torsor $\widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)$.

Let (x, f) be an S-point of $\widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)_{\mathcal{O}_E}$. The map q is defined by sending (x, f) to the inverse image $f^{-1}(\mathcal{F}) \subset V^\vee_{\mathcal{O}_{\mathbf{E}_{(v)}}} \otimes_{\mathcal{O}_{\mathbf{E}(v)}} \mathcal{O}_S$ of the Hodge filtration $\mathcal{F} \subset \mathcal{V}_x$. This gives us a map $\widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)_{\mathcal{O}_E} \to \mathrm{Gr}(V^\vee_{\mathbb{Z}_p}) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ which factors through $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$ by the argument of [KP18, Theorem 4.2.7], which also shows that q is smooth.

Now for a general (not necessarily neat) K^p, we let $\mathrm{K}^p_1 \subset \mathrm{K}^p$ be a neat compact open subgroup which is normal in K^p. The action of $\mathrm{K}^p/\mathrm{K}^p_1$ on $\mathscr{S}_{\widetilde{\mathrm{K}}_p\mathrm{K}^p_1}(\mathbf{G}, X)$

naturally extends to $\widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}_p\mathrm{K}_1^p}(\mathbf{G},X)$, and the map

$$q_1 : \widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}_p\mathrm{K}_1^p}(\mathbf{G},X)_{\mathcal{O}_E} \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$$

is compatible with this action. We thus obtain a diagram of stacks
(5.1.10.1)

$$\begin{array}{ccccc} & \widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}_p\mathrm{K}_1^p}(\mathbf{G},X)_{\mathcal{O}_E} & \xrightarrow{\widetilde{p}} & \widetilde{\mathscr{S}}_{\widetilde{\mathrm{K}}}(\mathbf{G},X)_{\mathcal{O}_E} & \\ {}^{\pi_1}\swarrow & & {}^{\pi}\swarrow & & \searrow^{q} \\ \mathscr{S}_{\widetilde{\mathrm{K}}_p\mathrm{K}_1^p}(\mathbf{G},X)_{\mathcal{O}_E} & \xrightarrow{p} & \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G},X)_{\mathcal{O}_E} & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}} \end{array}$$

as desired. □

5.1.11. We now use the above to study integral models for parahoric level structure. Let $\mathbf{G}^{\mathrm{sc}}$ denote the simply connected cover of $\mathbf{G}^{\mathrm{der}}$ and we set $\mathbf{C} := \ker(\mathbf{G}^{\mathrm{sc}} \to \mathbf{G}^{\mathrm{der}})$. For $c \in \mathrm{H}^1(\mathbb{Q},\mathbf{C})$ and ℓ a finite prime, we write c_ℓ for the image of c in $\mathrm{H}^1(\mathbb{Q}_\ell,\mathbf{C})$. We introduce the following assumption.

(5.1.11.1) If $c \in \mathrm{H}^1(\mathbb{Q},\mathbf{C})$ satisfies $c_\ell = 0$ for all $\ell \neq p$, then $c_p = 0$.

There is a natural finite map of Shimura varieties $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X) \to \mathrm{Sh}_{\widetilde{\mathrm{K}}}(\mathbf{G},X)$ and we define the integral model for parahoric level $\mathscr{S}_{\mathrm{K}}(\mathbf{G},X)$ to be the normalization of $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G},X)$ inside $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G},X)$. We similarly write $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G},X)$ for the inverse limit over the prime-to-p levels. The discussion in [KP18, §4.3] extends verbatim to the current situation and we obtain the following proposition; cf. [KP18, Proposition 4.3.7, Corollary 4.3.9].

Proposition 5.1.12. *Assume (5.1.11.1) is satisfied.*

(1) The covering $\mathscr{S}_{\mathrm{K}}(\mathbf{G},X) \to \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G},X)$ is étale, and for K^p sufficiently small, this covering splits over an unramified extension.
(2) The geometrically connected components of $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G},X)$ are defined over the maximal extension $\mathbf{E}^p$ of $\mathbf{E}$ unramified at all primes above p.

□

5.2. **Integral models for Shimura varieties of abelian type.** We now use the previous results to construct integral models for Shimura varieties of abelian type. In particular, this will allow us to construct integral models for general Hodge-type Shimura varieties without the assumption $p \nmid |\pi_1(G^{\mathrm{der}})|$. This last case is all that is needed for our main application on ℓ-independence. However, since the general abelian type case is no more difficult, we also include this case for completeness. As many of the arguments are exactly the same as in [KP18, §4], in what follows we will refer to relevant statements in [KP18] if the argument in *loc. cit.* carries over directly and only give details for those points which do not.

5.2.1. We keep the notation of §5.1, so that $(\mathbf{G},X)$ is a Shimura datum of Hodge type and $p > 2$ is a fixed prime; we set $G = \mathbf{G}_{\mathbb{Q}_p}$. As before, we let $\mathcal{G} = \mathcal{G}_x$ be a parahoric corresponding to a point $x \in \mathcal{B}(G,\mathbb{Q}_p)$, and $\widetilde{\mathcal{G}} = \widetilde{\mathcal{G}}_x$ the associated Bruhat–Tits stabilizer. Assume that $(\mathbf{G},X)$ satisfies the following conditions.

- $p \nmid |\pi_1(G^{\mathrm{der}})|$ and $(\mathbf{G},X)$ is acceptable.
- $\mathbf{G}$ satisfies (5.1.11.1).

- Both the center Z of $G := \mathbf{G}_{\mathbb{Q}_p}$ and the centralizer of a maximal $\breve{\mathbb{Q}}_p$-split torus in G are R-smooth tori.

Let $(\mathbf{G}_2, X_2)$ be a Shimura datum which is equipped with a central isogeny $\alpha : \mathbf{G}^{\mathrm{der}} \to \mathbf{G}_2^{\mathrm{der}}$ inducing an isomorphism $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$, in particular $(\mathbf{G}_2, X_2)$ is acceptable. The parahoric $\mathcal{G}$ determines a parahoric $\mathcal{G}_2$ for $G_2 := \mathbf{G}_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and we set $\mathrm{K}_{2,p} := \mathcal{G}_2(\mathbb{Z}_p)$. We write $\mathbf{E}_2$ for the reflex field of $(\mathbf{G}_2, X_2)$ and we let $\mathbf{E}' := \mathbf{E}.\mathbf{E}_2$. Our choice of embedding $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ induces a place v' (resp. v_2) of $\mathbf{E}'$ (resp. $\mathbf{E}_2$) and we set $E' := \mathbf{E}'_{v'}$ and $E_2 := \mathbf{E}_{2,v_2}$ to be the completions.

Fix a connected component $X^+ \subset X$. By real approximation, upon modifying the isomorphism $\mathbf{G}^{\mathrm{ad}} \cong \mathbf{G}_2^{\mathrm{ad}}$ by an element of $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})$, we may assume that the image of $X_2 \subset X_2^{\mathrm{ad}}$ contains the image of X^+. We write X_2^+ for X^+ viewed as a subset of X_2. We denote by $\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \subset \mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)$ and $\mathrm{Sh}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+ \subset \mathrm{Sh}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ the geometrically connected components corresponding to X^+ and X_2^+. These are defined over extensions of $\mathbf{E}$ and $\mathbf{E}'$ respectively, which are unramified at primes above p by Proposition 5.1.12 and our assumption on $\mathbf{G}$. The identification $X_2^+ \simeq X^+$ induces a finite map

$$\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \to \mathrm{Sh}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+ \tag{5.2.1.1}$$

Let x^{ad} be the image of x in $\mathcal{B}(G^{\mathrm{ad}}, \mathbb{Q}_p)$ and we denote by $\mathcal{G}^{\mathrm{ad}}$ the parahoric model of G^{ad} corresponding to x^{ad}. We then have the following generalization of [KP18, Corollary 4.6.18].

Proposition 5.2.2. *Under the assumptions above, there is a $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant extension of $\mathrm{Sh}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ to an $\mathcal{O}_{E'}$-scheme with $\mathbf{G}_2(\mathbb{A}_f^p)$-action $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ such that*

(1) *For any discrete valuation ring R of mixed characteristic the map*

$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)(R) \to \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X)(R[1/p])$$

is a bijection.

(2) *The map (5.2.1.1) induces a finite map of $\mathcal{O}_{E'^{\mathrm{ur}}}$-schemes*

$$\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \to \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+,$$

where $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+$ denotes the closure of $\mathrm{Sh}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+$ in the $\mathcal{O}_{E'^{\mathrm{ur}}}$-scheme $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'^{\mathrm{ur}}}}$, and similarly for $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+$.

(3) *If $\tilde{\mathcal{G}} = \mathcal{G}$, then there exists a diagram*

$$\begin{array}{ccc} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2) & \\ \swarrow^{\pi} & & \searrow^{q} \\ \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2) & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'} \end{array} \tag{5.2.2.1}$$

where π is a $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant $\mathcal{G}^{\mathrm{ad}}$-torsor and q is smooth of relative dimension $\dim \mathbf{G}^{\mathrm{ad}}$, and $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant, when $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}}$ is equipped with the trivial $\mathbf{G}_2(\mathbb{A}_f^p)$-action.

Proof. This can be deduced from Theorem 5.1.10, as in [KP18, §4.4-4.6], noting that we have an isomorphism $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'} \cong \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'}$ by Corollary

3.3.11 applied to the two morphisms $G \to G^{\mathrm{ad}}$ and $G_2 \to G_2^{\mathrm{ad}}$. We explain only how the assumption of R-smoothness of Z is used.

Let $\mathbf{G}_{\mathbb{Z}_{(p)}}$ (resp. $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$) denote the $\mathbb{Z}_{(p)}$-model of $\mathbf{G}$ (resp. $\mathbf{G}^{\mathrm{ad}}$) associated to $\mathcal{G}$ (resp. $\mathcal{G}^{\mathrm{ad}}$) via the construction in [KP18, §4.6.1]. Note that these groups are denoted $G^{\circ}_{\mathbb{Z}_{(p)}}$ and $G^{\mathrm{ad}\circ}_{\mathbb{Z}_{(p)}}$ respectively in [KP18, §4.6]. Let $\mathbf{Z}$ denote the center of $\mathbf{G}$ and $\mathbf{Z}_{\mathbb{Z}_{(p)}}$ the closure of $\mathbf{Z}$ in $\mathbf{G}_{\mathbb{Z}_{(p)}}$. By Proposition 2.4.14, the assumption of R-smoothness on $Z = \mathbf{Z}_{\mathbb{Q}_p}$ and descent implies that the natural map $\mathbf{G}_{\mathbb{Z}_{(p)}}/\mathbf{Z}_{\mathbb{Z}_{(p)}} \to \mathbf{G}^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}$ is an isomorphism. This gives us the analogue of [KP18, Lemma 4.6.2(2)], and allows us to carry out the constructions of §4.6 of *loc. cit.* □

Let $\mathrm{K}_2^p \subset \mathbf{G}_2(\mathbb{A}_f^p)$ be a compact open subgroup, and we write $\mathrm{K}_2 := \mathrm{K}_{2,p}\mathrm{K}_2^p \subset \mathbf{G}_2(\mathbb{A}_f)$. Taking the quotient of the diagram (5.2.2.1) by K_2^p, we obtain

$$q : \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}_2}(\mathbf{G}_2, X_2) \to \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}} \otimes_{\mathcal{O}_E} \mathcal{O}_{E'},$$

a smooth morphism of $\mathcal{O}_{E'}$-stacks of relative dimension $\dim \mathbf{G}^{\mathrm{ad}}$.

5.2.3. We recall some features of the construction in Proposition 5.2.2 which will be needed later. For a subgroup $H \subset \mathbf{G}(\mathbb{R})$, we write H_+ for the preimage of $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$, the connected component of the identity in $\mathbf{G}^{\mathrm{ad}}(\mathbb{R})$. We write $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+$ (resp. $\mathbf{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$) for $\mathbf{G}^{\mathrm{ad}}(\mathbb{Q}) \cap \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$ (resp. $\mathbf{G}^{\mathrm{ad}}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)}) \cap \mathbf{G}^{\mathrm{ad}}(\mathbb{R})^+$). We let $\mathbf{Z}(\mathbb{Q})^-$ and $\mathbf{G}(\mathbb{Q})_+^-$ denote the closures of $\mathbf{Z}(\mathbb{Q})$ and $\mathbf{G}(\mathbb{Q})_+$ in $\mathbf{G}(\mathbb{A}_f)$, respectively. We let $\mathbf{Z}(\mathbb{Z}_{(p)})^-$ and $\mathbf{G}(\mathbb{Z}_{(p)})_+^-$ denote the closures of $\mathbf{Z}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})$ and $\mathbf{G}_{\mathbb{Z}_{(p)}}(\mathbb{Z}_{(p)})_+$ in $\mathbf{G}(\mathbb{A}_f^p)$, respectively. As in [KP18, §4.5.6], we set

$$\mathscr{A}(\mathbf{G}) := \mathbf{G}(\mathbb{A}_f)/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Q})_+/\mathbf{Z}(\mathbb{Q})} \mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+$$

$$\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}}) := \mathbf{G}(\mathbb{A}_f^p)/\mathbf{Z}(\mathbb{Z}_{(p)})^- *_{\mathbf{G}(\mathbb{Z}_{(p)})_+/\mathbf{Z}(\mathbb{Z}_{(p)})} \mathbf{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+,$$

and as in [KP18, §4.6.3], we set

$$\mathscr{A}(\mathbf{G})^\circ := \mathbf{G}(\mathbb{Q})_+^-/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Q})_+/\mathbf{Z}(\mathbb{Q})} \mathbf{G}^{\mathrm{ad}}(\mathbb{Q})^+$$

$$\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ := \mathbf{G}(\mathbb{Z}_{(p)})_+^-/\mathbf{Z}(\mathbb{Z}_{(p)})^- *_{\mathbf{G}(\mathbb{Z}_{(p)})_+/\mathbf{Z}(\mathbb{Z}_{(p)})} \mathbf{G}^{\mathrm{ad}}(\mathbb{Z}_{(p)})^+.$$

We refer to *loc. cit.* §4.5.6 for the definition of the $*$ product. We obtain an $\mathscr{A}(\mathbf{G})$-action (resp. $\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})$-action) on $\mathrm{Sh}(\mathbf{G}, X)$ (resp. $\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)$). Here, the assumption that the center of G is an R-smooth torus implies that the $\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})$-action on $\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)$ extends to an $\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})$-action on $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)$. As in [KP18, §4.6.12], the natural map

$$\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ \backslash \mathscr{A}(\mathbf{G}_{2,\mathbb{Z}_{(p)}}) \to \mathscr{A}(\mathbf{G})^\circ \backslash \mathscr{A}(\mathbf{G}_2)/\mathrm{K}_{2,p} \tag{5.2.3.1}$$

is an injection. We fix a set $J \subset \mathbf{G}_2(\mathbb{Q}_p)$ which maps bijectively to a set of coset representatives for the image of $\mathscr{A}(\mathbf{G}_{2,\mathbb{Z}_{(p)}})$ in $\mathscr{A}(\mathbf{G})^\circ \backslash \mathscr{A}(\mathbf{G}_2)/\mathrm{K}_{2,p}$. A calculation shows that J is a finite set. Then $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ is constructed as

$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2) = \left[[\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \times \mathscr{A}(\mathbf{G}_{2,\mathbb{Z}_{(p)}})]/\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ \right]^{|J|}. \tag{5.2.3.2}$$

5.2.4. Let H be a simple, adjoint, reductive group over $\mathbb{R}$, which is of classical type, and is associated to a Hermitian symmetric domain; in particular $H(\mathbb{R})$ is not compact. Thus H is of type $A, B, C, D^{\mathbb{R}}, D^{\mathbb{H}}$ in the classification of [Del79, 1.3.9], with the type A case including unitary groups of any signature $U(p,q)$ with $p, q \neq 0$. We set $H^\sharp = H^{\mathrm{sc}}$, the simply connected cover of H, unless H is of type $D^{\mathbb{H}}$, in which case we set $H^\sharp$ equal to the image of H^{sc} in the representation corresponding to the standard representation of the orthogonal group.

Now let F be a totally real field, and $\mathbf{H}$ a simple, adjoint reductive group of classical type over F. Assume that

- for every embedding $\sigma : \mathrm{F} \hookrightarrow \mathbb{R}$, $\mathbf{H} \otimes_{\sigma,\mathrm{F}} \mathbb{R}$ is either compact or associated to a Hermitian symmetric domain.
- $\mathbf{H} \otimes_{\sigma,\mathrm{F}} \mathbb{R}$ is non-compact for some σ.
- If $\mathbf{H}$ is of type D, then for those σ such that $\mathbf{H} \otimes_{\sigma,F} \mathbb{R}$ is non-compact, the associated Hermitian symmetric domain does not depend on σ. That is, it is always of type $D^{\mathbb{R}}$ or always of type $D^{\mathbb{H}}$.

We define $\mathbf{H}^\sharp$ to be $\mathbf{H}^{\mathrm{sc}}$ unless $\mathbf{H}$ is of type D, in which case we define $\mathbf{H}^\sharp$ to be the unique quotient of $\mathbf{H}^{\mathrm{sc}}$ such that $\mathbf{H}^\sharp \otimes_{\sigma,F} \mathbb{R} = (\mathbf{H} \otimes_{\sigma,F} \mathbb{R})^\sharp$ whenever $\mathbf{H} \otimes_{\sigma,F} \mathbb{R}$ is non-compact.

Now suppose $\mathbf{H}$ is a reductive group over F, with $\mathbf{H}^{\mathrm{ad}} = \prod_{i=1}^{s} \mathbf{H}_i$ where each $\mathbf{H}_i$ is a simple, adjoint reductive group of classical type over F satisfying the three conditions above. Then we set $\mathbf{H}^\sharp = \prod_{i=1}^{s} \mathbf{H}_i^\sharp$.

Now let $(\mathbf{H}, Y)$ be a Shimura datum such that $(\mathbf{H}^{\mathrm{ad}}, Y^{\mathrm{ad}})$ is of abelian type. Recall [Del79, 1.3.10, 2.3.10] that in this case the three conditions above are satisfied, so $\mathbf{H}^\sharp$ is well defined [3], and $(\mathbf{H}, Y)$ is of abelian type if and only if $\mathbf{H}^{\mathrm{der}}$ is a quotient of $\mathbf{H}^\sharp$.

5.2.5. Proposition 5.2.2 shows that we can construct good integral models for Shimura data $(\mathbf{G}_2, X_2)$ of abelian type provided we can relate it to a Shimura datum $(\mathbf{G}, X)$ of Hodge type satisfying good properties. The following lemma is the analogue of [KP18, Lemma 4.6.22], and is the key input that will allow us to deduce the existence of good integral models in the abelian-type case.

Proposition 5.2.6. *Let $(\mathbf{G}_2, X_2)$ be an acceptable Shimura datum of abelian type and $\mathcal{G}_2$ a parahoric of G_2. Then there exists a Shimura datum $(\mathbf{G}, X)$ of Hodge type together with a central isogeny $\mathbf{G}^{\mathrm{der}} \to \mathbf{G}_2^{\mathrm{der}}$ which induces an isomorphism $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$. Moreover, $(\mathbf{G}, X)$ may be chosen to satisfy the following conditions.*

(1) $\pi_1(G^{\mathrm{der}})$ is a 2-group and is trivial if $(\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$. Moreover $\mathbf{G}$ satisfies assumption (5.1.11.1).
(2) Any prime $v_2|p$ of $\mathbf{E}_2$ splits in the composite $\mathbf{E}' := \mathbf{E}.\mathbf{E}_2$.
(3) The center Z of $G := \mathbf{G}_{\mathbb{Q}_p}$ is an R-smooth torus over $\mathbb{Q}_p$.
(4) $X_*(G^{\mathrm{ab}})_I$ is torsion free.
(5) $p \nmid |\pi_1(G^{\mathrm{der}})|$ and the centralizer of a maximal $\breve{\mathbb{Q}}_p$-split torus in G is R-smooth.

[3] In [KP18, 4.6.21] it is incorrectly asserted that $\mathbf{H}^\sharp$ is defined for any (H, Y) with H of classical type, however H may not satisfy the third condition above. This is however satisfied if $(\mathbf{H}^{\mathrm{ad}}, Y^{\mathrm{ad}})$ is of abelian type.

Proof. We follow the proof of [KP18, Lemma 4.6.22]. Let $\mathbf{G}_2^{\mathrm{ad}} \cong \prod_{j=1}^s \mathrm{Res}_{\mathrm{F}_j/\mathbb{Q}}\mathbf{H}_j$, where F_j is a totally real field and $\mathbf{H}_j$ is an absolutely simple F_j-group. By [Del79, 2.3.10], we may choose $(\mathbf{G}, X)$ a Shimura datum of Hodge type with $\mathbf{G}^{\mathrm{der}} \cong \mathbf{G}_2^{\mathrm{ad}\sharp}$, and such that the central isogeny $\mathbf{G}^{\mathrm{der}} \to \mathbf{G}_2^{\mathrm{der}}$ induces an isomorphism of Shimura data $(\mathbf{G}^{\mathrm{ad}}, X^{\mathrm{ad}}) \cong (\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$. Then $\mathbf{G}^{\mathrm{der}}$ has the form $\prod_{j=1}^s \mathrm{Res}_{\mathrm{F}_j/\mathbb{Q}}\mathbf{H}_j^\sharp$. As in [KP18, Lemma 4.6.22], it follows that $(\mathbf{G}, X)$ satisfies (1).

We now explain how to choose $(\mathbf{G}, X)$ satisfying (2). We first assume $s = 1$ so that $\mathbf{G}_2^{\mathrm{ad}} \cong \mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbf{H}$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_d$ denote the primes of F above p and write F_i for the completion of F at $\mathfrak{p}_i$. Then $\mathbf{G}_{2,\mathbb{Q}_p}^{\mathrm{ad}} \cong \prod_{i=1}^d \mathrm{Res}_{F_i/\mathbb{Q}_p}\mathbf{H}_{F_i}$, and our assumptions imply that $H_i := \mathbf{H}_{F_i}$ splits over a tame extension of F_i. We choose K/F a quadratic imaginary extension of F such that all primes of F above p split in K. We fix a set T of embeddings $\mathrm{K} \to \mathbb{C}$ satisfying the same conditions as in [KP18, Lemma 4.6.22]. The construction of [Del79, Proposition 2.3.10] then gives a Shimura datum $(\mathbf{G}, X)$ of Hodge type such that any prime $v_2|p$ of $\mathbf{E}_2$ splits in $\mathbf{E}'$. The group $\mathbf{G}$ is G_1 in the notation of [Del79] and it arises from an auxiliary group $\mathbf{G}' = \mathrm{Res}_{F/\mathbb{Q}}\mathbf{H}'$ (G_3 in Deligne's notation) via the construction of [Del79, Corollaire 2.3.3]. The group $\mathbf{G}'$ splits over the composite of K and the splitting field of $\mathbf{G}$. It follows that $\mathbf{G}'_{\mathbb{Q}_p} \cong \prod_{i=1}^d \mathrm{Res}_{F_i/\mathbb{Q}_p} H'_i$ where H'_i splits over a tamely ramified extension of F_i. For $s > 1$, we apply the above construction of each of the individual factors.

We now show that we can arrange so that (3) is satisfied. Let $(\mathbf{G}, X) \to (\mathbf{GSp}(V), S^\pm)$ be the Hodge embedding arising from the (product of the) construction in [Del79, Proposition 2.3.10], and $\mathbf{G}' = \prod_{i=1}^s \mathrm{Res}_{\mathrm{F}_i/\mathbb{Q}}\mathbf{H}'_i$ the group containing $\mathbf{G}$ above. We let $\mathbf{G}^{\mathrm{Sp}}$ denote the identity component of $\mathbf{Sp}(V) \cap \mathbf{G}$ where $\mathbf{Sp}(V)$ is the symplectic group. Then by the construction of [Del79, Corollaire 2.3.3], $\mathbf{G}^{\mathrm{Sp}}$ is generated by $\mathbf{G}'^{\mathrm{der}}$ and the maximal compact subtorus of the center $\mathbf{Z}'$ of $\mathbf{G}'$ (which is necessarily defined over $\mathbb{Q}$ since $\mathbf{Z}'$ splits over a CM field). It follows that $\mathbf{G}_{\mathbb{Q}_p}^{\mathrm{Sp}} \cong \prod_{i=1}^s \mathrm{Res}_{\mathrm{F}_i/\mathbb{Q}}\mathbf{H}_i^{\mathrm{Sp}}$ where $\mathbf{H}_i^{\mathrm{Sp}}$ is a reductive group over F_i whose base change to the completion at any p-adic place of F splits over a tame extension. Let T be the centralizer of a maximal $\breve{\mathbb{Q}}_p$-split torus in $G := \mathbf{G}_{\mathbb{Q}_p}$, and $T^{\mathrm{Sp}} = T \cap \mathbf{G}_{\mathbb{Q}_p}^{\mathrm{Sp}}$. Thn T^{Sp} is a product of Weil-restrictions of tame tori, and hence is R-smooth by Proposition 2.4.6. It follows that T is R-smooth by Corollary 2.4.7, since it is an extension of $\mathbb{G}_m$ by a group whose identity component is T^{Sp}.

Arguing as in [Kis10, Proof of Prop 2.2.4], we may choose a maximal torus $\mathbf{T}$ of $\mathbf{G}$ such that $\mathbf{T}_{\mathbb{Q}_p}$ is $\mathbf{G}(\mathbb{Q}_p)$-conjugate to T, and there exists $h \in X$ such that h factors through $\mathbf{T}_\mathbb{R}$. We set $\mathbf{G}_1 := (\mathbf{G} \times \mathbf{T})/\mathbf{Z}$, where $\mathbf{Z}$ is the center of $\mathbf{G}$. Then the center Z_1 of $G_1 := \mathbf{G}_{1,\mathbb{Q}_p}$ is isomorphic to T and hence an R-smooth torus. We let X_1 denote the conjugacy class of Deligne homomorphisms for $\mathbf{G}_1$ determined by $h \times 1$ for $h \in X$. As in [KP18, Lemma 4.6.22], we let W denote the $\mathbf{G}_1$-representation $\mathrm{Hom}_{\mathbf{Z}}(V, V)$, and we may equip W with an alternating form such that there is a Hodge embedding $(\mathbf{G}_1, X_1) \to (\mathbf{GSp}(W), S_1^\pm)$. Thus upon replacing $(\mathbf{G}, X)$ by $(\mathbf{G}_1, X_1)$ and V by W, we may assume $(\mathbf{G}, X)$ satisfies (3).

To show we can arrange so that (4) and (5) are satisfied, first note that in the construction of $(\mathbf{G}, X)$ satsifying (3) above, the identity component $\mathbf{G}^{\mathrm{Sp}}$ of $\mathbf{G} \cap \mathrm{Sp}(V)$ remains of the form $\mathbf{G}^{\mathrm{Sp}} = \prod_{i=1}^s \mathrm{Res}_{\mathrm{F}_i/\mathbb{Q}}\mathbf{H}_i^{\mathrm{Sp}}$ where $\mathbf{H}_i$ are reductive groups over F_i whose base change at p-adic places split over a tame extension. We now apply the same construction as in [KP18, Lemma 4.6.22] to $(\mathbf{G}, X)$. This gives a Shimura datum $(\mathbf{G}_1, X_1)$ of Hodge type with connected center and such that

$X_*(\mathbf{G}^{\mathrm{ab}}_{1,\mathbb{Q}_p})_I$ is torsion free, i.e. condition (4) is satisfied. It is easy to check that the property concerning $\mathbf{G}^{\mathrm{Sp}}$ described above is also satisfied for $\mathbf{G}_1^{\mathrm{Sp}}$. Hence as before, both the center Z_1 of $\mathbf{G}_{1,\mathbb{Q}_p}$ and the centralizer of a maximal $\breve{\mathbb{Q}}_p$-split torus in $\mathbf{G}_1$ are extensions of $\mathbb{G}_m$ by a group whose identity component is an R-smooth torus, and hence are R-smooth tori by Corollary 2.4.7. Finally, since we have assumed $p > 2$, condition (1) implies $p \nmid |\pi_1(G^{\mathrm{der}})|$. Thus condition (5) is satisfied. □

5.2.7. For later applications to constructing canonical liftings, we introduce the following additional condition on the parahoric.

Definition 5.2.8. Let $(\mathbf{G}_2, X_2)$ be a Shimura datum of abelian type and $\mathcal{G}_2$ a parahoric group scheme of $G_2 = \mathbf{G}_{2,\mathbb{Q}_p}$. We say the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is *strongly acceptable* if $(\mathbf{G}, X)$ is acceptable and we can choose a Shimura datum as in Proposition 5.2.6 such that the corresponding parahoric $\mathcal{G}$ of $G = \mathbf{G}_{\mathbb{Q}_p}$ is connected, i.e. x can be chosen so that $\widetilde{\mathcal{G}} = \mathcal{G}$.

Corollary 5.2.9. *Let $(\mathbf{G}_2, X_2)$ be an acceptable Shimura datum of abelian type and $\mathcal{G}_2$ any parahoric group scheme of G_2. Assume $(\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$. Then the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is strongly acceptable.*

Proof. Let $(\mathbf{G}, X)$ be as in Proposition 5.2.6 and $\mathcal{G}$ the corresponding parahoric group scheme of G. Since $\pi_1(G^{\mathrm{der}})$ is trivial, we have $\pi_1(G) \cong X_*(G^{\mathrm{ab}})$. Thus $\pi_1(G)_I \cong X_*(G^{\mathrm{ab}})_I$ is torsion free and hence the Kottwitz map $\widetilde{\kappa}_G$ is trivial on $\widetilde{\mathcal{G}}(\breve{\mathbb{Z}}_p)$. It follows that $\mathcal{G}$ is a connected parahoric. □

Remark 5.2.10.
(1) The assumption of strong acceptability on the triple above is what is needed to construct canonical liftings in §5.4. We remark that it is possible for a triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ to be strongly acceptable even if $(\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has factors of type $D^{\mathbb{H}}$, cf. Proposition 7.2.4.
(2) The only place in Proposition 5.2.6 where the running assumption that $p > 2$ is used is in the verification of the condition $p \nmid |\pi_1(G^{\mathrm{der}})|$ in (5). In particular Corollary 5.2.9 also holds in the case $p = 2$.

5.2.11. Proposition 5.2.6 shows that if $(\mathbf{G}_2, X_2)$ is acceptable, it can be related to a Hodge-type Shimura datum $(\mathbf{G}, X)$ satisfying the assumptions in §5.2.1. We thus obtain the following theorem.

Theorem 5.2.12. *Let $p > 2$ and let $(\mathbf{G}_2, X_2)$ be an acceptable Shimura datum of abelian type. Let $\mathcal{G}_2$ be a parahoric group scheme of G_2 and set $\mathrm{K}_{2,p} := \mathcal{G}_2(\mathbb{Z}_p)$.*

Then there exists a Shimura datum of Hodge type $(\mathbf{G}, X)$ such that the conditions of Proposition 5.2.2 are satisfied and such that all primes $v_2|p$ of $\mathbf{E}_2$ split completely in $\mathbf{E}' = \mathbf{E}.\mathbf{E}_2$. In particular for any prime $v_2|p$ of $\mathbf{E}_2$, we obtain a $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant $\mathcal{O}_{E_2}$-scheme $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ with the following properties.

1. *$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ is étale locally isomorphic to $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}}$.*
2. *For any discrete valuation ring R of mixed characteristic the map*
$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)(R) \to \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)(R[1/p])$$
is a bijection.
3. *If the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is strongly acceptable, then $(\mathbf{G}, X)$ can be chosen so that for any compact open subgroup $\mathrm{K}_2 = \mathrm{K}_{2,p}\mathrm{K}_2^p \subset \mathbf{G}_2(\mathbb{A}_f)$, there exists*

a diagram of $\mathcal{O}_{E_2}$-stacks

$$\begin{array}{ccccc} & & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}_2}(\mathbf{G}_2, X_2) & & \\ & \swarrow^{\pi} & & \searrow^{q} & \\ \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2) & & & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}} \end{array}$$

where $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2) := \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)/\mathrm{K}_2^p$, π is a $\mathcal{G}^{\mathrm{ad}}$-torsor and the map q is smooth of relative dimension $\dim \mathbf{G}^{\mathrm{ad}}$*. In particular, such a diagram exists if $(\mathbf{G}_2^{\mathrm{ad}}, X_2^{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$.*

Proof. The argument is the same as [KP18, Theorem 4.6.23]. Note that the assumptions in (3) imply we may choose $x \in \mathcal{B}(G, \mathbb{Q}_p)$ such that $\widetilde{\mathcal{G}}_x = \mathcal{G}_x$, and hence Proposition 5.2.2 (3) can be applied. □

Remark 5.2.13.
(1) If $p > 3$, then every Shimura datum $(\mathbf{G}_2, X_2)$ of abelian type is acceptable (see §3.3.1). Thus this Theorem essentially completes the construction of integral models for abelian-type Shimura varieties with parahoric level over primes $p > 3$. Moreover for $p = 3$, only the case when G_2^{ad} has a factor of type D_4 needs to be excluded.
(2) The local model diagram we obtain is almost the expected one, with the only difference being that we have a $\mathcal{G}^{\mathrm{ad}}$-torsor instead of a $\mathcal{G}_2$-torsor. For our applications, the important property is that the torsor is for a group scheme with *connected* special fiber.

5.3. **Some functorial properties of integral models.** In this subsection we prove some functorial properties of the integral models that we have constructed. In particular, we show that in the Hodge type case, the integral models are independent of the choice of good embedding. The main result is Proposition 5.3.8 which will be used in the next subsection to define the μ-ordinary locus.

5.3.1. Let $f : (\mathbf{G}, X) \to (\mathbf{G}', X')$ be a morphism between Shimura data of Hodge type satisfying (5.1.4.1) and (5.1.4.2). Let $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}'$ be Bruhat–Tits stabilizer schemes of $G = \mathbf{G}_{\mathbb{Q}_p}$ and $G' = \mathbf{G}'_{\mathbb{Q}_p}$ respectively, and we let $\mathcal{G}$ and $\mathcal{G}'$ denote their respective parahorics. We assume that $f_{\mathbb{Q}_p} : G \to G'$ extends to a morphism $\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$, and that condition (5.1.11.1) holds. As always, we assume $p > 2$.

Let $\mathrm{K} = \mathrm{K}_p\mathrm{K}^p$, $\mathrm{K}' = \mathrm{K}'_p\mathrm{K}'^p$, be compact open subgroups of $\mathbf{G}(\mathbb{A}_f)$ and $\mathbf{G}'(\mathbb{A}_f)$ respectively, with $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$ and $\mathrm{K}'_p = \mathcal{G}'(\mathbb{Z}_p)$. We assume $f(\mathrm{K}) \subset \mathrm{K}'$. As usual, we fix a prime $v|p$ of the reflex field $\mathbf{E}$ of $(\mathbf{G}, X)$, and write $E = \mathbf{E}_v$, and k for an algebraic closure of the residue field of E.

Using Lemma 5.1.3, we fix good Hodge embeddings

$$\iota : (\mathbf{G}, X) \to (\mathbf{GSp}(V), S^{\pm}) \quad \text{and} \quad \iota' : (\mathbf{G}', X') \to (\mathbf{GSp}(V'), S'^{\pm}),$$

and $\mathbb{Z}_p$-lattices $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ and $V'_{\mathbb{Z}_p} \subset V'_{\mathbb{Q}_p}$ so that ι and ι' are good with respect to $V_{\mathbb{Z}_p}$ and $V'_{\mathbb{Z}_p}$ respectively. We write $V_{\mathbb{Z}_{(p)}} = V_{\mathbb{Z}_p} \cap V$, and similarly for $V'_{\mathbb{Z}_{(p)}}$.

Proposition 5.3.2. *The morphism of Shimura varieties induced by f extends to a morphism of integral models over $\mathcal{O}_E$*

$$f_{\mathscr{S}} : \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)_{\mathcal{O}_E} \to \mathscr{S}_{\mathrm{K}'}(\mathbf{G}', X')_{\mathcal{O}_E},$$

associated to ι and ι'.

Moreover, if f induces an isomorphism of derived groups $\mathbf{G}^{\mathrm{der}} \cong \mathbf{G}'^{\mathrm{der}}$ and the parahorics $\mathcal{G}$ and $\mathcal{G}'$ are associated, then for K and K' neat and $\overline{x} \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)_{\mathcal{O}_E}(k)$ with image $\overline{y} \in \mathscr{S}_{\mathrm{K}}(\mathbf{G}', X')_{\mathcal{O}_E}(k)$, the morphism $f_{\mathscr{S}}$ induces an isomorphism of completions $\widehat{U}_{\overline{x}} \cong \widehat{U}_{\overline{y}}$ at $\overline{x}$ and $\overline{y}$.

Proof. By Proposition 5.1.12, it suffices to prove the result for $\mathcal{G}$ and $\mathcal{G}'$ replaced by $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}'$. We let $\widetilde{\mathrm{K}}_p, \widetilde{\mathrm{K}}'_p, \widetilde{\mathrm{K}}, \widetilde{\mathrm{K}}'$ denote the corresponding compact open subgroups. For notational convenience, we assume the integral models are base changed to $\mathcal{O}_E$, and so we omit this notation from the subscripts in the proof.

We first prove the proposition in the special case, when $V'_{\mathbb{Z}_{(p)}}$ is a direct summand of $V_{\mathbb{Z}_{(p)}}$ as a $\widetilde{\mathcal{G}}$-representation. Thus, there is an idempotent $\varpi \in \mathrm{End}(V_{\mathbb{Z}_{(p)}})$, whose action on $V_{\mathbb{Z}_p}$ commutes with $\widetilde{\mathcal{G}}$, and with $\varpi(V_{\mathbb{Z}_{(p)}}) \simeq V'_{\mathbb{Z}_{(p)}}$. Now recall that, by construction, ι induces

$$\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X) \to \mathscr{S}_{\mathrm{K}_1}(\mathbf{GSp}(V), S^{\pm}) \tag{5.3.2.1}$$

for a suitable level structure K_1, where the scheme on the right is a moduli space for tuples (A, λ, ϵ^p), where ϵ^p is a prime-to-p level structure, and λ is a weak polarization. Let $\mathscr{S}_{\varpi}$ be the moduli space over $\mathcal{O}_E$ of tuples $(A, \lambda, \epsilon^p, \varpi_A)$ where $\varpi_A \in \mathrm{End}(A)$ is an endomorphism compatible with ϖ via the level structure. Then the map

$$\mathscr{S}_{\varpi} \to \mathscr{S}_{\mathrm{K}_1}(\mathbf{GSp}(V), S^{\pm}) \tag{5.3.2.2}$$

is finite. Indeed, for K_1 sufficiently small ϖ_A, if it exists, is determined by ϖ and ϵ^p, and (5.3.2.2) is an embedding. It follows that (5.3.2.1) factors through $\mathscr{S}_{\varpi}$, as this holds on generic fibers.

We also have the analogue of (5.3.2.1) for ι'

$$\mathscr{S}_{\widetilde{\mathrm{K}}'}(\mathbf{G}', X) \to \mathscr{S}_{\mathrm{K}'_1}(\mathbf{GSp}(V'), S'^{\pm}). \tag{5.3.2.3}$$

One can choose $\mathrm{K}_1, \mathrm{K}'_1$, so that there is a map $\mathscr{S}_{\varpi} \to \mathscr{S}_{\mathrm{K}'_1}(\mathbf{GSp}(V'), S'^{\pm})$, which is given by sending the tuple $(A, \lambda, \epsilon^p, \varpi_A)$ to $\varpi_A(A)$ with the induced weak polarization and level structure. Now the composite

$$\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X) \to \mathscr{S}_{\varpi} \to \mathscr{S}_{\mathrm{K}'_1}(\mathbf{GSp}(V'), S'^{\pm}) \tag{5.3.2.4}$$

factors through $\mathscr{S}_{\widetilde{\mathrm{K}}'}(\mathbf{G}', X)$, as this holds on generic fibers. This gives the map $f_{\mathscr{S}}$.

We now assume f induces an isomorphism of derived groups and that $\mathcal{G}$ and $\mathcal{G}'$ are associated. We prove the second part of the proposition, still assuming $V'_{\mathbb{Z}_{(p)}}$ is a direct summand in $V_{\mathbb{Z}_{(p)}}$. We let $\mathscr{G}_{\overline{x}}$ and $\mathscr{G}_{\overline{y}}$ denote the p-divisible groups associated to $\overline{x}$ and $\overline{y}$. Applying the construction of §5.1.7 to $\widetilde{\mathcal{G}}'$ and ι', we obtain formal local models $\widehat{\mathbb{M}}^{\mathrm{loc}} = \mathrm{Spf} A$, $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}'} = \mathrm{Spf} A_{\widetilde{\mathcal{G}}'}$ associated to $\overline{y}$. Similarly, we have a local model $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}} = \mathrm{Spf} A_{\widetilde{\mathcal{G}}}$ associated to $\overline{x}$. For notational simplicity, we use the same notation for the base change of these objects to $\mathcal{O}_{\breve{E}}$. Our assumptions imply that we have an isomorphism of adjoint local model triples $(G^{\mathrm{ad}}, \{\mu_h^{\mathrm{ad}}\}, \mathcal{G}^{\mathrm{ad}}) \cong G'^{\mathrm{ad}}, \{\mu_{h'}^{\mathrm{ad}}\}, \mathcal{G}'^{\mathrm{ad}})$. Hence by Corollary 3.3.13 we have a canonical isomorphism $f^{\mathrm{loc}} : \widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}} \simeq \widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}'}$.

As in the proof of Lemma 4.1.7, we have a $\widetilde{\mathcal{G}}$-torsor $\mathcal{T}$ over $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}}$, and a $\widetilde{\mathcal{G}}'$-torsor $\mathcal{T}'$, over $\widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}'} \simeq \widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}}$. Note that we can take ϖ to be one of the s_α in the construction of Lemma 4.1.7, from which one sees that $\mathcal{T}' = (\mathcal{T} \times \widetilde{\mathcal{G}}')/\widetilde{\mathcal{G}}$. Now fix rigid sections $\Psi_{A_{\tilde{\mathcal{G}}}}$, and Ψ, as in Proposition 4.1.7, and let $\Psi_{A_{\tilde{\mathcal{G}}'}}$ be the (rigid) section of $\mathcal{T}'$ induced by $\Psi_{A_{\tilde{\mathcal{G}}}}$. With these choices, and using Proposition 5.1.8, we obtain maps

$$\begin{array}{ccccc} \widehat{U}_{\overline{x}} & \xrightarrow[s_{\overline{x}}]{\sim} & \widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}} & & \\ {\scriptstyle f_{\mathscr{S}}}\downarrow & & \cong\downarrow{\scriptstyle f^{\mathrm{loc}}} & & \\ \widehat{U}_{\overline{y}} & \xrightarrow[s_{\overline{y}}]{\sim} & \widehat{\mathbb{M}}^{\mathrm{loc}}_{\mathcal{G}'} & \xrightarrow{\iota} & \widehat{\mathbb{M}}^{\mathrm{loc}} \end{array}$$

where the map on the left is induced by $f_{\mathscr{S}}$.

It suffices to show that the square on the left commutes, or that $\iota \circ f^{\mathrm{loc}} \circ s_{\overline{x}} = \iota \circ s_{\overline{y}} \circ f_{\mathscr{S}}$. To see this, we may check on points with values in a finite extension $\mathcal{O}_K$ of $W(k)$. Such a point x, corresponds to a deformation $\mathscr{G}_x$ of $\mathscr{G}_{\overline{x}}$, which is equipped with an idempotent endomorphism ϖ_x. Then both maps send x to $\varpi_x(\mathscr{G}_x)$, as a deformation of $\varpi_{\overline{x}}(\mathscr{G}_{\overline{x}}) = \mathscr{G}_{\overline{y}}$. For $\iota \circ f^{\mathrm{loc}} \circ s_{\overline{x}}(x)$ this follows from the definitions, and our choice of $\Psi_{A_{\tilde{\mathcal{G}}'}}$. For $\iota \circ s_{\overline{y}} \circ f_{\mathscr{S}}$, this follows from the construction of the map $f_{\mathscr{S}}$, as the composite (5.3.2.4).

Finally, we drop our assumption on $V'_{\mathbb{Z}_{(p)}}$, and consider the general case. Let $V''_{\mathbb{Z}_{(p)}} = V'_{\mathbb{Z}_{(p)}} \oplus V_{\mathbb{Z}_{(p)}}$, equipped with the direct sum of the symplectic forms on $V_{\mathbb{Z}_{(p)}}$ and $V'_{\mathbb{Z}_{(p)}}$. Then

$$\iota'' = \iota \times \iota' : \mathbf{G} \to \mathbf{GSp}(V'')$$

is a good Hodge embedding with respect to $V''_{\mathbb{Z}_p}$, and we obtain a corresponding integral model $\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)''$. Using the projections from $V''_{\mathbb{Z}_{(p)}}$ to $V'_{\mathbb{Z}_{(p)}}$ and $V_{\mathbb{Z}_{(p)}}$, and the special case considered above, we obtain maps

$$\mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X) \leftarrow \mathscr{S}_{\widetilde{\mathrm{K}}}(\mathbf{G}, X)'' \to \mathscr{S}_{\widetilde{\mathrm{K}}'}(\mathbf{G}', X'),$$

and what we have already shown implies that the first map is an isomorphism. Inverting this map, we obtain the desired morphism $f_{\mathscr{S}}$.

If f induces an isomorphism of derived groups and $\mathcal{G}$ and $\mathcal{G}'$ are associated, the second map induces isomorphisms on complete local rings at k-points and hence the second part of the proposition follows. □

Corollary 5.3.3. *The integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ is independent of the choice of good embedding ι.*

□

Remark 5.3.4. When $\mathcal{G}$ is a connected parahoric, this follows from [PRb, Theorem 1.3.4]; see also [Pap22, Theorem 7.1.7].

5.3.5. We now assume that f induces an isomorphism of derived groups and that the parahorics $\mathcal{G}$ and $\mathcal{G}'$ are associated. As in §5.2.1, the connected component $X^+ \subset X$ determines neutral connected components $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+$ and $\mathscr{S}_{\mathrm{K}'_p}(\mathbf{G}', X')^+$, which for notational convenience we assume are base changed to $\mathcal{O}_{E^{\mathrm{ur}}}$.

Corollary 5.3.6. *The morphism $f_{\mathscr{S}}$ induces an isomorphism of $\mathcal{O}_{E^{\mathrm{ur}}}$-schemes.*

$$\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \to \mathscr{S}_{\mathrm{K}_p}(\mathbf{G}', X')^+.$$

Proof. We will consider neat compact open subgroups $\mathrm{K}^{1,p}, \mathrm{K}^{2,p} \subset \mathbf{G}'(\mathbb{A}_f^p)$, and we write $\mathrm{K}^1 = \mathrm{K}'_p\mathrm{K}^{1,p}$ and $\mathrm{K}^2 = \mathrm{K}'_p\mathrm{K}^{2,p}$. Since the morphism $\mathbf{G} \to \mathbf{G}'$ induces an isomorphism of derived groups, the map $\mathrm{Sh}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \to \mathrm{Sh}_{\mathrm{K}'_p}(\mathbf{G}', X')^+$ is an isomorphism. Thus for any sufficiently small neat compact open $\mathrm{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$, there exist $\mathrm{K}^{1,p}, \mathrm{K}^{2,p} \subset \mathbf{G}'(\mathbb{A}_f^p)$ such that f induces maps

$$\mathrm{Sh}_{\mathrm{K}^2}(\mathbf{G}', X')^+ \to \mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)^+ \to \mathrm{Sh}_{\mathrm{K}^1}(\mathbf{G}', X')^+. \tag{5.3.6.1}$$

Let $\mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+$ be the normalization of $\mathscr{S}_{\mathrm{K}^1}(\mathbf{G}', X')^+$ in $\mathrm{Sh}_{\mathrm{K}}(\mathbf{G}, X)^+$. Then (5.3.6.1) extends to a sequence of morphisms

$$\mathscr{S}_{\mathrm{K}^2}(\mathbf{G}', X')^+ \to \mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+ \to \mathscr{S}_{\mathrm{K}^1}(\mathbf{G}', X')^+$$

whose composite is finite étale. It follows that both maps in the sequence are finite, and since all the schemes are normal, both maps are finite étale. Passing to the limit with $\mathrm{K}^{2,p}$ and K^p we obtain a commutative diagram

$$\begin{array}{ccc} \mathscr{S}_{\mathrm{K}_p^2}(\mathbf{G}', X')^+ & \longrightarrow & \mathscr{S}_{\mathrm{K}_p}^\dagger(\mathbf{G}, X)^+ \\ \downarrow & & \downarrow \\ \mathscr{S}_{\mathrm{K}^2}(\mathbf{G}', X')^+ & \longrightarrow & \mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+ \end{array}$$

Since the map on the left is pro-finite étale, and the bottom map is finite étale, the map on the right is pro-finite étale.

By Proposition 5.3.2 and the normality of $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)^+$, there is also a morphism $\alpha : \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)^+ \to \mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+$, whose composite with $\mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+ \to \mathscr{S}_{\mathrm{K}^1}(\mathbf{G}', X')^+$ is étale, and hence α is étale. Again, passing to the limit with K^p, we obtain a commutative diagram

$$\begin{array}{ccc} \mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+ & \longrightarrow & \mathscr{S}_{\mathrm{K}_p}^\dagger(\mathbf{G}, X)^+ \\ \downarrow & & \downarrow \\ \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)^+ & \xrightarrow{\alpha} & \mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+ \end{array}$$

where the vertical maps are pro-finite étale. For any finite extension K of $W(k)[1/p]$, a point $x^\dagger \in \mathscr{S}_{\mathrm{K}}^\dagger(\mathbf{G}, X)^+(\mathcal{O}_K)$ lifts to a point of $\tilde{x}^\dagger \in \mathscr{S}_{\mathrm{K}_p}^\dagger(\mathbf{G}, X)^+(\mathcal{O}_K)$, and hence to a point $\tilde{x} \in \mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+(K)$. By Theorem 5.2.12 (2), $\tilde{x}$ extends to a point in $\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+(\mathcal{O}_K)$. This implies that α is surjective.

Thus α is a surjective étale birational morphism between normal schemes, hence an isomorphism. We thus obtain a morphism $\mathscr{S}_{\mathrm{K}^2}(\mathbf{G}', X')^+ \to \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)^+$ which, after taking the inverse limit, gives an inverse for the morphism

$$\mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+ \to \mathscr{S}_{\mathrm{K}'_p}(\mathbf{G}', X')^+$$

induced by $f_{\mathscr{S}}$. □

5.3.7. We now use the notation of §5.2. Thus, we let $(\mathbf{G}_2, X_2)$ be an acceptable Shimura datum of *Hodge type* and we write $\mathrm{K}_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ for a parahoric $\mathcal{G}_2$ corresponding to $x_2 \in \mathcal{B}(G_2, \mathbb{Q}_p)$. By Theorem 5.2.12, we may construct an integral model $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)$ over $\mathcal{O}_{E_2}$ for $\mathrm{Sh}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)$ by viewing $(\mathbf{G}_2, X_2)$ as a Shimura datum of abelian type and using an auxiliary Shimura datum $(\mathbf{G}, X)$ of Hodge type as in the conclusion of Proposition 5.2.6. In particular $(\mathbf{G}, X)$ satisfies the conditions in §5.2.1 as well as (5.1.11.1). We fix such a $(\mathbf{G}, X)$ and a good Hodge embedding $\iota : (\mathbf{G}, X) \to (\mathbf{GSp}(V), S^\pm)$ for the rest of this section.

Now let $\iota_2 : (\mathbf{G}_2, X_2) \to (\mathbf{GSp}(V_2), S_2^\pm)$ be a Hodge embedding. By the main theorem of [Lan00], ι_2 induces a $G_2(\mathbb{Q}_p^{\mathrm{ur}})$ and $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ur}}/\mathbb{Q}_p)$-equivariant embedding of buildings. Upon replacing ι_2 with a new Hodge embedding we may assume there is a $\mathbb{Z}_p$-lattice $V_{2,\mathbb{Z}_p} \subset V_{2,\mathbb{Q}_p}$ with $V_{2,\mathbb{Z}_p} \subset V_{2,\mathbb{Z}_p}^\vee$ such that $G_2 \to \mathrm{GSp}(V_{2,\mathbb{Q}_p})$ extends to a morphism of Bruhat–Tits stabilizer schemes $\widetilde{\mathcal{G}}_2 \to \mathcal{GSP}$, where $\mathcal{GSP}$ is the group scheme stabilizer of $V_{2,\mathbb{Z}_p}$ (cf. [BT84, Proposition 1.7.6]). We set $\mathrm{K}'_{2,p} := \mathcal{GSP}(\mathbb{Z}_p)$ and we let $\mathrm{K}_2'^p \subset \mathbf{GSp}(V_{2,\mathbb{A}_f^p})$ be a compact open subgroup containing K_2^p.

Proposition 5.3.8. *Assume $p > 2$. There is a map of $\mathcal{O}_{E_2}$-stacks*

$$\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2) \to \mathscr{S}_{\mathrm{K}_2'}(\mathbf{GSp}(V_2), S_2^\pm)_{\mathcal{O}_{E_2}} \tag{5.3.8.1}$$

extending the natural map on the generic fiber.

5.3.9. To prove this proposition, we make use of the following auxiliary construction. Let $\mathbf{G}_3$ be the identity component of $\mathbf{G}_2 \times_{\mathbf{G}^{\mathrm{ad}}, \mathbb{G}_m} \mathbf{G}$, where the projections onto $\mathbb{G}_m$ are given by composing ι, ι_2 with the symplectic multipliers. There are natural morphisms $\mathbf{G}_3 \to \mathbf{G}_2$ and $\mathbf{G}_3 \to \mathbf{G}$, the latter of which induces an isomorphism $\mathbf{G}_3^{\mathrm{der}} \xrightarrow{\sim} \mathbf{G}^{\mathrm{der}}$. Let $h \in X^+$. By assumption, the isomorphism $\mathbf{G}^{\mathrm{ad}} \cong \mathbf{G}_2^{\mathrm{ad}}$ allows us to view X^+ as a subset of X_2, and we let $h_2 \in X_2$ denote the element determined by h. The homomorphism

$$h_3 := (\iota \circ h, \iota_2 \circ h_2) : \mathbb{S} \to \mathbf{GL}(V_{\mathbb{R}}) \times \mathbf{GL}(V_{2,\mathbb{R}})$$

factors through $\mathbf{G}_3$, and we denote by X_3 the $\mathbf{G}_{3,\mathbb{R}}$-orbit of h_3. The pair $(\mathbf{G}_3, X_3)$ forms a Shimura datum of Hodge type.

5.3.10. We would like to construct an integral model for $(\mathbf{G}_3, X_3)$ using the procedure in §5.1, however it is not clear that (5.1.4.2) is satisfied. We therefore consider a modification of $(\mathbf{G}_3, X_3)$.

Let L be a CM field with maximal totally real subfield L_0, and set

$$\mathbf{S}^{\mathrm{L}} = \mathrm{Res}_{\mathrm{L}/\mathbb{Q}}\mathbb{G}_m / \ker(N_{\mathrm{L}_0/\mathbb{Q}} : \mathrm{Res}_{L_0/\mathbb{Q}}\mathbb{G}_m \to \mathbb{G}_m).$$

Then $\mathbf{S}^{\mathrm{L}}$ is an extension of $\mathrm{Res}_{\mathrm{L}_0/\mathbb{Q}}\mathbf{T}$ by $\mathbb{G}_m$, where $\mathbf{T} = (\mathrm{Res}_{\mathrm{L}/\mathrm{L}_0}\mathbb{G}_m)/\mathbb{G}_m$. We denote by $w : \mathbb{G}_m \to \mathbf{S}^{\mathrm{L}}$ the inclusion. Since $p > 2$, $\mathbf{T}$ is tamely ramified at any p-adic place of L_0. Hence $(\mathrm{Res}_{\mathrm{L}_0/\mathbb{Q}}\mathbf{T})_{\mathbb{Q}_p}$ is R-smooth by Proposition 2.4.6 (1), and it follows that the base change of $\mathbf{S}^{\mathrm{L}}$ to $\mathbb{Q}_p$ is R-smooth by Proposition 2.4.6 (2).

Let $\mu^{\mathrm{ab}} \in X_*(\mathbf{G}_3^{\mathrm{ab}})$ be the image of μ_{h_3}. Then as in [Del82b, §A, (a)] (cf. also [KSZ, §6.3.1]), we may choose L, so that there exists a $\mu^{\mathrm{L}} \in X_*(\mathbf{S}^{\mathrm{L}})$, mapping to μ^{ab}, and satisfying $\mu^{\mathrm{L}} + c(\mu^{\mathrm{L}}) = w$, where c is the complex conjugation on L.

We set $\mathbf{G}_4 := \mathbf{G}_3 \times_{\mathbf{G}_3^{\mathrm{ab}}} \mathbf{S}^{\mathrm{L}}$ and write $G_4 := \mathbf{G}_{4,\mathbb{Q}_p}$. Then $\mathbf{G}_4$ is an extension of $\mathbf{S}^{\mathrm{L}}$ by $\mathbf{G}^{\mathrm{der}}$, and hence $\mathbf{G}_4$ is acceptable and $p \nmid |\pi_1(G_4^{\mathrm{der}})|$. Moreover the centralizer T_4 of a maximal $\breve{\mathbb{Q}}_p$-split torus in G_4 is an extension of $\mathbf{S}^{\mathrm{L}}_{\mathbb{Q}_p}$ by the centralizer T^{der} of a maximal $\breve{\mathbb{Q}}_p$-split torus in G^{der}. Now T^{der} is R-smooth by Proposition 2.4.6 (1) and the assumption of acceptability, and hence T_4 is R-smooth by Proposition 2.4.6 (2). As explained in [KSZ, Lemma 6.3.2], we may extend $(\mathbf{S}^{\mathrm{L}}, \mu^{\mathrm{L}})$ to a Shimura datum $(\mathbf{S}^{\mathrm{L}}, h^{\mathrm{L}})$, which is of Hodge type. Then for $h_3 \in X_3$, its image in $\mathbf{G}_3^{\mathrm{ab}}$ agrees with the image of h^L and we obtain a morphism $h_4 : \mathbb{S} \to \mathbf{G}_{4,\mathbb{R}}$. Let X_4 be the conjugacy class of h_4. Then $(\mathbf{G}_4, X_4)$ is a Shimura datum of Hodge type (cf. [KSZ, Lemma 6.3.2]). It follows that $(\mathbf{G}_4, X_4)$ satisfies the assumptions (5.1.4.1)–(5.1.4.2), and so we may fix a choice of good Hodge embedding $\iota_4 : (\mathbf{G}_4, X_4) \to (\mathbf{GSp}(V_4), S_4^\pm)$.

Proof of Proposition 5.3.8. It suffices to construct a map

$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2) \to \mathscr{S}_{\mathrm{K}'_{2,p}}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E_2}} \tag{5.3.10.1}$$

which is $\mathbf{G}_2(\mathbb{A}_f^p)$-equivariant. Let $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)'$ be the closure of

$$\mathrm{Sh}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2) \to \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2) \times \mathscr{S}_{\mathrm{K}'_{2,p}}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E_2}} \tag{5.3.10.2}$$

Then the existence of (5.3.10.1) is equivalent to requiring that

$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)' \to \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$$

is an isomorphism. We may check this over $\mathcal{O}_{E'}$, where $E' \supset E_2$, is any complete, discretely valued extension of E_2. In particular, we may assume that the connected components of $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)$ are defined over $\mathcal{O}_{E'}$.

Let $\mathscr{S} \to \mathscr{S}'$ be a map of connected components induced by (5.3.10.1). Then the explicit description given by (5.2.3.2) shows that one may identify the diagrams

$$\begin{array}{ccc} \mathscr{S}[1/p] & \longrightarrow & \mathscr{S}'[1/p] \\ \big\downarrow & & \big\downarrow \\ \mathscr{S} & & \mathscr{S}' \end{array}$$

coming from different choices of $\mathscr{S}$. Thus, it suffices to construct the map

$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_{2,p}}(\mathbf{GSp}(V_2), S_2^{\pm})^+_{\mathcal{O}_{E'}} \tag{5.3.10.3}$$

where $\mathscr{S}_{\mathrm{K}'_{2,p}}(\mathbf{GSp}(V_2), S_2^{\pm})^+_{\mathcal{O}_{E'}}$ is the connected component corresponding to the connected component of $S_2^{\pm}$ containing the image of X_2^+.

To do this we make use of the Shimura data $(\mathbf{G}_4, X_4)$ constructed above. This is equipped with morphisms of Shimura data

$$(\mathbf{G}, X) \longleftarrow (\mathbf{G}_4, X_4) \longrightarrow (\mathbf{G}_2, X_2) \longrightarrow (\mathbf{GSp}(V_2), S_2^{\pm}),$$

where the leftmost morphism induces an isomorphism on derived groups. Let $\mathcal{G}$ and $\mathcal{G}_4$ denote the parahoric group schemes for G and G_4 corresponding to $\mathcal{G}_2$, and set $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$, $\mathrm{K}_{4,p} = \mathcal{G}_4(\mathbb{Z}_p)$. For a compact open subgroup $\mathrm{K}_4^p \subset \mathbf{G}_4(\mathbb{A}_f^p)$, and setting $\mathrm{K}_4 = \mathrm{K}_{4,p}\mathrm{K}_4^p$, we may construct an integral model $\mathscr{S}_{\mathrm{K}_4}(\mathbf{G}_4, X_4)$ for $\mathrm{Sh}_{\mathrm{K}_4}(\mathbf{G}_4, X_4)$ as in §5.1, using the Hodge embedding ι_4.

We assume that E' is large enough that the connected components of $\mathrm{Sh}_{\mathrm{K}_4}(\mathbf{G}_4, X_4)$ are defined over E'. By Corollary 5.3.6, we have an isomorphism

$$\mathscr{S}_{\mathrm{K}_{4,p}}(\mathbf{G}_4, X_4)^+_{\mathcal{O}_{E'}} \xrightarrow{\sim} \mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+_{\mathcal{O}_{E'}}. \tag{5.3.10.4}$$

By Proposition 5.3.2, there is a morphism of integral models

$$\mathscr{S}_{\mathrm{K}_{4,p}}(\mathbf{G}_4, X_4)_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_2}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E'}}.$$

Taking inverse limits and restricting to neutral connected components, we obtain a morphism

$$\mathscr{S}_{\mathrm{K}_{4,p}}(\mathbf{G}_4, X_4)^+_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_{2,p}}(\mathbf{GSp}(V_2), S_2^{\pm})^+_{\mathcal{O}_{E'}}. \tag{5.3.10.5}$$

By the construction of $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'}}$, we have

$$\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)^+_{\mathcal{O}_{E'}} = \mathscr{S}_{\mathrm{K}_p}(\mathbf{G}, X)^+_{\mathcal{O}_{E'}}/\Delta(\mathbf{G}, \mathbf{G}_2) \cong \mathscr{S}_{\mathrm{K}_{4,p}}(\mathbf{G}_4, X_4)^+_{\mathcal{O}_{E'}}/\Delta(\mathbf{G}, \mathbf{G}_2),$$

where $\Delta(\mathbf{G},\mathbf{G}_2) := \ker(\mathscr{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ \to \mathscr{A}(\mathbf{G}_{2,\mathbb{Z}_{(p)}}))$. The map (5.3.10.5) factors through the action of $\Delta(\mathbf{G},\mathbf{G}_2)$, since it does so on the generic fiber. We thus obtain a map $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2,X_2)^+_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_{2,p}}(\mathbf{GSp}(V_2),S_2^\pm)^+_{\mathcal{O}_{E'}}$ as desired. □

5.4. μ-ordinary locus and canonical liftings.

5.4.1. We keep the notation of the previous subsection. Thus we let $(\mathbf{G}_2,X_2)$ be an acceptable Shimura datum of Hodge type and $\mathrm{K}_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ where $\mathcal{G}_2$ is a parahoric group scheme of $G_2 := \mathbf{G}_{2,\mathbb{Q}_p}$. In this subsection, we study the μ-ordinary locus in the special fiber of integral models and show that when $(\mathbf{G}_2,X_2,\mathcal{G}_2)$ is strongly acceptable (cf. Definition 5.2.8), each μ-ordinary point lifts to a special point. Similar to the construction of integral models, the result is deduced from the corresponding statement in some special Hodge-type cases.

We have the integral $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)$ over $\mathcal{O}_{E_2}$ which is constructed from an auxiliary Hodge-type Shimura datum $(\mathbf{G},X)$ satisfying the conditions in §5.2.1 and a choice of good Hodge embedding ι. Let $\iota_2 : (\mathbf{G}_2,X_2) \to (\mathbf{GSp}(V_2),S_2^\pm)$ be a Hodge embedding and $V_{2,\mathbb{Z}_p} \subset V_{2,\mathbb{Q}_p}$ a lattice as in §5.3.7. Then by Proposition 5.3.8, there is a morphism of integral models

$$\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2) \to \mathscr{S}_{\mathrm{K}'_2}(\mathbf{GSp}(V_2),S_2^\pm)_{\mathcal{O}_{E_2}}. \tag{5.4.1.1}$$

Let $h : \mathcal{A}_2 \to \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)$ denote the pullback of the universal abelian variety along (5.4.1.1). Let $s_\alpha \in V_2^\otimes$ be a collection of tensors whose stabilizer is $\mathbf{G}_2$. Then as in §5.1.5, these give rise to tensors $s_{\alpha,B} \in V_B := R^1h_{\mathrm{an}*}\mathbb{Q}$, $s_{\alpha,\ell} \in \mathcal{V}_\ell(\mathcal{A}_2) := R^1h_{\text{ét}*}\mathbb{Q}_\ell$ for all $\ell \neq p$ and $s_{\alpha,p} \in \mathcal{V}_p(\mathcal{A}_2) := R^1h_{\eta,\text{ét}*}\mathbb{Q}_p$. For any $\mathcal{O}_{E_2}$-scheme T and $x \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)(T)$, we write $\mathcal{A}_{2,x}$ for the pullback of $\mathcal{A}_2$ to x.

For $K/\breve{\mathbb{Q}}_p$ finite and $\widetilde{x} \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)(\mathcal{O}_K)$ with special fiber $\overline{x}$, we let $s_{\alpha,0,\widetilde{x}} \in \mathbb{D}(\mathcal{A}_{2,\overline{x}}[p^\infty])[1/p]^\otimes$ denote the images of $s_{\alpha,p,\widetilde{x}}$ under the p-adic comparison isomorphism. As in §5.1.6, these tensors depend only on $\overline{x}$ and not on $\widetilde{x}$; we thus write $s_{\alpha,0,\overline{x}}$ for these tensors. Note that [KPS, Prop. 1.3.7] applies here since the morphism $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2) \to \mathscr{S}_{\mathrm{K}'_2}(\mathbf{GSp}(V_2),S_2^\pm)_{\mathcal{O}_{E_2}}$ factors through the normalization of its scheme theoretic image, and all objects are pulled back from this.

5.4.2. Let $\overline{x} \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)(k)$, and set $\mathbb{D} := \mathbb{D}(\mathcal{A}_{2,\overline{x}}[p^\infty])$. We fix an isomorphism

$$V_{2,\mathbb{Z}_p}^\vee \otimes_{\mathbb{Z}_p} \breve{\mathbb{Q}}_p \cong \mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p,$$

taking s_α to $s_{\alpha,0,\overline{x}}$; such an isomorphism exists by Steinberg's theorem (cf. [KPS, 1.3.8]). Then the Frobenius on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p$ is given by $b\sigma$ for some $b \in G_2(\breve{\mathbb{Q}}_p)$. By [KPS, Lemma 1.3.9], we have $[b] \in B(G_2,\{\mu_2\})$ where $\{\mu_2\} = \{\mu_{h_2}^{-1}\}$. We write $\mathcal{S}_{\mathrm{K}_2}$ (resp. $\mathcal{S}_{\mathrm{K}_{2,p}}$) for the special fiber of $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)$ (resp. $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2,X_2)$) over the residue field k_{E_2} of $\mathcal{O}_{E_2}$. The map $\mathcal{S}_{\mathrm{K}_2}(k) \to B(G_2,\{\mu_2\})$ sending $\overline{x}$ to the σ-conjugacy class $[b]$ of the associated element b induces the Newton stratification of $\mathcal{S}_{\mathrm{K}_2,k} := \mathcal{S}_{\mathrm{K}_2} \otimes_{k_{E_2}} k$. Let $[b] \in B(G_2,\{\mu_2\})$, we write $\mathcal{S}_{\mathrm{K}_2,[b]} \subset \mathcal{S}_{\mathrm{K}_2,k}$ for the strata corresponding to $[b]$; if K_2^p is neat, it is a locally closed subscheme of $\mathcal{S}_{\mathrm{K}_2,k}$. Similarly, we write

$$\mathcal{S}_{\mathrm{K}_{2,p},[b]} = \lim_{\leftarrow \mathrm{K}_2^p} \mathcal{S}_{\mathrm{K}_{2,p}\mathrm{K}_2^p,[b]},$$

which makes sense since $\mathcal{S}_{\mathrm{K}_2,[b]}$ is compatible with the prime-to-p level. For the rest of §5.4 we assume the existence of the class $[b]_{\mu_2} \in B(G_2,\{\mu_2\})$ in Definition 2.2.4.

Definition 5.4.3. We define the μ_2-ordinary locus of $\mathcal{S}_{\mathrm{K}_2,k}$ to be $\mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}$.

5.4.4. We say that a parahoric subgroup $\mathrm{K}_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ is very special if $\mathcal{G}_2(\breve{\mathbb{Z}}_p)$ is a special parahoric subgroup of $G_2(\breve{\mathbb{Q}}_p)$. Note that such a parahoric exists if and only if G_2 is quasi-split (cf. [Zhu14, Lemma 6.1]). The following is deduced easily from [KPS, Corollary 1.3.16].

Theorem 5.4.5. *Assume G_2 is quasi-split, $\mathrm{K}_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ is a very special parahoric subgroup and K_2^p is neat. Then*

(1) $\mathcal{S}_{\mathrm{K}_2}$ is normal.
(2) The μ_2-ordinary locus $\mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}$ is Zariski open and dense in $\mathcal{S}_{\mathrm{K}_2,k}$.

Proof. To show (1), it suffices by Theorem 5.2.12 to show that the special fiber of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G}_2,\{\mu_{h_2}\}}$ is normal. For this, it suffices by Theorem 3.1.6 to show that the special fiber is integral. This follows from the argument in [PZ13, Corollary 9.4], noting that as in *loc. cit.* the μ-admissible set $\mathrm{Adm}(\{\mu\})_J$ has a single extremal element when $J \subset \mathbb{S}$ corresponds to a very special standard parahoric of $G(\breve{\mathbb{Q}}_p)$.

(2) follows from (1) by [KPS, Corollary 1.3.16]. □

5.4.6. Let $\overline{x} \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(k)$. Define $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{2,\overline{x}})$ to be the $\mathbb{Q}$-group whose points in a $\mathbb{Q}$-algebra R are given by

$$\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{2,\overline{x}})(R) = (\mathrm{End}(\mathcal{A}_{2,\overline{x}}) \otimes_{\mathbb{Z}} R)^{\times}$$

By functoriality, $\mathrm{Aut}_{\mathbb{Q}}\mathcal{A}_{2,\overline{x}}$ acts on $T_\ell\mathcal{A}_{2,\overline{x}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for $\ell \neq p$ and on $\mathbb{D} \otimes_{\breve{\mathbb{Z}}_p} \breve{\mathbb{Q}}_p$, and we write $I_{\overline{x}}$ for the closed subgroup of $\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{2,\overline{x}})$ consisting of automorphisms which preserve $s_{\alpha,\ell,\overline{x}}$ and $s_{\alpha,0,\overline{x}}$. There is a canonical inclusion $I_{\overline{x}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \subset J_b$, where J_b is the σ-centralizer group for $b \in G_2(\breve{\mathbb{Q}}_p)$.

The goal of the rest of this section is to prove the following theorem.

Theorem 5.4.7. *Let $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ be a strongly acceptable triple of Hodge type. Let $\overline{x} \in \mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}(k)$. Then $\overline{x}$ admits a lifting to a special point $\widetilde{x} \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(K)$ for some $K/\breve{\mathbb{Q}}_p$ finite such that the action of $I_{\overline{x}}(\mathbb{Q})$ on $\mathcal{A}_{2,\overline{x}}$ lifts to an action (in the isogeny category) on $\mathcal{A}_{2,\widetilde{x}}$.*

Remark 5.4.8. (1) Recall that $x \in \mathrm{Sh}_{\mathrm{K}_2}(\mathbf{G}_2, X)(\mathbb{C})$ is said to be *special* if there exists a torus $\mathbf{T} \subset \mathbf{G}_2$ such that under the identification

$$\mathrm{Sh}_{\mathrm{K}_2}(\mathbf{G}_2, X)(\mathbb{C}) \cong \mathbf{G}_2(\mathbb{Q})\backslash X_2 \times \mathbf{G}_2(\mathbb{A}_f)/\mathrm{K}_2,$$

the point x corresponds to an element $(h, g) \in \mathbf{G}_2(\mathbb{Q})\backslash X_2 \times \mathbf{G}_2(\mathbb{A}_f)/\mathrm{K}$, with $h(\mathbb{C}^\times) \subset \mathbf{T}(\mathbb{R})$. More generally, if K is a field of characteristic 0 which contains $\mathbf{E}_2$ and $x \in \mathrm{Sh}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(K)$, we say x is a special point if for some (equivalently any) $\mathbf{E}_2$-algebra embedding $K \hookrightarrow \mathbb{C}$, the induced $\mathbb{C}$-point of $\mathrm{Sh}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)$ is a special point.

(2) The integral model $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)$ and the associated objects such as the μ-ordinary locus and the groups $I_{\overline{x}}$ all implicitly depend on the choice of auxiliary Shimura datum $(\mathbf{G}, X)$ and the choice of Hodge embedding ι_2. It is possible to show that they are independent of choices: If $(\mathbf{G}', X')$ is a different choice of auxiliary Hodge-type Shimura datum, we can find another auxiliary Hodge-type Shimura datum $(\mathbf{G}'', X'')$ equipped with maps $(\mathbf{G}', X') \leftarrow (\mathbf{G}'', X'') \rightarrow (\mathbf{G}, X)$ by considering (a suitable modification of) $\mathbf{G}' \times_{\mathbf{G}^{\mathrm{ad}},\mathbb{G}_m} \mathbf{G}$. The independence then follows from Corollary 5.3.6. If ι_2' is a different choice of Hodge embedding for $(\mathbf{G}_2, X_2)$, we can show that

the strata $\mathcal{S}_{\mathrm{K}_2,[b]}$ and the groups $I_{\overline{x}}$ are independent of this choice by considering the product Hodge embedding $\iota_2 \times \iota_2'$. Since these independence properties are not essential for applications, we omit the details.

(3) The assumption that $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is a strongly acceptable triple of Hodge type is weaker than requiring that $\widetilde{\mathcal{G}}_2 = \mathcal{G}_2$. It only requires the existence of the auxiliary triple $(\mathbf{G}, X, \mathcal{G})$ of Hodge type with $\widetilde{\mathcal{G}} = \mathcal{G}$.

5.4.9. Note that $(\mathbf{G}, X, \mathcal{G})$ is also a strongly acceptable triple with $(\mathbf{G}, X)$ of Hodge type. Theorem 5.4.7 will be reduced to the following special case.

Proposition 5.4.10. *Assume* $(\mathbf{G}_2, X_2, \mathcal{G}_2) = (\mathbf{G}, X, \mathcal{G})$ *and the Hodge embeddings* ι *and* ι_2 *coincide. Then Theorem 5.4.7 holds.*

Proof. Under these assumptions, we have $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X) = \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)$ and the integral model is constructed as in §5.1.4. Moreover $\mathcal{G}_2$ is a connected parahoric and we have $\widetilde{\mathcal{G}}_2 = \mathcal{G}_2$. Since the definition of $I_{\overline{x}}$ is independent of the prime-to-p level, it suffices to consider the case of neat K_2^p. Applying the construction in §4.3, we obtain a parahoric model $\mathcal{M}$ of a Levi subgroup $M \subset G_2$, and an M-valued cocharacter $\widetilde{\lambda}$ lying in the G_2-conjugacy class of μ_{h_2} and such that $\widetilde{\lambda}$ is central in M. Let $\mathscr{G}$ be the $(\mathcal{M}, \widetilde{\lambda})$-adapted deformation to $\mathcal{O}_K$ constructed in Theorem 4.3.5. By Proposition 5.1.8, $\mathscr{G}$ corresponds to a point $\widetilde{x} \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ lifting $\overline{x}$ and hence to an abelian variety $\mathcal{A}_{2,\widetilde{x}}$ over K. By Theorem 4.3.5, the action of $J_b(\mathbb{Q}_p)$ on $\mathscr{G}_{\overline{x}}$ lifts to $\mathscr{G}$. Since $I_{\overline{x}}(\mathbb{Q}) \subset J_b(\mathbb{Q}_p)$, by the Serre–Tate theorem, the action of $I_{\overline{x}}$ lifts to $\mathcal{A}_{2,\widetilde{x}}$ in the isogeny category.

We now show $\widetilde{x}$ is a special point. Since $I_{\overline{x}}$ fixes the tensors $s_{\alpha,0,\overline{x}}$, it also fixes $s_{\alpha,p,\widetilde{x}}$, and hence it fixes $s_{\alpha,B}$. Thus we may consider $I_{\overline{x}}$ as a subgroup of $\mathbf{G}_2$. By [KPS, Theorem 6], the absolute rank of $I_{\overline{x}}$ is equal to the absolute rank of $\mathbf{G}_2$. Let $\mathbf{T}$ be a maximal torus of $I_{\overline{x}}$, which is therefore a maximal torus of $\mathbf{G}_2$. The Mumford–Tate group of $\mathcal{A}_{2,\widetilde{x}}$ is a subgroup of $\mathbf{G}_2$ which commutes with $\mathbf{T}$ hence must be contained in $\mathbf{T}$. Therefore $\widetilde{x}$ is a special point. □

5.4.11. To prove Theorem 5.4.7 in general, we use the auxiliary construction from Proposition 5.3.8. For notational convenience, we write $(\mathbf{G}_1, X_1)$ for $(\mathbf{G}, X)$ and $\iota_1 : (\mathbf{G}_1, X_1) \to (\mathbf{GSp}(V_1), S_1^{\pm})$ for the Hodge embedding ι. Then $\mathbf{G}_3$ is defined to be the identity component of $\mathbf{G}_1 \times_{\mathbf{G}_{1,\mathrm{ad}},\mathbb{G}_m} \mathbf{G}_2$. We obtain a Shimura datum $(\mathbf{G}_3, X_3)$ together with morphisms $(\mathbf{G}_1, X_1) \leftarrow (\mathbf{G}_3, X_3) \to (\mathbf{G}_2, X_2)$ and a Hodge embedding $\iota_3 : (\mathbf{G}_3, X_3) \to (\mathbf{GSp}(V_3), S_3^{\pm})$, where $V_3 = V_1 \oplus V_2$.

For $i = 1, 2, 3$, let $\mathbf{E}_i$ denote the reflex field of $(\mathbf{G}_i, X_i)$; then we have $\mathbf{E}_3 \subset \mathbf{E}' := \mathbf{E}_1\mathbf{E}_2$. We let v_i (resp. v') denote the place of $\mathbf{E}_i$ (resp. $\mathbf{E}'$) induced by the embedding i_p and we let E_i (resp. E') denote the completion. By construction, we have $E' = E_2$. Set $G_i := \mathbf{G}_{i,\mathbb{Q}_p}$, and let $\mathcal{G}_1$ (resp. $\mathcal{G}_3$) denote the parahoric subgroup of G_1 (resp. G_3) determined by $\mathcal{G}_2$. For $i = 1, 2, 3$, we set $\mathrm{K}_{i,p} := \mathcal{G}_i(\mathbb{Z}_p)$ and we fix compact open subgroups $\mathrm{K}_i^p \subset \mathbf{G}_i(\mathbb{A}_f^p)$ such that K_3^p maps to K_1^p and K_2^p. We set $\mathrm{K}_i := \mathrm{K}_{i,p}\mathrm{K}_i^p$. We then have integral models $\mathscr{S}_{\mathrm{K}_i}(\mathbf{G}_i, X_i)$, where $\mathscr{S}_{\mathrm{K}_1}(\mathbf{G}_1, X_1)$ is constructed from the good Hodge embedding ι_1 and $\mathscr{S}_{\mathrm{K}_i}(\mathbf{G}_i, X_i)$ for $i = 2, 3$ is constructed from $\mathscr{S}_{\mathrm{K}_1}(\mathbf{G}_1, X_1)$ by viewing $(\mathbf{G}_i, X_i)$ as Shimura data of abelian type and $(\mathcal{G}_1, X_1)$ as the auxiliary Hodge type datum.

5.4.12. Let $\mathbf{H}$ denote the subgroup of $\mathbf{GSp}(V_1) \times \mathbf{GSp}(V_2)$ consisting of elements (g_1, g_2) such that $c_1(g_1) = c_2(g_2)$. Then the natural map $\mathbf{G}_3 \to \mathbf{GSp}(V_1) \times \mathbf{GSp}(V_2)$ factors through $\mathbf{H}$ and we let S_H denote the $\mathbf{H}_{\mathbb{R}}$-conjugacy class of homomorphisms

$\mathbb{S} \to \mathbf{H}_{\mathbb{R}}$ induced by X_3. There are natural morphisms of Shimura data $(\mathbf{H}, S_H) \to (\mathbf{GSp}(V_i), S_i^{\pm})$ for $i = 1, 2, 3$.

We let $V_{1,\mathbb{Z}_p} \subset V_{1,\mathbb{Q}_p}$ be a $\mathbb{Z}_p$-lattice such that ι_1 is good with respect to $V_{1,\mathbb{Z}_p}$ and we set $V_{3,\mathbb{Z}_p} := V_{1,\mathbb{Z}_p} \oplus V_{2,\mathbb{Z}_p} \subset V_{3,\mathbb{Q}_p}$. For $i = 1, 2, 3$, we let $\mathrm{K}'_{i,p}$ denote the stabilizer of $V_{i,\mathbb{Z}_p}$ inside $\mathbf{GSp}(V_{i,\mathbb{Q}_p})$ and let H_p denote the stabilizer of $V_{3,\mathbb{Z}_p}$ inside $\mathbf{H}(\mathbb{Q}_p)$. We also fix compact open subgroups $\mathrm{K}_i'^p \subset \mathbf{GSp}(V_{i,\mathbb{A}_f^p})$ containing the image of K_i^p for $i = 1, 2, 3$, $\mathrm{H}^p \subset \mathbf{H}(\mathbb{A}_f^p)$ containing the image of K_3^p, and we set $\mathrm{K}'_i = \mathrm{K}'_{i,p}\mathrm{K}_i'^p$, $\mathrm{H} = \mathrm{H}_p\mathrm{H}^p$. Then the Shimura variety $\mathrm{Sh}_{\mathrm{H}}(\mathbf{H}, S_H)$ admits a moduli interpretation as pairs of tuples $(A_i, \lambda_i, \epsilon_i)$, $i = 1, 2$, where A_i are abelian varieties of $\dim(V_i)/2$, λ_i is a weak polarization, and ϵ_i are level $\mathrm{Im}(\mathrm{H}^p \to \mathbf{GSp}(V_{i,\mathbb{A}_f^p}))$-structures which preserve symplective pairings up to the same $\mathbb{A}_f^{p\times}$-scalar (cf. [Zho20, 7.2]). This moduli problem extends to $\mathbb{Z}_{(p)}$, hence we obtain an integral model $\mathscr{S}_{\mathrm{H}}(\mathbf{H}, S_H)/\mathbb{Z}_{(p)}$.

Proposition 5.4.13. *There is a commutative diagram of $\mathcal{O}_{E'}$-stacks*
(5.4.13.1)

$$\begin{array}{ccccc}
\mathscr{S}_{\mathrm{K}_1}(\mathbf{G}_1, X_1)_{\mathcal{O}_{E'}} & \xleftarrow{j_1} & \mathscr{S}_{\mathrm{K}_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}} & \xrightarrow{j_2} & \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'}} \\
\downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_2} \\
\mathscr{S}_{\mathrm{K}'_1}(\mathbf{GSp}(V_1), S_1^{\pm})_{\mathcal{O}_{E'}} & \longleftarrow & \mathscr{S}_{\mathrm{K}'}(\mathbf{H}, S_H)_{\mathcal{O}_{E'}} & \longrightarrow & \mathscr{S}_{\mathrm{K}'}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E'}}
\end{array}.$$

Proof. It suffices to consider the case of neat prime-to-p level structure so that we may assume all objects are schemes. The existence of the bottom row follows from the moduli interpretations of the integral models. The morphisms j_1 and j_2 are constructed in a similar way to Proposition 5.3.8, using that $\mathscr{S}_{\mathrm{K}_i}(\mathbf{G}_i, X_i)$ for $i = 2, 3$, are constructed via the auxiliary Shimura datum $(\mathbf{G}_1, X_1)$.

The morphism i_1 exists by construction of $\mathscr{S}_{\mathrm{K}_1}(\mathbf{G}_1, X_1)_{\mathcal{O}_{E'}}$ and i_2 is constructed in Proposition 5.3.8. For ι_3, note that there is a finite morphism

$$\mathscr{S}_{\mathrm{K}'}(\mathbf{H}, S_H)_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_1}(\mathbf{GSp}(V_1), S_1^{\pm})_{\mathcal{O}_{E'}} \times \mathscr{S}_{\mathrm{K}'_2}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E'}}.$$

The morphism

$$\mathscr{S}_{\mathrm{K}_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_1}(\mathbf{GSp}(V_1), S_1^{\pm})_{\mathcal{O}_{E'}} \times \mathscr{S}_{\mathrm{K}'_2}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E'}}$$

induced by $(i_1 \circ j_1, i_2, \circ j_2)$ factors through $\mathscr{S}_{\mathrm{K}'}(\mathbf{H}, S_H)_{\mathcal{O}_{E'}}$ on the generic fiber, and hence lifts to a morphism $i_3 : \mathscr{S}_{\mathrm{K}_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'}(\mathbf{H}, S_H)_{\mathcal{O}_{E'}}$ as desired. □

5.4.14. Let $\mathcal{A}_i \to \mathscr{S}_{\mathrm{K}_i}(\mathbf{G}_i, X_i)_{\mathcal{O}_{E'}}$, denote the pullback of the universal abelian variety along $\mathscr{S}_{\mathrm{K}_i}(\mathbf{G}_i, X_i)_{\mathcal{O}_{E'}} \to \mathscr{S}_{\mathrm{K}'_i}(\mathbf{GSp}(V_i, S_i^{\pm})_{\mathcal{O}_{E'}}$. For $i = 3$, this map factors through $\mathscr{S}_{\mathrm{H}}(\mathbf{H}, S_H)_{\mathcal{O}_{E'}}$ and there is an identification

$$\mathcal{A}_3 \cong j_1^*\mathcal{A}_1 \times j_2^*\mathcal{A}_2. \tag{5.4.14.1}$$

Let $\overline{x}_3 \in \mathscr{S}_{\mathrm{K}_3}(\mathbf{G}_3, X_3)(k)$ and write $\overline{x}_1 \in \mathscr{S}_{\mathrm{K}_1}(\mathbf{G}_1, X_1)(k)$, $\overline{x}_2 \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(k)$ for the image of $\overline{x}_3$ under j_1 and j_2. The isomorphism (5.4.14.1) implies we have an isomorphism $\mathcal{A}_{3,\overline{x}_3} \cong \mathcal{A}_{1,\overline{x}_1} \times \mathcal{A}_{2,\overline{x}_2}$. We let $I_{\overline{x}_3} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{3,\overline{x}_3})$, $I_{\overline{x}_1} \subset \mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{1,\overline{x}_1})$ denote the groups constructed in the same way as §5.4.6.

Proposition 5.4.15. *There are natural exact sequences:*

$$0 \longrightarrow \mathbf{C}_1 \longrightarrow I_{\overline{x}_3} \longrightarrow I_{\overline{x}_1} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{C}_2 \longrightarrow I_{\overline{x}_3} \longrightarrow I_{\overline{x}_2} \longrightarrow 0$$

where $\mathbf{C}_1$ *(resp.* $\mathbf{C}_2$*) is the kernel of the map* $f:\mathbf{G}_3\to\mathbf{G}_1$ *(resp.* $g:\mathbf{G}_3\to\mathbf{G}_2$*).*

Proof. Since $\mathbf{G}_3\subset\mathbf{H}$, we may assume that the set of tensors defining $\mathbf{G}_3\subset\mathbf{GL}(V_3)$ includes tensors corresponding to the projections of $V_{3,\mathbb{Z}_{(p)}}$ onto the direct summands $V_{i,\mathbb{Z}_{(p)}}\subset V_{3,\mathbb{Z}_{(p)}}$ for $i=1,2$. It follows that $I_{\overline{x}_3}$ respects the product decomposition $\mathcal{A}_{3,\overline{x}_3}\cong\mathcal{A}_{1,\overline{x}_1}\times\mathcal{A}_{2,\overline{x}_2}$ and hence we obtain a natural map $I_{\overline{x}_3}\to\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{1,\overline{x}_1})$. Similarly, by considering the pullback to V_3 of tensors defining $\mathbf{G}_1$, one can show that $I_{\overline{x}_3}\to\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{1,\overline{x}_1})$ factors through $I_{\overline{x}_1}$. We obtain a natural map $I_{\overline{x}_3}\to I_{\overline{x}_1}$.

Let $\widetilde{x}_3\in\mathscr{S}_{\mathrm{K}_3}(\mathbf{G}_3,X_3)(\mathcal{O}_K)$ denote a lift of $\overline{x}_3$. Since $\mathbf{C}_1$ lies in the center of $\mathbf{G}_3$, we have natural maps

$$\mathbf{C}_1\to\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{3,\widetilde{x}_3}\otimes_K\overline{K})\to\mathrm{Aut}_{\mathbb{Q}}(\mathcal{A}_{3,\overline{x}_3,k})$$

whose image lies in $I_{\overline{x}_3}$.

We thus obtain a sequence $\mathbf{C}_1\to I_{\overline{x}_3}\to I_{\overline{x}_1}$ and it suffices to check the exactness upon base changing to $\mathbb{Q}_\ell$ for some prime $\ell\neq p$. By [KPS, Theorem 6] there is a semisimple element $\gamma_\ell\in\mathbf{G}_3(\mathbb{Q}_\ell)$ such that the natural inclusion $I_{\overline{x}_3}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell\subset\mathbf{G}_{3,\mathbb{Q}_\ell}$ (resp. $I_{\overline{x}_1}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell\subset\mathbf{G}_{1,\mathbb{Q}_\ell}$) identifies $I_{\overline{x}_3}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$ (resp. $I_{\overline{x}_1}\otimes_{\mathbb{Q}}\mathbb{Q}_\ell$) with the centralizer of γ_ℓ in $\mathbf{G}_{3,\mathbb{Q}_\ell}$ (resp. $f(\gamma_\ell)$ in $\mathbf{G}_{1,\mathbb{Q}_\ell}$). We thus obtain the first exact sequence and the argument for $I_{\overline{x}_2}$ is analogous. □

5.4.16. We can now prove the general case of Theorem 5.4.7.

Proof of Theorem 5.4.7. It suffices to consider the case of neat prime-to-p level structure. For $i=1,2,3$, we write $\mathcal{S}_{\mathrm{K}_i}$ for the special fiber of the integral model $\mathscr{S}_{\mathrm{K}_i}(\mathbf{G}_i,X_i)$. Let $\overline{x}_2\in\mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}(k)$. We first assume $\overline{x}_2=j_2(\overline{x}_3)$ for some $\overline{x}_3\in\mathcal{S}_{\mathrm{K}_3}(k)$; by Lemma 2.2.8 we have $\overline{x}_3\in\mathcal{S}_{\mathrm{K}_3,[b]_{\mu_3}}(k)$. Let $\overline{x}_1\in\mathcal{S}_{\mathrm{K}_1,[b]_{\mu_1}}(k)$ denote the image of $\overline{x}_3$. By Proposition 5.4.10, there exists $K/\breve{\mathbb{Q}}_p$ finite and $\widetilde{x}_1\in\mathrm{Sh}_{\mathrm{K}_1}(\mathbf{G}_1,X_1)(K)$ lifting $\overline{x}_1$ such that the action of $I_{\overline{x}_1}(\mathbb{Q})$ lifts to $\mathcal{A}_{1,\widetilde{x}_1}$. Then we may consider $I_{\overline{x}_1}$ as a subgroup of $\mathbf{G}_1$ and we let $\mathbf{T}_1$ denote the connected component of the center of $I_{\overline{x}_1}$. The Mumford–Tate group of $\mathcal{A}_{1,\widetilde{x}_1}$ is a connected subgroup of $\mathbf{G}_1$ which commutes with $I_{\overline{x}_1}$, hence is contained in $\mathbf{T}_1$, as $I_{\overline{x}_1}$ and $\mathbf{G}_1$ have the same rank.

Let $\mathbf{T}_3\subset\mathbf{G}_3$ denote the identity component of the preimage of $\mathbf{T}_1$ in $\mathbf{G}_3$ and $\mathbf{T}_2$ the image of $\mathbf{T}_3$ in $\mathbf{G}_2$. By construction, the morphisms of integral models

$$\mathscr{S}_{\mathrm{K}_1}(\mathbf{G}_1,X_1)_{\mathcal{O}_{E'}}\leftarrow\mathscr{S}_{\mathrm{K}_3}(\mathbf{G}_3,X_3)_{\mathcal{O}_{E'}}\to\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)_{\mathcal{O}_{E'}}$$

induce isomorphisms of completions at geometric points in the special fiber. Thus let $\widetilde{x}_3$ (resp. $\widetilde{x}_2$) denote the point lifting $\overline{x}_3$ (resp. $\overline{x}_2$) corresponding to $\widetilde{x}_1$. Then the Mumford–Tate group for $\mathcal{A}_{3,\widetilde{x}_3}$ (resp. $\mathcal{A}_{2,\widetilde{x}_2}$) is contained in $\mathbf{T}_3$ (resp. $\mathbf{T}_2$). It follows from Proposition 5.4.15 that $I_{\overline{x}_3}$ (resp. $I_{\overline{x}_2}$) is contained in the centralizer of $\mathbf{T}_3$ in $\mathbf{G}_3$ (resp. $\mathbf{T}_2$ in $\mathbf{G}_2$), and hence the action of $I_{\overline{x}_2}(\mathbb{Q})$ lifts to one on $\mathcal{A}_{\widetilde{x}_2}$.

Now let $\overline{x}_2\in\mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}(k)$ be any point. It suffices to prove the result with $\mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2,X_2)$ in place of $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2,X_2)$, and with $\overline{x}_2$ replaced by a lift to a point of $\mathcal{S}_{\mathrm{K}_{2,p},[b]_{\mu_2}}(k)$, which we will again denote $\overline{x}_2$. Recall $J\subset\mathbf{G}_2(\mathbb{Q}_p)$ is a set mapping bijectively to a set of coset representatives for the image of (5.2.3.1). Then by the construction of $\mathscr{S}_{\mathrm{K}_2,p}(\mathbf{G}_2,X_2)$ via $\mathscr{S}_{\mathrm{K}_1,p}(\mathbf{G}_1,X_1)$ in §5.2.3, there exists $j\in J$ such that $\overline{x}_2\in[\mathscr{S}_{\mathrm{K}_{1,p}}(\mathbf{G}_1,X_1)^+\times\mathscr{A}(\mathbf{G}_{2,\mathbb{Z}_{(p)}})j]/\mathscr{A}(\mathbf{G}_{1,\mathbb{Z}_{(p)}})^\circ$. We let $\overline{x}_2'\in[\mathscr{S}_{\mathrm{K}_{1,p}}(\mathbf{G}_1,X_1)^+\times\mathscr{A}(\mathbf{G}_{2,\mathbb{Z}_{(p)}})]/\mathscr{A}(\mathbf{G}_{1,\mathbb{Z}_{(p)}})^\circ$ be the point corresponding to $\overline{x}_2$ under the isomorphism induced by j. Then upon modifying $\overline{x}_2$ by an element of

$\mathbf{G}_2(\mathbb{A}_f^p)$ which only changes the abelian variety $\mathcal{A}_{2,\overline{x}_2}$ up to prime-to-p isogeny, we may assume $\overline{x}_2' = j_2(\overline{x}_3')$ for some $\overline{x}_3' \in \mathscr{S}_{\mathrm{K}_{3,p}}(\mathbf{G}_3, X_3)(k)$.

Let $\widetilde{x}_2' \in \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ be a lift of of $\overline{x}_2'$, for some finite extension $K/\breve{\mathbb{Q}}_p$. By construction, corresponding to the element j, there is (after possibly increasing K) a point $\widetilde{x}_2 \in \mathscr{S}_{\mathrm{K}_{2,p}}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ lifting $\overline{x}_2$, and a p-power quasi-isogeny $\mathcal{A}_{2,\widetilde{x}_2} \to \mathcal{A}_{2,\widetilde{x}_2'}$ taking $s_{\alpha,0,\overline{x}_2}$ to $s_{\alpha,0,\overline{x}_2'}$ (resp. $s_{\alpha,\ell,\overline{x}_2}$ to $s_{\alpha,\ell,\overline{x}_2'}$ for $\ell \neq p$). By considering the reduction of this quasi-isogeny one sees that $\overline{x}_2' \in \mathcal{S}_{\mathrm{K}_{2,p},[b]_\mu}(k)$, and one also obtains an induced isomorphism $I_{\overline{x}_2} \cong I_{\overline{x}_2'}$. From what we saw above, it follows that we may choose $\widetilde{x}_2'$ such that the action of $I_{\overline{x}_2'}$ lifts to $\mathcal{A}_{2,\widetilde{x}_2'}$. Then the action of $I_{\overline{x}_2} \cong I_{\overline{x}_2'}$ lifts to $\mathcal{A}_{2,\widetilde{x}_2}$. □

5.4.17. We will use the above to deduce properties about the conjugacy class of Frobenius as in [Kis17, §2.3]. Assume $\overline{x} \in \mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}(k)$ arises from an $\mathbb{F}_q$-point $x \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(\mathbb{F}_q)$ where $\mathbb{F}_q$ is a finite extension of k_{E_2}. For $\ell \neq p$ a prime, let γ_ℓ denote the geometric q-Frobenius in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the dual of the ℓ-adic Tate module $T_\ell\mathcal{A}_{2,\overline{x}}^\vee$. Since the tensors $s_{\alpha,\ell,\overline{x}} \in T_\ell\mathcal{A}_{2,\overline{x}}^\otimes$ are Galois-invariant, we may consider γ_ℓ as an element of $\mathbf{G}_2(\mathbb{Q}_\ell)$ via the level structure $V_{\mathbb{Q}_\ell} \cong T_\ell\mathcal{A}_{2,\overline{x}} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Corollary 5.4.18. *Assume $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is a strongly acceptable triple of Hodge type. Suppose $\overline{x} \in \mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}(k)$ arises from $x \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(\mathbb{F}_q)$. There exists an element $\gamma_0 \in \mathbf{G}_2(\mathbb{Q})$, such that*

(1) For $\ell \neq p$, γ_0 is conjugate to γ_ℓ in $\mathbf{G}_2(\mathbb{Q}_\ell)$.
(2) γ_0 is elliptic in $\mathbf{G}_2(\mathbb{R})$.

Proof. The proof is the same as in [Kis17, Corollary 2.3.1]. Since $\mathcal{A}_{2,x}$ is defined over $\mathbb{F}_q$, the q-Frobenius γ lies in $I_{\overline{x}}(\mathbb{Q})$. Let $\widetilde{x} \in \mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)(K)$ denote the lifting constructed in Theorem 5.4.7. Then by considering the action of $I_{\overline{x}}(\mathbb{Q})$ on the Betti cohomology of $\mathcal{A}_{2,\widetilde{x}}$, we may consider $I_{\overline{x}}(\mathbb{Q})$ as a subgroup of $\mathbf{G}_2(\mathbb{Q})$. Defining γ_0 to be the image of γ inside $\mathbf{G}_2(\mathbb{Q})$, we have that γ_0 is conjugate to γ_ℓ in $\mathbf{G}_2(\mathbb{Q}_\ell)$ by the Betti-étale comparison isomorphism. If $\mathbf{T}$ is any torus in $I_{\overline{x}}$ containing γ_0, the positivity of the Rosati involution implies $\mathbf{T}(\mathbb{R})/w_{h_2}(\mathbb{R}^\times)$ is compact. Hence $\gamma_0 \in \mathbf{T}(\mathbb{Q})$ is elliptic in $\mathbf{G}_2(\mathbb{R})$. □

Remark 5.4.19. The elements γ_ℓ arise as the local Frobenii acting on the stalk of a $\mathbf{G}_2(\mathbb{Q}_\ell)$-local system $\mathbb{L}_\ell$ over $\mathcal{S}_{\mathrm{K}_2}$; see §6.1.1. Thus even though the proof of Corollary 5.4.18 uses the Hodge embedding ι_2 in order to define the abelian variety $\mathcal{A}_{2,\widetilde{x}}$, one can view it as proving a property of the local systems $\mathbb{L}_\ell$ over $\mathcal{S}_{\mathrm{K}_2,[b]_{\mu_2}}$, which are intrinsic to $\mathscr{S}_{\mathrm{K}_2}(\mathbf{G}_2, X_2)$; cf. Remark 5.4.8 (2). In particular, the image of γ_0 in $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ is independent of choices (cf. Remark 5.4.8 (2)).

6. Independence of ℓ for Shimura varieties

We now apply the results of the previous section to prove ℓ-independence for the conjugacy class of Frobenius at all points on the special fiber of Shimura varieties.

6.1. **Frobenius conjugacy classes.**

6.1.1. We keep the notation of the previous section but now $(\mathbf{G}, X)$ will be an acceptable Shimura datum of Hodge type. As before we let $\mathcal{G}$ be a parahoric group scheme of $G = \mathbf{G}_{\mathbb{Q}_p}$ and set $\mathrm{K}_p = \mathcal{G}(\mathbb{Z}_p)$. Then we have the integral model $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ over $\mathcal{O}_E$ constructed from a fixed auxiliary Hodge-type Shimura datum

$(\mathbf{G}_1, X_1)$ as in Proposition 5.2.6 and a good Hodge embedding ι_1. The auxiliary Shimura datum $(\mathbf{G}_1, X_1)$ plays a minor role in what follows.

Let $p > 2$ and $\ell \neq p$ be primes and suppose that in addition the compact open subgroup $\mathrm{K} \subset \mathbf{G}(\mathbb{A}_f)$ is of the form $\mathrm{K}_\ell \mathrm{K}^\ell$, with $\mathrm{K}_\ell \subset \mathbf{G}(\mathbb{Q}_\ell)$ and $\mathrm{K}^\ell \subset \mathbf{G}(\mathbb{A}_f^\ell)$. We let $\widetilde{\mathbb{L}}_\ell$ denote the $\mathbf{G}(\mathbb{Q}_\ell)$-local system on $\mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ arising from the pro-étale covering

$$\mathscr{S}_{\mathrm{K}^\ell}(\mathbf{G}, X) := \lim_{\substack{\leftarrow \\ \mathrm{K}'_\ell \subset \mathrm{K}_\ell}} \mathscr{S}_{\mathrm{K}'_\ell \mathrm{K}^\ell}(\mathbf{G}, X) \to \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$$

and we write $\mathbb{L}_\ell$ for the induced local system on the special fiber $\mathcal{S}_{\mathrm{K}}$ over k_E. If $\iota : (\mathbf{G}, X) \to (\mathbf{GSp}(V), S^\pm)$ is a Hodge embedding as in §5.4.1 then we have an identification

$$\mathbb{L}_\ell = \underline{\mathrm{Isom}}_{(s_\alpha, s_{\alpha,\ell})}(V_{\mathbb{Q}_\ell}, \mathcal{V}_\ell^\vee) \tag{6.1.1.1}$$

where the scheme classifies $\mathbb{Q}_\ell$-linear isomorphisms taking s_α to $s_{\alpha,\ell}$; here the notation is as in §5.4.1.

6.1.2. Let $y \in \mathcal{S}_{\mathrm{K}}(\mathbb{F}_q)$ and we write $\overline{y}$ for the induced geometric point of $\mathcal{S}_{\mathrm{K}}$. We let $\mathcal{S}^0_{\mathrm{K}}$ denote the connected component of $\mathcal{S}_{\mathrm{K}}$ containing y and $\overline{x} \in \mathcal{S}^0_{\mathrm{K}}(k)$ a fixed geometric point. Over $\mathcal{S}^0_{\mathrm{K}}$, the $\mathbf{G}(\mathbb{Q}_\ell)$-local system $\mathbb{L}_\ell$ corresponds to a homomorphism

$$\rho^0_\ell : \pi_1(\mathcal{S}^0_{\mathrm{K}}, \overline{x}) \to \mathbf{G}(\mathbb{Q}_\ell).$$

We have a map

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to \pi_1(\mathcal{S}^0_{\mathrm{K}}, \overline{y}) \xrightarrow{\sim} \pi_1(\mathcal{S}^0_{\mathrm{K}}, \overline{x}),$$

where the isomorphism $\pi_1(\mathcal{S}^0_{\mathrm{K}}, \overline{y}) \xrightarrow{\sim} \pi_1(\mathcal{S}^0_{\mathrm{K}}, \overline{x})$ is well-defined up to conjugation. We thus obtain a well defined conjugacy class in $\pi_1(\mathcal{S}^0_{\mathrm{K}}, \overline{x})$ corresponding to the image of the geometric q-Frobenius and we write Frob_y for a representative of this conjugacy class.

6.1.3. For a reductive group H over a field F of characteristic 0, we write Conj_H for the variety of conjugacy classes in H. Explicitly, if $H = \mathrm{Spec}\ R$, the action of H on itself via conjugation induces an action of H on R, and we have $\mathrm{Conj}_H = \mathrm{Spec}\ R^H$. Then Conj_H is an F-variety which is a universal categorical quotient for this action, and the set $\mathrm{Conj}_H(\overline{F})$ can be identified with the set of semisimple $H(\overline{F})$ conjugacy classes in $H(\overline{F})$ (see [MF82, Chapters 0,1]). We write $\chi_H : H \to \mathrm{Conj}_H$ for the projection map. For example if $H = \mathrm{GL}_n$, $\mathrm{Conj}_{\mathrm{GL}_n}$ is the variety $\mathbb{A}^{n-1}_F \times \mathbb{G}_{m,F}$ and the map χ takes an element of GL_n to its associated characteristic polynomial.

In our setting, we thus obtain for each prime $\ell \neq p$, a well-defined element $\gamma_{y,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ corresponding to $\chi_{\mathbf{G}}(\rho^0_\ell(\mathrm{Frob}_y))$. Our main theorem concerning the ℓ-independence property of Shimura varieties is the following.

Theorem 6.1.4. *Let $p > 2$ and $(\mathbf{G}, X, \mathcal{G})$ a strongly acceptable triple of Hodge type. Assume that $G = \mathbf{G}_{\mathbb{Q}_p}$ is quasi-split and $\mathcal{G}$ is a very special parahoric group scheme. Let $y \in \mathcal{S}_{\mathrm{K}}(\mathbb{F}_q)$ where $\mathbb{F}_q/k_E$ is a finite extension. Then there exists an element $\gamma_0 \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that $\gamma_0 = \gamma_{y,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.*

Remark 6.1.5. A group theoretic argument shows that if we assume in addition that $\mathbf{G}^{\mathrm{der}}$ is simply-connected, γ can be lifted to an element of $\mathbf{G}(\mathbb{Q})$ (cf. Corollary 7.3.4). See also Remark 7.3.5 about the expectations surrounding liftability of γ.

The rest of §6 will be devoted to the proof of Theorem 6.1.4.

6.2. **Explicit curves in the special fiber of local models.**

6.2.1. We begin by recalling the local model diagram and properties of the Kottwitz–Rapoport stratification. By Theorem 5.2.12 (3), there exists a diagram of stacks

(6.2.1.1)
$$\begin{array}{ccc} & \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}}(\mathbf{G}, X) & \\ {\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle q} \\ \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X) & & \mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}} \end{array}$$

where $\pi : \widetilde{\mathscr{S}}^{\mathrm{ad}}_{\mathrm{K}}(\mathbf{G}, X) \to \mathscr{S}_{\mathrm{K}}(\mathbf{G}, X)$ is a $\mathcal{G}^{\mathrm{ad}}$-torsor.

Let $\mathcal{M}$ denote the special fiber of $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}}$; it is a scheme over k_E. Recall that $\mathbb{M}^{\mathrm{loc}}_{\mathcal{G},\{\mu_h\}} = \mathrm{M}^{\mathrm{loc}}_{\mathcal{G}',\{\mu\}}$ is defined using a group $G' \cong \prod_{i=1}^r \mathrm{Res}_{F_i/\mathbb{Q}_p} H$ such that there is an isomorphism $G'^{\mathrm{ad}} \cong G^{\mathrm{ad}}$, and the parahoric group scheme $\mathcal{G}'$ of G' is determined by $\mathcal{G}$. Then the geometric special fiber $\mathcal{M}_k$ has a stratification indexed by $\mathrm{Adm}_{G'}(\{\mu\})_{J'}$ (cf §3.1.11). Here we consider $\mathrm{Adm}_{G'}(\{\mu\})_{J'} \subset W'_{J'}\backslash W'/W'_{J'}$ where W' is the Iwahori Weyl group for G' and $J' \subset \mathbb{S}'$ is the subset of simple reflections for G' determined by $\mathcal{G}'$. We write $\mathcal{M}_k^w$ for the strata corresponding to $w \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$. It follows formally from the existence of the diagram (6.2.1.1) and the fact that $\mathcal{G}^{\mathrm{ad}}$-orbits on $\mathcal{M}_k$ and $\mathcal{G}'$-orbits on $\mathcal{M}_k$ agree, that $\mathcal{S}_{\mathrm{K},k}$ admits a stratification by $\mathrm{Adm}_{G'}(\{\mu\})_{J'}$. This is known as the Kottwitz–Rapoport stratification and we write $\mathcal{S}^w_{\mathrm{K},k}$ for the strata corresponding to $w \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$. From the definition of this stratification, for $\overline{x} \in \mathcal{S}_{\mathrm{K}}(k)$ the complete local ring of $\mathcal{S}^w_{\mathrm{K},k}$ at $\overline{x}$ is identified with the complete local ring at a point $\overline{x}' \in \mathcal{M}_k^w(k)$. The closure relations for this stratification are given by the Bruhat order on $W'_{J'}\backslash W'/W'_{J'}$.

6.2.2. For the rest of §6, we assume $(\mathbf{G}, X, \mathcal{G})$ satisfies the assumptions in Theorem 6.1.4. In this case, $\mathcal{M}_k$ and $\mathcal{S}_{\mathrm{K},k}$ are normal schemes; cf. Theorem 5.4.5.

We let $\mathfrak{s} \in \mathcal{B}(G, \breve{\mathbb{Q}}_p)$ denote the special vertex associated to $\mathcal{G}$. This determines a special vertex $\mathfrak{s}' \in \mathcal{B}(G', \breve{\mathbb{Q}}_p)$. In this case the set $\mathrm{Adm}_{G'}(\{\mu\})_{J'}$ has the following alternative description. Let S' denote a maximal $\breve{\mathbb{Q}}_p$-split torus of G' defined over $\mathbb{Q}_p$ such that $\mathfrak{s}' \in \mathcal{A}(G', S', \breve{\mathbb{Q}}_p)$ and T' the centralizer of S'. Fix a Borel subgroup of G' defined over $\mathbb{Q}_p$ and assume we have identified $X_*(T')_I \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathcal{A}(G', S', \breve{\mathbb{Q}}_p)$ via the choice of special vertex $\mathfrak{s}'$. We let $\mu \in X_*(T')_I$ be the image of a dominant representative of $\{\mu\}$ in $X_*(T')$. For $\lambda, \lambda' \in X_*(T')_I^+$, we write $\lambda \preccurlyeq \lambda'$ if $\lambda' - \lambda$ is an *integral* linear combination of positive coroots in the reduced root system Σ' associated to G'; we write $\lambda \prec \lambda'$ if in addition $\lambda \neq \lambda'$. Then there is an identification

$$W'_{J'}\backslash W'/W'_{J'} \cong X_*(T')_I^+,$$

and the ordering $\preccurlyeq$ agrees with the Bruhat order on $W'_{J'}\backslash W'/W'_{J'}$ under this identification (cf. [Lus83]). It follows that we have an identification

$$\mathrm{Adm}_{G'}(\{\mu\})_{J'} = \{t_\lambda | \lambda \in X_*(T')_I^+,\ \lambda \preccurlyeq \mu\}.$$

We will write $\mathcal{M}_k^\lambda$ (resp. $\mathcal{S}^\lambda_{\mathrm{K},k}$) for the strata $\mathcal{M}_k^{t_\lambda}$ (resp. $\mathcal{S}^{t_\lambda}_{\mathrm{K},k}$).

6.2.3. For notational simplicity, we will use $\underline{\mathcal{G}}$ to denote the group $\underline{\mathcal{G}}'_{\mathbb{F}_p[[t]]}$ defined in §3.1.9. Its generic fiber will be denoted $\underline{G}$ and the Iwahori Weyl group $W_{\underline{G}}$ may be identified with the Iwahori Weyl group for G'. As in Theorem 3.1.12, we may identify $\mathcal{M}_k$ with a union of Schubert varieties corresponding to $\mathrm{Adm}_{G'}(\{\mu\})_{J'}$ in $\mathcal{FL}_{\underline{\mathcal{G}}}$. The strata $\mathcal{M}_k^\lambda$ may be identified with the $\underline{\mathcal{G}}(k[[t]])$-orbit of the element $\dot{\underline{t}}_\lambda$ considered as an element in $\mathcal{FL}_{\underline{\mathcal{G}}}$ and by the above discussion, the closure relations between the strata are given by the partial ordering $\preccurlyeq$. Since $t_\mu \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$ is the unique maximal element, it follows that $\mathcal{M}_k^\mu$ is contained in the smooth locus of $\mathcal{M}$ and hence $\mathcal{S}_{\mathrm{K},k}^\mu$ is contained in the smooth locus of $\mathcal{S}_{\mathrm{K},k}$.

The strata $\mathcal{M}_k^\lambda$ and $\mathcal{S}_{\mathrm{K},k}^\lambda$ are both defined over the field of definition of $\lambda \in W'_{J'}\backslash W'/W'_{J'}$. In other words, if n is the smallest positive integer such that $\sigma^n(\lambda) = \lambda$, then $\mathcal{M}_k^\lambda$ and $\mathcal{S}_{\mathrm{K},k}^\lambda$ are both defined over $\mathbb{F}_{p^n}$; we write $\mathcal{M}^\lambda$ and $\mathcal{S}_{\mathrm{K}}^\lambda$ for the models over $\mathbb{F}_{p^n}$.

6.2.4. The key geometric property of the Kottwitz–Rapoport stratification on $\mathcal{M}_k$ that we will need is the following.

Proposition 6.2.5. *Let $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$ with $\lambda \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$ and $\lambda \neq \mu$. There exists a smooth, geometrically connected curve C over $\mathbb{F}_q$ and a map $\phi : C \to \mathcal{M}_{\mathbb{F}_q}$ such that*

(i) There exists $y' \in C(\mathbb{F}_q)$ such that $\phi(y') = y$.

(ii) $\phi^{-1}(\mathcal{M}_k^{\lambda'})$ is open and dense in C for some $\lambda' \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$ with $\lambda \prec \lambda'$.

Remark 6.2.6. Using an ampleness argument, it is easy to show that such a map always exists if we replace $\mathbb{F}_q$ by its algebraic closure k. The key property is that for $\mathcal{M}$, this map exists without extending the residue field. By [Dri12, §6], there are normal and Cohen–Macaulay schemes where this property fails.

Proof of Proposition 6.2.5. The statement depends only on $\mathcal{G}'$ and not on $\mathcal{G}$, so we may assume (for notational simplicity) that $G = G'$. We first show using the $\underline{\mathcal{G}}$-action on $\mathcal{M}$ that it suffices to consider the case

$$y = \dot{\underline{t}}_\lambda \in \underline{G}(k((t)))/\underline{\mathcal{G}}(k[[t]]).$$

Let σ_q denote the q-Frobenius; then since $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$, we have $\sigma_q(\lambda) = \lambda$. Therefore we may choose the lift $\dot{\underline{t}}_\lambda \in \underline{G}(\mathbb{F}_q((t)))$ so that $\dot{\underline{t}}_\lambda \in \mathcal{M}^\lambda(\mathbb{F}_q)$. By Lemma 6.2.7 below, there exists $g \in \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$ such that $g\dot{\underline{t}}_\lambda = y$ in $\mathcal{FL}_{\underline{\mathcal{G}}}$. Therefore if C satisfies the conditions (i) and (ii) for the point $\dot{\underline{t}}_\lambda$, gC satisfies (i) and (ii) for the point y. It therefore suffices to prove the case $y = \dot{\underline{t}}_\lambda$; we make this assumption from now on.

Now since $\lambda \prec \mu$, by Stembridge's Lemma [Rap00, Lemma 2.3], there exists a positive root $\alpha \in \Sigma$ such that $\lambda + \alpha^\vee \preccurlyeq \mu$. Since $\lambda, \mu \in X_*(T)_I^{\sigma_q}$, it follows that

$$\lambda + \sigma_q^i(\alpha^\vee) \preccurlyeq \mu$$

for all i. If $\{\alpha, \sigma_q(\alpha), \dots, \sigma_q^{m-1}(\alpha)\}$ denotes the orbit of α under σ_q, it follows that

$$\lambda' := \lambda + \sum_{i=0}^{m-1} \sigma_q^i(\alpha^\vee) \preccurlyeq \mu,$$

and hence $\lambda' \in \mathrm{Adm}_G(\{\mu\})_J$. Now α determines a relative root $\widetilde{\alpha}$ of $\underline{G}$ over $\mathbb{F}_q((t))$ which we always take to be the short root; then either $2\widetilde{\alpha}$ is a relative root, or no rational multiple of $\widetilde{\alpha}$ is a relative root. We let $U_{\widetilde{\alpha}}$ denote the relative root subgroup corresponding to $\widetilde{\alpha}$ and $\underline{G}_{\widetilde{\alpha}}$ the simply connected covering of the (semi-simple) group generated by $U_{\widetilde{\alpha}}$ and $U_{-\widetilde{\alpha}}$; it is a reductive group over $\mathbb{F}_q((t))$. We will identify $U_{\widetilde{\alpha}}$ with the corresponding unipotent subgroup of $\underline{G}_{\widetilde{\alpha}}$. The parahoric $\underline{\mathcal{G}}$ determines a parahoric model $\underline{\mathcal{G}}_{\widetilde{\alpha}}$ of $\underline{G}_{\widetilde{\alpha}}$ and there is a morphism

$$\iota_{\widetilde{\alpha}} : \mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}} \to \mathcal{FL}_{\underline{\mathcal{G}},\mathbb{F}_q}$$

defined over $\mathbb{F}_q$, where $\mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}$ is the partial affine flag variety associated to $\underline{\mathcal{G}}_{\widetilde{\alpha}}$. Then $\iota_{\widetilde{\alpha}}$ factors as $\mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}} \to \mathcal{FL}_{\underline{\mathcal{G}}'_{\widetilde{\alpha}}} \to \mathcal{FL}_{\underline{\mathcal{G}},\mathbb{F}_q}$, where $\mathcal{FL}_{\underline{\mathcal{G}}'_{\widetilde{\alpha}}}$ is the corresponding partial affine flag variety for the group generated by $U_{\widetilde{\alpha}}$ and $U_{-\widetilde{\alpha}}$. The first map identifies $\mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}$ with the neutral connected component of $\mathcal{FL}_{\underline{\mathcal{G}}'_{\widetilde{\alpha}}}$ by [PR08, §6.a.1] and the second is a proper monomorphism when restricted to a connected component. It follows that $\iota_{\widetilde{\alpha}}$ is a closed immersion. We write $\mathcal{U}_{\widetilde{\alpha}}$ (resp. $\mathcal{U}_{-\widetilde{\alpha}}$) for the group schemes over $\mathbb{F}_q[[t]]$ corresponding to $U_{\widetilde{\alpha}}(\mathbb{F}_q((t))) \cap \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$ (resp. $U_{-\widetilde{\alpha}}(\mathbb{F}_q((t))) \cap \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$). Then we claim that for each positive α, there exists a morphism

$$f : \mathbb{A}^1_{\mathbb{F}_q} \to \mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}$$

defined over $\mathbb{F}_q$ satisfying the following two conditions

(i') $f(0) = \dot{e}$, where $\dot{e}$ is the base point in $\mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}$.

(ii') $f(\mathbb{A}^1_{\mathbb{F}_q} \backslash \{0\}) \subset L^+\mathcal{U}_{\widetilde{\alpha}} \dot{\underline{t}}_{\alpha^\vee} L^+\underline{\mathcal{G}}_{\widetilde{\alpha}}/L^+\underline{\mathcal{G}}_{\widetilde{\alpha}}$.

Assuming the claim we may prove the proposition as follows. We consider the morphism

$$\phi : \mathbb{A}^1_{\mathbb{F}_q} \to \mathcal{FL}_{\underline{\mathcal{G}}}, \quad x \mapsto \dot{\underline{t}}_\lambda(\iota_{\widetilde{\alpha}} \circ f)(x),$$

in other words we translate the composition $\iota_{\widetilde{\alpha}} \circ f$ by $\dot{\underline{t}}_\lambda$. Then condition (i) follows from (i') and condition (ii) follows from (ii') using the fact that λ is dominant.

It remains to prove the existence of f satisfying (i') and (ii'). We will construct f explicitly using a presentation of the group $\underline{G}_{\widetilde{\alpha}}$; it turns out that by [BT84, §4.1.4] there are essentially three distinct cases to consider which we now describe.

If $2\widetilde{\alpha}$ is not a relative root then there is an identification

$$\underline{G}_{\widetilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SL}_2$$

where K is some finite separable extension of $\mathbb{F}_q((t))$ and the parahoric $\underline{\mathcal{G}}_{\widetilde{\alpha}}$ is characterized by the property

$$\underline{\mathcal{G}}_{\widetilde{\alpha}}(k[[t]]) = \mathrm{SL}_2(\mathcal{O}_K \otimes_{\mathbb{F}_q[[t]]} k[[t]]).$$

If $2\widetilde{\alpha}$ is also a relative root, then there is an identification

$$\underline{G}_{\widetilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SU}_3$$

where $K/\mathbb{F}_q((t))$ is finite separable and SU_3 is the special unitary group associated to a hermitian space over a (separable)[4] quadratic extension K'/K. We recall the presentation of the K-group SU_3 in [Tit79, Example 1.15]. We let $\tau \in \mathrm{Gal}(K'/K)$ denote the non-trivial element and we consider the hermitian form on K'^3 given by

$$\langle (x_{-1}, x_0, x_1), (y_{-1}, y_0, y_1) \rangle = \tau(x_{-1})y_1 + \tau(x_0)y_0 + \tau(x_1)y_{-1}.$$

The group SU_3 is the special unitary group attached to this form. For $c, d \in K'$ such that $\tau(c)c + d + \tau(d) = 0$, we define $\mathbf{u}_+(c,d), \mathbf{u}_-(c,d) \in \mathrm{SU}_3(K)$ by

$$\mathbf{u}_\pm(c,d) = I_3 + (g_{rs})$$

[4] Since we have assumed $p > 2$, this is automatic.

where I_3 is the identity matrix and (g_{rs}) is the matrix with entries $g_{\mp 1,0} = -\tau(c)$, $g_{0,\pm 1} = c$, $g_{\mp 1,\pm 1} = d$ and $g_{rs} = 0$ otherwise. The root subgroups $U_{\pm\widetilde{\alpha}}$ are then given by

$$U_{\pm\widetilde{\alpha}}(K) = \{\mathbf{u}_\pm(c,d) | c,d \in K', \tau(c)c + \tau(d) + d = 0\}.$$

Then we may consider the parahoric

$$\underline{\mathcal{G}}_{\widetilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap \mathrm{GL}_3(\mathcal{O}_{K'});$$

we call this the standard parahoric.

When K'/K is unramified this is the only very special parahoric (up to conjugacy). When K'/K is ramified, there is another conjugacy class of very special parahorics in addition to the standard parahoric which we shall call the non-standard parahoric. We let u' be a uniformizer of K' such that $\tau(u') = -u'$ and we define $s \in \mathrm{GL}_3(K')$ to be the element $\mathrm{diag}(1,1,u')$. Then the non-standard parahoric $\underline{\mathcal{G}}_{\widetilde{\alpha}}$ is given by

$$\underline{\mathcal{G}}_{\widetilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap s\mathrm{GL}_3(\mathcal{O}_{K'})s^{-1}.$$

We label the cases as follows.

Case (1): $2\widetilde{\alpha}$ is not a root, $\underline{G}_{\widetilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SL}_2$ and $\underline{\mathcal{G}}_{\widetilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SL}_2(\mathcal{O}_K)$.

Case (2): $2\widetilde{\alpha}$ is a root, $\underline{G}_{\widetilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SU}_3$ and $\underline{\mathcal{G}}_{\widetilde{\alpha}}$ is the standard parahoric.

Case (3): $2\widetilde{\alpha}$ is a root, $\underline{G}_{\widetilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SU}_3$ with K'/K ramified and $\underline{\mathcal{G}}_{\widetilde{\alpha}}$ is the non-standard parahoric.

We now proceed with the construction of f in each of the three cases.

Case (1). In this case the isomorphism $\underline{G}_{\widetilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SL}_2$ induces identifications

$$\mathbf{u}_\pm : \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathbb{G}_a \xrightarrow{\sim} U_{\pm\widetilde{\alpha}}.$$

Let u be a uniformizer of K; then we may define a map

$$f : \mathbb{A}^1_{\mathbb{F}_q} \to \mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}, \quad x \mapsto \mathbf{u}_-(u^{-1}x).$$

Clearly (i') is satisfied, and a simple calculation in SL_2 shows that for $0 \neq x$, we have

$$\mathbf{u}_-(u^{-1}x) \in \mathbf{u}_+(ux^{-1})\dot{\underline{t}}_{\alpha^\vee} L^+\underline{\mathcal{G}}_{\widetilde{\alpha}}$$

so that (ii') also holds.

Case (2). Recall in this case, the parahoric $\underline{\mathcal{G}}_{\widetilde{\alpha}}$ is characterized by $\underline{\mathcal{G}}_{\widetilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap \mathrm{GL}_3(\mathcal{O}_{K'})$. We define

$$f : \mathbb{A}^1_{\mathbb{F}_q} \to \mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}, \quad x \mapsto \mathbf{u}_-(0, u'^{-1}x),$$

where we recall that $u' \in K'$ is a uniformizer with $\tau(u') = -u'$. A calculation using the presentation recalled above shows that for $x \neq 0$, we have

$$\mathbf{u}_-(0, u'^{-1}x) \in \mathbf{u}_+(0, u'x^{-1})\dot{\underline{t}}_{\alpha^\vee} L^+\underline{\mathcal{G}}_{\widetilde{\alpha}};$$

as in Case (1), it follows that (i') and (ii') are satisfied.

Case (3). Recall K'/K is ramified and $\underline{\mathcal{G}}_{\widetilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap s\mathrm{GL}_3(\mathcal{O}_{K'})s^{-1}$. We consider the map

$$\mathbb{A}^1_{\mathbb{F}_q} \to \mathcal{FL}_{\underline{\mathcal{G}}_{\widetilde{\alpha}}}, \quad x \mapsto \mathbf{u}_-(x, -\frac{x^2}{2}).$$

Then in the presentation above, we have that $\dot{t}_{\alpha^\vee}^{-1}\mathbf{u}_+(-2x^{-1}, 2x^{-2})^{-1}\mathbf{u}_-(x, -x^2/2)$ is equal to

$$\begin{pmatrix} u'^{-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -u' \end{pmatrix} \begin{pmatrix} 1 & -2x^{-1} & -2x^{-2} \\ 0 & 1 & 2x^{-1} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ -x^2/2 & -x & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -u'^{-1}2x^{-2} \\ 0 & 1 & -2x^{-1} \\ u'x^2/2 & u'x & -u' \end{pmatrix}.$$

This lies in the parahoric $\underline{\mathcal{G}}_{\widetilde{\alpha}}$, and hence we have

$$\mathbf{u}_-(x, -\frac{x^2}{2}) \in \mathbf{u}_+(-2x^{-1}, 2x^{-2})\dot{t}_{\alpha^\vee} L^+\underline{\mathcal{G}}_{\widetilde{\alpha}}.$$

As in the previous two cases it follows that (i') and (ii') are satisfied. $\square$

Lemma 6.2.7. *Let $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$ and assume $\dot{\underline{t}}_\lambda \in \underline{G}(\mathbb{F}_q[[t]])$. Then there exists $g \in \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$ such that $g\dot{\underline{t}}_\lambda L^+\underline{\mathcal{G}} = y$ in $\mathcal{FL}_{\underline{\mathcal{G}}}$.*

Proof. By definition, there exists $h \in \underline{\mathcal{G}}(k[[t]])$ such that $h\dot{\underline{t}}_\lambda = y$. We consider the subgroup

$$\underline{\mathcal{G}}(k[[t]]) \cap \dot{\underline{t}}_\lambda \underline{\mathcal{G}}(k[[t]])\dot{\underline{t}}_\lambda^{-1} \subset \underline{G}(k((t)));$$

it is the intersection of the kernel of the Kottwitz homomorphism $\widetilde{\kappa}_{\underline{G}}$ and the stabilizer of a bounded subset of the building $\mathcal{B}(\underline{G}, k((t)))$. Thus by [HR08, Prop. 3 and Remark 4], it arises as the k-points of a smooth group scheme $\underline{\mathcal{K}}_\lambda$ defined over $\mathbb{F}_q[[t]]$ with connected special fiber.

The element h is defined up to right multiplication by $\underline{\mathcal{K}}_\lambda(k[[t]])$; hence since $\sigma_q(y) = y$, we have $\sigma_q(h) = hk$ for some $k \in \underline{\mathcal{K}}_\lambda(k[[t]])$. By Lang's theorem applied to $\underline{\mathcal{K}}_\lambda$, there exists $k_1 \in \underline{\mathcal{K}}_\lambda(k[[t]])$ such that $g := hk_1$ is fixed by σ_q, and we have $g\dot{t}_\lambda = y$ in $\mathcal{FL}_{\underline{\mathcal{G}}}$. $\square$

6.2.8. Using Theorem 6.2.1.1, we may deduce the following result about the local structure of the Shimura stack $\mathcal{S}_{\mathrm{K}}$.

Corollary 6.2.9. *Let $x \in \mathcal{S}_{\mathrm{K}}^\lambda(\mathbb{F}_q)$ with $\lambda \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$ and $\lambda \neq \mu$. There exists a smooth, geometrically connected curve C' over $\mathbb{F}_q$ and a map $\phi' : C' \to \mathcal{S}_{\mathbb{F}_q}$ such that*

- *(i) There exists $x' \in C'(\mathbb{F}_q)$ such that $\phi'(x') = x$.*
- *(ii) $\phi'^{-1}(\mathcal{S}_{\mathrm{K},k}^{\lambda'}) \subset C'$ is an open dense subscheme for some $\lambda' \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$ with $\lambda \prec \lambda'$.*

Proof. We write

$$\begin{array}{ccccc} & & \widetilde{\mathcal{S}_{\mathrm{K}}} & & \\ & \swarrow^{\pi_{k_E}} & & \searrow^{q_{k_E}} & \\ \mathcal{S}_{\mathrm{K}} & & & & \mathcal{M} \end{array}$$

for the special fiber of (6.2.1.1). Since π_{k_E} is a torsor for the smooth group scheme $\mathcal{G}_{\mathrm{ad},k_E}$ with connected special fiber, the point x lifts to a point $\widetilde{x} \in \widetilde{\mathcal{S}_{\mathrm{K}}}(\mathbb{F}_q)$ and we write y for its image in $\mathcal{M}(\mathbb{F}_q)$. By definition of the stratification on $\mathcal{S}_{\mathrm{K}}$, we have $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$. We apply Proposition 6.2.5 to y to obtain a map $\phi' : C \to \mathcal{M}_{\mathbb{F}_q}$ satisfying (i) and (ii) in Proposition 6.2.5 for some $\lambda' \in \mathrm{Adm}_{G'}(\{\mu\})_{J'}$ with $\lambda \prec \lambda'$; we let $y' \in C(\mathbb{F}_q)$ mapping to y.

Consider the pullback $\widetilde{\mathcal{S}}_{\mathrm{K},\mathbb{F}_q} \times_{\mathcal{M}_{\mathbb{F}_q}} C$ which is a smooth stack over $\mathbb{F}_q$. By [LMB00, Théorème 6.3], there exists a smooth scheme $Y/\mathbb{F}_q$ and a smooth map $Y \to \widetilde{\mathcal{S}}_{\mathrm{K},\mathbb{F}_q} \times_{\mathcal{M}_{\mathbb{F}_q}} C$ defined over $\mathbb{F}_q$ such that $\widetilde{x}$ lies in the image of a point $\widetilde{y} \in Y(\mathbb{F}_q)$. Now let $Y^{\lambda'}$ denote the preimage of $\mathcal{M}^{\lambda'}$ in Y; by the assumption on C, it is a dense open subscheme of Y. By [Poo04, Corollary 3.4], there exists a smooth geometrically connected curve $C' \subset Y$ such that $\widetilde{y} \in C'(\mathbb{F}_q)$ and $C' \cap Y^{\lambda'} \neq \emptyset$ so that the preimage of $Y^{\lambda'}$ in C' is open and dense. We write $\phi' : C' \to \mathcal{S}_{\mathrm{K},\mathbb{F}_q}$ for the composition

$$C' \to Y \to \widetilde{\mathcal{S}}_{\mathrm{K},\mathbb{F}_q} \times_{\mathcal{M}_{\mathbb{F}_q}} C \to \widetilde{\mathcal{S}}_{\mathrm{K},\mathbb{F}_q} \to \mathcal{S}_{\mathrm{K},\mathbb{F}_q}.$$

Then setting $x' = \widetilde{y} \in C'(\mathbb{F}_q)$, we have $\phi'(x') = x$, so (i) is satisfied, and property (ii) follows by the construction. □

6.3. **Compatible local systems and ℓ-independence.**

6.3.1. We recall the theory of compatible local systems. Let X be a normal scheme over $\mathbb{F}_q$ where q is a power of p and let $\mathcal{L}_\ell$ be a $\overline{\mathbb{Q}}_\ell$-local system (lisse sheaf) on X. For $x \in X(\mathbb{F}_{q^n})$, we write Frob_x for the local Frobenius automorphism acting on the stalk $\mathcal{L}_{\ell,\overline{x}}$ of $\mathcal{L}_\ell$ at a geometric point $\overline{x}$ lying over x. Suppose that for every closed point $x \in X(\mathbb{F}_{q^n})$ the characteristic polynomial $\det(1-\mathrm{Frob}_x t|\mathcal{L}_{\ell,\overline{x}})$, has coefficients in a number field $E \subset \overline{\mathbb{Q}}_\ell$ (this is conjectured to be the case if $\mathcal{L}_\ell$ has determinant of finite order). Let ℓ' be a prime not equal to p and $\lambda' : E \hookrightarrow \overline{\mathbb{Q}}_{\ell'}$ an embedding of fields. A $\overline{\mathbb{Q}}_{\ell'}$-local system $\mathcal{K}_{\ell'}$ is said to be λ'-*compatible* for $\mathcal{L}_\ell$ if for every closed point $x \in X(\mathbb{F}_{q^n})$, the characteristic polynomial $\det(1-\mathrm{Frob}_x t|\mathcal{K}_{,\ell',\overline{x}})$ has coefficients in E and there is an equality

$$\det(1-\mathrm{Frob}_x t|\mathcal{L}_{\ell,\overline{x}}) = \det(1-\mathrm{Frob}_x t|\mathcal{K}_{\ell',\overline{x}}) \in E[t].$$

The existence of λ'-compatible local systems over smooth curves is due to Lafforgue [Laf02, Théorème VII.6] (under the assumption of finite determinant), and the case of smooth schemes is due to Drinfeld [Dri12, Theorem 1.1].

6.3.2. We now continue with the notations of §6.1. For the rest of this section, it will be convenient to fix a Hodge embedding $\iota : (\mathbf{G}, X) \to (\mathbf{GSp}(V), S^{\pm})$ as in §5.4.1.

The element $\gamma_{y,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ arises as an element of $\mathrm{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$. Indeed the image of $\gamma_{y,\ell}$ in $\mathrm{Conj}_{\mathbf{GL}(V)}(\mathbb{Q}_\ell)$ under the map induced by ι lies in $\mathrm{Conj}_{\mathbf{GL}(V)}(\mathbb{Q})$ since it corresponds to the action of Frobenius on the ℓ-adic Tate module of an abelian variety. Since $\mathrm{Conj}_{\mathbf{G}} \to \mathrm{Conj}_{\mathbf{GL}(V)}$ is a finite map, $\gamma_{y,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$. Similarly if $\ell' \nmid p\ell$ is another prime, $\gamma_{y,\ell'}$ arises as an element of $\mathrm{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$.

We let F be a finite extension of $\mathbb{Q}$ such that $\gamma_{y,\ell}, \gamma_{y,\ell'} \in \mathrm{Conj}_{\mathbf{G}}(F)$; such an extension exists since $\mathrm{Conj}_{\mathbf{G}}$ is a $\mathbb{Q}$-variety. Let λ, λ' be the two places over F induced by the fixed embeddings $i_\ell : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_\ell$ and $i_{\ell'} : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_{\ell'}$. We take $\vartheta : \mathbf{G}_F \to \mathbf{GL}_{\mathbf{n}F}$ to be an arbitrary representation over F (not necessarily coming from the Hodge embedding ι); then the $\mathbf{G}(\mathbb{Q}_\ell)$-local system $\mathbb{L}_\ell$ induces an F_λ-adic local system $\mathcal{L}_\ell$ over $\mathcal{S}_{\mathrm{K}}$. Similarly we obtain an $F_{\lambda'}$-adic local system $\mathcal{L}_{\ell'}$.

Lemma 6.3.3. *For any closed point $x \in \mathcal{S}_{\mathrm{K}}(\mathbb{F}_q)$, the eigenvalues of* Frob_x *acting on $\mathcal{L}_{\ell,\overline{x}}$ are ℓ-adic units.*

Proof. It suffices to prove this for a single faithful representation of $\mathbf{G}$. For the representation $\mathbf{G} \to \mathbf{GL}(V)$ induced by ι, the action of Frob_x on $\mathcal{L}_{\ell,\overline{x}}$ corresponds to the action of Frobenius on the ℓ-adic Tate module of an abelian variety and hence its eigenvalues are all ℓ-adic units. □

6.3.4. We let $\vartheta(\gamma_{y,\ell}) \in \mathrm{Conj}_{\mathbf{GL_n}}(F) \subset \mathrm{Conj}_{\mathbf{GL_n}}(F_\lambda)$ denote the image of the conjugacy class of Frob_y under ϑ and we similarly define $\vartheta(\gamma_{y,\ell'}) \in \mathrm{Conj}_{\mathbf{GL_n}}(F) \subset \mathrm{Conj}_{\mathbf{GL_n}}(F_{\lambda'})$.

Proposition 6.3.5. *For any representation $\vartheta : \mathbf{G}_F \to \mathbf{GL}_{nF}$, we have*

$$\vartheta(\gamma_{y,\ell}) = \vartheta(\gamma_{y,\ell'})$$

in $\mathrm{Conj}_{\mathbf{GL_n}}(F)$.

Proof. Let C be a smooth geometrically connected curve and $\psi : C \to \mathcal{S}_{\mathrm{K},\mathbb{F}_q}$ a morphism defined over $\mathbb{F}_q$ such that there exists a point $x \in C(\mathbb{F}_q)$ with $\psi(x) = y$. We first show that if the proposition holds for the image under ψ of a Zariski open and dense set $U \subset C$, then it holds for y.

We write $\mathcal{L}_\ell^C$ (resp. $\mathcal{L}_{\ell'}^C$) for the pullback $\psi^*\mathcal{L}_\ell$ of $\mathcal{L}_\ell$ (resp. $\psi^*\mathcal{L}_{\ell'}$ of $\mathcal{L}_{\ell'}$) to C. By Lemma 6.3.3, $\mathcal{L}_\ell^C$ satisfies the conditions in Chin's refinement of Lafforgue's Theorem [Chi04, Theorem 4.6]. Thus upon enlarging F, there exists a $\overline{\mathbb{Q}}_{\ell'}$-local system $\mathcal{K}_{\ell'}^C$ over C which is λ'-compatible for $\mathcal{L}_\ell^C$.

For any closed point $x \in C(\mathbb{F}_{q^s})$,

$$\det(1 - \mathrm{Frob}_x t | \mathcal{L}_{\ell,\bar{x}}^C) = \det(1 - \mathrm{Frob}_x t | \mathcal{K}_{\ell',\bar{x}}^C) \in F[t].$$

By assumption, for any closed point $x \in U(\mathbb{F}_{q^s})$, we have

$$\det(1 - \mathrm{Frob}_x t | \mathcal{L}_{\ell',\bar{x}}^C) = \det(1 - \mathrm{Frob}_x t | \mathcal{L}_{\ell,\bar{x}}^C) = \det(1 - \mathrm{Frob}_x t | \mathcal{K}_{\ell',\bar{x}}^C).$$

Therefore, by the Chebotarev density Theorem, the semisimplifications of $\mathcal{K}_{\ell'}^C$ and $\mathcal{L}_{\ell'}^C$ are isomorphic, and hence

$$\vartheta(\gamma_{y,\ell}) = \det(1 - \mathrm{Frob}_y t | \mathcal{L}_{\ell,\bar{y}}^C) = \det(1 - \mathrm{Frob}_y t | \mathcal{L}_{\ell',\bar{y}}^C) = \vartheta(\gamma_{y,\ell'})$$

as desired.

We now show that the Proposition holds for $y \in \mathcal{S}_\mathrm{K}^\mu(\mathbb{F}_q)$; we recall that $\mathcal{S}_\mathrm{K}^\mu$ is the open Kottwitz–Rapoport stratum and is smooth. Using the same argument as in the proof of 6.2.9 (i.e. applying [LMB00, Théorème 6.3] and [Poo04, Corollary 3.4]), we may find a smooth geometrically connected curve C over $\mathbb{F}_q$ and a map $\psi : C \to \mathcal{S}_{\mathrm{K},\mathbb{F}_q}^\mu$ defined over $\mathbb{F}_q$ such that there exists a point $x \in C(\mathbb{F}_q)$ with $\psi(x) = y$ and such that the preimage $U := \psi^{-1}(\mathcal{S}_{\mathrm{K},[b]_\mu}) \subset C$ of the μ-ordinary locus is open and dense. By Corollary 5.4.18, the Proposition holds for points y' lying in the image of U, and hence it holds for y by the above argument.

Finally we show that the Proposition holds for all $y \in \mathcal{S}_\mathrm{K}(\mathbb{F}_q)$. We assume $y \in \mathcal{S}_\mathrm{K}^\nu(\mathbb{F}_q)$ and we proceed by descending induction on ν; the case of the maximal element $\nu = \mu$ was proved above. Now suppose the result is true for all $\nu' \succ \nu$. Let $\psi : C \to \mathcal{S}_{\mathrm{K},\mathbb{F}_q}$ be a map as in Corollary 6.2.9 where C is a smooth geometrically connected curve over $\mathbb{F}_q$. We let $U \subset C$ denote the preimage of $\bigcup_{\nu \prec \nu'} \mathcal{S}_{\mathrm{K},\mathbb{F}_q}^{\nu'}$ which is Zariski open and dense. By induction hypothesis, the proposition holds for the image of U, hence it holds for y.

□

6.3.6. We may now prove Theorem 6.1.4.

Proof of Theorem 6.1.4. For all $\ell, \ell' \neq p$, and ϑ as above, we have $\vartheta(\gamma_{y,\ell}) = \vartheta(\gamma_{y,\ell'})$ by Proposition 6.3.5. This implies that $\gamma_{y,\ell} = \gamma_{y,\ell'} \in \mathrm{Conj}_\mathbf{G}(\overline{\mathbb{Q}})$, by a result of Steinberg [Ste65, 6.6]. Hence, there exists $\gamma_y \in \mathrm{Conj}_\mathbf{G}(\overline{\mathbb{Q}})$ such that $\gamma_y = \gamma_{y,\ell}$ for all $\ell \neq p$. It suffices to show γ_y is defined over $\mathbb{Q}$.

Since $\mathrm{Conj}_{\mathbf{G}}$ is a $\mathbb{Q}$-variety, the residue field of the point γ_y is a finite extension $F/\mathbb{Q}$. Since $\gamma_y \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ for all ℓ, each finite prime of $\mathbb{Q}$ has a split prime in F above it; hence the Chebotarev density theorem implies $\gamma_y \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$. Indeed let $F'/\mathbb{Q}$ be the Galois closure of F. Then for every prime $\ell \neq p$, there exists l a prime of F' above ℓ such that the Frobenius Frob_l lies in $\mathrm{Gal}(F'/F) \subset \mathrm{Gal}(F'/\mathbb{Q})$. It follows that $\mathrm{Gal}(F'/F)$ intersects every conjugacy class of $\mathrm{Gal}(F'/\mathbb{Q})$ and hence these groups are equal. □

Remark 6.3.7. The application of [Ste65, 6.6] in the previous theorem is one of the reasons we obtain γ as an element of $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$, as opposed to an element of $\mathbf{G}(\mathbb{Q})$.

7. Conjugacy class of Frobenius for abelian varieties

We apply the results of §6 to prove our main result concerning abelian varieties.

7.1. **Mumford–Tate groups.**

7.1.1. Let A be an abelian variety over a number field E. Recall we have fixed an embedding $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$; using this we may consider E as a subfield of $\mathbb{C}$. We write V_B for the Betti cohomology $\mathrm{H}^1_B(A(\mathbb{C}), \mathbb{Q})$ which is equipped with a Hodge structure of type $((0,-1),(-1,0))$. This Hodge structure is induced by a morphism

$$h : \mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to \mathrm{GL}(V_B)$$

We write

$$\mu : \mathbb{C}^\times \xrightarrow{z\mapsto(z,1)} \mathbb{C}^\times \times c^*(\mathbb{C}^\times) \xrightarrow{h} \mathrm{GL}(V_B \otimes \mathbb{C})$$

for the Hodge cocharacter.

Definition 7.1.2. The Mumford–Tate group $\mathbf{G}$ of A is the smallest algebraic subgroup of $\mathrm{GL}(V_B)$ defined over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{C})$ contains the image of μ.

The group $\mathbf{G}$ can also be characterized as the algebraic subgroup of $\mathrm{GL}(V_B)$ that stabilizes all Hodge cycles of type (0,0) on the tensor spaces $V_B^{\otimes r} \otimes (V_B^\vee)^{\otimes r}$ for $r \in \mathbb{Z}_{\geq 0}$; it is known that $\mathbf{G}$ is a connected reductive group.

We remark that $\mathbf{G}$ depends on the embedding $\mathrm{E} \hookrightarrow \mathbb{C}$; if $\mathbf{G}_1$ is the group defined by a different embedding then there is a canonical inner twisting $\mathbf{G}_{\overline{\mathbb{Q}}} \cong \mathbf{G}_{1,\overline{\mathbb{Q}}}$ induced by the torsor of tensor preserving isomorphisms between the Betti cohomology groups (see [Del82a, Proof of Theorem 3.8] for the construction of this torsor).

7.1.3. For a prime number ℓ, we write $T_\ell A$ for the Tate module of A. The action of the absolute Galois group $\Gamma_{\mathrm{E}} := \mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E})$ on $T_\ell A^\vee$ gives rise to a representation $\rho_\ell : \Gamma_{\mathrm{E}} \to \mathbf{GL}(T_\ell A^\vee)$ and the Betti-étale comparison gives us a canonical isomorphism

$$\mathrm{H}^1_B(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell x \cong T_\ell A^\vee \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Deligne's theorem that Hodge cycles are absolutely Hodge [Del82a], implies that upon replacing E by a finite extension, the map ρ_ℓ factors through $\mathbf{G}(\mathbb{Q}_\ell)$; see [Noo09, Remarque 1.9]. In fact this condition does not depend on ℓ.

Lemma 7.1.4. *The representation ρ_ℓ factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for some prime ℓ, if and only if it factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for all primes ℓ.*

Proof. The subgroup $\mathbf{G} \subset \mathrm{GL}(V_B)$ is the stabilizer of a collection of Hodge cycles $(s_\alpha)_\alpha$. We consider the ℓ-adic components $(s_{\alpha,\ell})_\ell$, as in §5.1.5. For $\sigma \in \Gamma_{\mathrm{E}}$, $(\sigma(s_{\alpha,\ell}))_\ell$, is again a Hodge cycle, by Deligne's theorem [Del82a, Theorem 2.11]. In particular, if $(\sigma(s_{\alpha,\ell}))_\ell$, and $(s_{\alpha,\ell})_\ell$ have equal components at some prime ℓ, then they are equal. □

The lemma shows that the condition that Γ_{E} fixes $(s_{\alpha,\ell})_\alpha$ pointwise does not depend on ℓ. This condition is equivalent to asking that Γ_{E} maps to $\mathbf{G}(\mathbb{Q}_\ell)$.

7.1.5. We replace E by the smallest extension such that Γ_{E} maps to $\mathbf{G}(\mathbb{Q}_\ell)$, and we write $\rho_\ell^{\mathbf{G}}$ for the induced map $\Gamma_{\mathrm{E}} \to \mathbf{G}(\mathbb{Q}_\ell)$ and ι_ℓ for the inclusion $\mathbf{G}(\mathbb{Q}_\ell) \to \mathbf{GL}(T_\ell A^\vee)$.

Let v be a prime of E lying above a prime p such that A has good reduction at v. Upon modifying the embedding $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ fixed in §5.1.1, we may assume that v is induced by i_p. We write $E = \mathrm{E}_v$, and we let $\mathbb{F}_q$ denote the residue field of E at v. For $\ell \neq p$ a prime, ρ_ℓ is unramified at v. Let Fr_v be a geometric Frobenius element at v, we write $\gamma_\ell(v) = \chi_{\mathbf{G}}(\rho_\ell^{\mathbf{G}}(\mathrm{Fr}_v)) \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ for the conjugacy class of $\rho_\ell^{\mathbf{G}}(\mathrm{Fr}_v)$ which only depends on v and not the choice of Frobenius element. We write $P_{v,\ell}(t)$ for the characteristic polynomial of Fr_v acting on $T_\ell A^\vee$, which has coefficients in $\mathbb{Z}$ and is independent of ℓ.

7.1.6. We will make use of the following auxiliary construction. Let $\mathrm{F}/\mathbb{Q}$ be a totally real field, and let $\mathbf{H}' := \mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbf{G}_{\mathrm{F}}$. There is a canonical inclusion $\mathbf{G} \hookrightarrow \mathbf{H}'$. We let (V, ψ) be the symplectic space corresponding to $\mathrm{H}_1(A(\mathbb{C}), \mathbb{Q})$ where ψ is a Riemann form for A and $\mathbf{G} \to \mathbf{GSp}(V)$ is the natural map. We let W denote the symplectic space over $\mathbb{Q}$ whose underlying vector space is $V \otimes_{\mathbb{Q}} \mathrm{F}$ and whose alternating form ψ' is given by the composition

$$W \times W \xrightarrow{\psi \otimes_{\mathbb{Q}} \mathrm{F}} \mathrm{F} \xrightarrow{\mathrm{Tr}_{\mathrm{F}/\mathbb{Q}}} \mathbb{Q}.$$

Let $c_{\mathbf{G}} : \mathbf{G} \to \mathbb{G}_m$ denote the restriction of the multiplier homomorphism $c : \mathbf{GSp}(V) \to \mathbb{G}_m$ to $\mathbf{G}$. We form the fiber product

$$\begin{array}{ccc} \mathbf{H}'' & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \Delta \\ \mathbf{H}' & \xrightarrow{\mathrm{Res}_{\mathrm{F}/\mathbb{Q}} c_{\mathbf{G}}} & \mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbb{G}_m \end{array}$$

where the map Δ is the diagonal map and we let $\mathbf{H}$ denote the neutral connected component of $\mathbf{H}''$. Thus $\mathbf{H}$ is a connected reductive group over $\mathbb{Q}$. The inclusion $\mathbf{G} \hookrightarrow \mathbf{H}'$ factors through $\mathbf{H}$ and we let h' denote the composition

$$\mathbb{S} \xrightarrow{h} \mathbf{G}_{\mathbb{R}} \to \mathbf{H}_{\mathbb{R}}.$$

Write X for the $\mathbf{G}(\mathbb{R})$-conjugacy class of h and $X_{\mathbf{H}}$ for the $\mathbf{H}(\mathbb{R})$-conjugacy class of h'.

Consider the composition

$$\iota' : \mathbf{H}' \xrightarrow{\mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\iota} \mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbf{GSp}(V) \xrightarrow{f} \mathbf{GL}(W)$$

where f is induced by the forgetful functor from F-vector spaces to $\mathbb{Q}$-vector spaces. It is easy to see that the restriction of ι' to $\mathbf{H}$ factors through $\mathbf{GSp}(W)$, and we also denote by ι' the induced map. We write $S'^{\pm}$ for the Siegel half space corresponding to W. One checks easily that $(\mathbf{G}, X)$, and $(\mathbf{H}, X_{\mathbf{H}})$ are Shimura data, and that we have embeddings of Shimura data

$$(\mathbf{G}, X) \hookrightarrow (\mathbf{H}, X_{\mathbf{H}}) \hookrightarrow (\mathbf{GSp}(W), S'^{\pm}).$$

7.1.7. The following lemma will be used to show that for the ℓ-independence of $\gamma_\ell(v)$ in $\mathrm{Conj}_{\mathbf{G}}$, it suffices to show the ℓ-independence in $\mathrm{Conj}_{\mathbf{H}}$.

Lemma 7.1.8. *The natural inclusion* $\mathbf{G} \to \mathbf{H}$ *induces a* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$*-equivariant injection*

$$\mathrm{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}}) \to \mathrm{Conj}_{\mathbf{H}}(\overline{\mathbb{Q}}).$$

Proof. Let $h, h' \in \mathbf{G}(\overline{\mathbb{Q}})$ such that there exists $g \in \mathbf{H}(\overline{\mathbb{Q}})$ such that $g^{-1}hg = h'$. We consider $\mathbf{H}$ as a subgroup of $\mathbf{H}'$. Then under the identification

$$\mathbf{H}'_{\overline{\mathbb{Q}}} \cong \prod_{\iota:\mathrm{F}\to\overline{\mathbb{Q}}} \mathbf{G}_{\overline{\mathbb{Q}}},$$

h, h' correspond to the elements $(h, \dots, h), (h', \dots, h')$ respectively and we write $g = (g_1, \dots, g_n)$. Then $g^{-1}hg = h'$ implies $g_1 h g_1^{-1} = h'$. Thus h and h' have the same image in $\mathrm{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$. The $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariance follows from the fact that $\mathbf{G} \to \mathbf{H}$ is defined over $\mathbb{Q}$. □

7.2. **The main theorem.** We now prove our main theorem (cf. Theorem 1.1). We need the following preliminary result.

Lemma 7.2.1. *Let G be a connected reductive group over $\mathbb{Q}_p$. If $g \in G(\mathbb{Q}_p)$ lies in some compact open subgroup of $G(\mathbb{Q}_p)$, then there exists a finite extension $F/\mathbb{Q}_p$ over which G splits and such that g lies in the parahoric subgroup of $G(F)$ associated to a special vertex in the building $\mathcal{B}(G, F)$.*

Remark 7.2.2. Note that if G splits over F, the notion of special vertex, very special vertex, and hyperspecial vertex in $\mathcal{B}(G, F)$ all coincide.

Proof. Write $g = g_s g_u$ for the Jordan decomposition of g so that g_s is semisimple and g_u is unipotent. Since g lies in a compact open subgroup of $G(\mathbb{Q}_p)$, g is power bounded and hence g_s and g_u are power pounded. Let $T \subset G$ be a maximal torus defined over $\mathbb{Q}_p$ such that $g_s \in T(\mathbb{Q}_p)$. We will take F to be the splitting field of T.

Since $g_s \in T(F)$ is power bounded, it is contained in $\mathcal{T}_{F,0}(\mathcal{O}_F)$ where $\mathcal{T}_{F,0}$ is the connected Néron model for the base change T_F. If we let $\mathcal{A}(G, T, F) \subset \mathcal{B}(G, F)$ be the apartment corresponding to T_F, then g_s acts trivially on $\mathcal{A}(G, T, F)$.

Now $g_u \in U(F)$ where U is the unipotent radical of some Borel subgroup B of G_F containing T. Let $\mathfrak{s} \in \mathcal{A}(G, T, F)$ be any special vertex and we use this vertex to identify $\mathcal{A}(G, T, F)$ with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Since each affine root subgroup of G_F fixes a half apartment in $\mathcal{A}(G, T, F)$, there exists a sufficiently dominant (with respect to the choice of Borel B) special vertex $\mathfrak{s}'$ which is fixed by g_u. It follows that $\mathfrak{s}'$ is fixed by g. We write $\widetilde{\mathcal{G}}$ for the Bruhat–Tits stabilizer scheme over $\mathcal{O}_F$ corresponding to $\mathfrak{s}'$; by the above discussion we have $g \in \widetilde{\mathcal{G}}(\mathcal{O}_F)$. Since G is split over F, $\widetilde{\mathcal{G}}$ is equal to the parahoric group scheme $\mathcal{G}$ associated to $\mathfrak{s}'$. □

7.2.3. We now return to the assumptions and notation of §7.1. Thus we have an abelian variety A/E, such that $\rho_\ell : \Gamma_{\mathrm{E}} \to \mathrm{GL}(T_\ell A^\vee)$ factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for all ℓ. Recall $E = \mathrm{E}_v$ and $\mathbb{F}_q$ is its residue field. The map $i_p : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ determines an inclusion

$$\mathrm{Gal}(\overline{E}/E) \to \mathrm{Gal}(\overline{\mathrm{E}}/\mathrm{E}). \tag{7.2.3.1}$$

We let $\widetilde{\sigma}_q \in \Gamma_{\mathrm{E}}$ be the image under (7.2.3.1) of a lift of the geometric Frobenius in $\mathrm{Gal}(\overline{E}/E)$.

Proposition 7.2.4. *Let $p > 2$. There exists a totally real field* F *such that if $(\mathbf{H}, X_{\mathbf{H}})$ denotes the Shimura datum of Hodge type coming from the construction in §7.1.6, we have $H := \mathbf{H}_{\mathbb{Q}_p}$ is quasi-split and there exists a very special parahoric group scheme $\mathcal{H}$ for H such that*

1. *The image of $\rho_p^{\mathbf{G}}(\widetilde{\sigma}_q)$ in $H(\mathbb{Q}_p)$ lies in $\mathcal{H}(\mathbb{Z}_p)$.*
2. *The triple $(\mathbf{H}, X_{\mathbf{H}}, \mathcal{H})$ is strongly acceptable (Definition 5.2.8).*

Proof. Let $G = \mathbf{G}_{\mathbb{Q}_p}$. By Lemma 7.2.1 applied to the element $\rho_p^{\mathbf{G}}(\widetilde{\sigma}_q) \in G(\mathbb{Q}_p)$, there exists a finite extension $F/\mathbb{Q}_p$ such that G_F is split and there exists a special parahoric $\mathcal{G}$ of G_F such that the image of $\rho_p^{\mathbf{G}}(\widetilde{\sigma}_q)$ in $G(F)$ lies in $\mathcal{G}(\mathcal{O}_F)$. We let F be a totally real field such that $\mathrm{F}_w \cong F$ for all places $w|p$ of F. By construction $\mathbf{H} \subset \mathbf{H}' = \mathrm{Res}_{\mathrm{F}/\mathbb{Q}}\mathbf{G}$ and we have an isomorphism

$$H' := \mathbf{H}'_{\mathbb{Q}_p} \cong \prod_{w|p} \mathrm{Res}_{\mathrm{F}_w/\mathbb{Q}_p} \mathbf{G}_{\mathrm{F}_w} \cong \prod_{w|p} \mathrm{Res}_{F/\mathbb{Q}_p} G_F.$$

Then H' is quasi-split since it is a product of the restriction of scalars of a split group, and hence $H := \mathcal{H}_{\mathbb{Q}_p}$ is quasi-split.

We let $\mathcal{H}'$ denote the parahoric group scheme of H' corresponding to $\prod_{w|p} \mathcal{G}$. Then $\mathcal{H}' \cong \prod_{w|p} \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}_p} \mathcal{G}$, and since G splits over F, $\mathcal{H}'$ is a very special parahoric. It follows that $\mathcal{H}'(\mathbb{Z}_p) \cap H(\mathbb{Q}_p)$ arises as the $\mathbb{Z}_p$-points of a very special parahoric group scheme $\mathcal{H}$ for H. Since $G(\mathbb{Q}_p) \subset H(\mathbb{Q}_p)$, the image of $\rho_p^{\mathbf{G}}(\widetilde{\sigma}_q)$ in $H(\mathbb{Q}_p)$ lies in $\mathcal{H}(\mathbb{Z}_p)$ so that (1) is satisfied. Moreover $(\mathbf{H}, X)$ is acceptable by construction.

To show (2) is satisfied, we let $(\mathbf{H}_1, X_1)$ be an auxiliary Shimura datum of Hodge type as constructed in Proposition 5.2.6 so that there is a central extension $\mathbf{H}_1^{\mathrm{der}} \to \mathbf{H}^{\mathrm{der}}$ and we write $H_1 := \mathbf{H}_{1,\mathbb{Q}_p}$. The parahoric $\mathcal{H}$ of H determines a very special parahoric $\mathcal{H}_1$ of H_1. It suffices to show $\mathcal{H}_1$ is a connected parahoric.

Note that there is an isomorphism $H^{\mathrm{ad}} \cong H_1^{\mathrm{ad}} \cong \prod_{i=1}^r \mathrm{Res}_{F_i/\mathbb{Q}_p} G_i$ where G_i is a *split* reductive group over F_i. It follows that any parahoric of H^{ad} is connected. There is a natural map $\widetilde{\mathcal{H}}_1 \to \widetilde{\mathcal{H}}^{\mathrm{ad}}$ and a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{H}}_1(\breve{\mathbb{Z}}_p) & \longrightarrow & \widetilde{\mathcal{H}}^{\mathrm{ad}}(\breve{\mathbb{Z}}_p) \\ \downarrow{\scriptstyle \widetilde{\kappa}_{H_1}} & & \downarrow{\scriptstyle \widetilde{\kappa}_{H^{\mathrm{ad}}}} \\ \pi_1(H_1)_I & \longrightarrow & \pi_1(H^{\mathrm{ad}})_I. \end{array}$$

Therefore $\widetilde{\mathcal{H}}_1(\breve{\mathbb{Z}}_p)$ maps to $\ker(\pi_1(H_1)_I \to \pi_1(H^{\mathrm{ad}})_I)$ and it suffices to show this group is torsion free.

We have a commutative diagram with exact rows.

$$\begin{array}{ccccccccc} & & \pi_1(H_1^{\mathrm{der}})_I & \longrightarrow & \pi_1(H_1)_I & \longrightarrow & X_*(H_1^{\mathrm{ab}})_I & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(H^{\mathrm{ad}})_I & \xrightarrow{\sim} & \pi_1(H^{\mathrm{ad}})_I & \longrightarrow & \{1\} & \longrightarrow & 0 \end{array}$$

Since $\pi_1(H_1^{\mathrm{der}}) \to \pi_1(H^{\mathrm{ad}})$ is injective and these are induced modules, it follows that $\pi_1(H_1^{\mathrm{der}})_I \to \pi_1(H^{\mathrm{ad}})_I$ is injective. By construction, $X_*(H_1^{\mathrm{ab}})_I$ is torsion free, and hence so is $\ker(\pi_1(H_1)_I \to \pi_1(H^{\mathrm{ad}})_I)$ by the snake Lemma.

□

Theorem 7.2.5. *Let $p > 2$ be a prime and $v|p$ a place of* E *where A has good reduction. Then there exists an element $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that for all $\ell \neq p$, we have $\gamma = \gamma_\ell(v)$ in $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_l)$.*

Remark 7.2.6. As remarked above, the group $\mathbf{G}$ depends on the embedding $\mathrm{E} \hookrightarrow \mathbb{C}$ up to inner automorphism. If $\mathbf{G}_1$ is the group associated to a different embedding $\mathrm{E} \hookrightarrow \mathbb{C}$, the inner twisting $\mathbf{G}_{\overline{\mathbb{Q}}} \cong \mathbf{G}_{1,\overline{\mathbb{Q}}}$ induces a canonical isomorphism $\mathrm{Conj}_{\mathbf{G}} \cong \mathrm{Conj}_{\mathbf{G}_1}$ and it can be checked that the statement of the theorem is independent of the choice of embedding.

Proof of 7.2.5. We may assume that $\mathbf{G}$ is not a torus as in this case A has complex multiplication and the result is a theorem of Shimura–Taniyama. We choose a totally real field F as in Proposition 7.2.4 and let $(\mathbf{H}, X_{\mathbf{H}})$ be the associated Shimura datum of Hodge type arising from the construction in §7.1.6. By construction, there is a very special parahoric $\mathcal{H}$ of $\mathbf{H}_{\mathbb{Q}_p}$ such that the image of $\rho_p^{\mathbf{G}}(\widetilde{\sigma}_q)$ inside $\mathbf{H}(\mathbb{Q}_p)$ lies in $\mathrm{K}_p := \mathcal{H}(\mathbb{Z}_p)$. Hence, there exists a finite extension E' of E such that $\rho_p^{\mathbf{G}}|_{\Gamma_{\mathrm{E}'}}$ factors through K_p, and such that there is a prime $v'|v$ of E' such that $\mathrm{E}'_{v'}$ has residue field $\mathbb{F}_q$. We may thus replace E by E', without changing the statement of the theorem, and assume that the image of $\rho_p^{\mathbf{G}}$ in $\mathbf{H}(\mathbb{Q}_p)$ factors through K_p.

Now let $(s_{\alpha,\ell})_{\ell\neq p} \in \widehat{V}^p(A)^{\otimes}$ denote the ℓ-adic realizations of the absolute Hodge cycles for A. By our assumption on E, the representation $\rho^p : \Gamma_{\mathrm{E}} \to \mathrm{GL}(\widehat{V}^p(A))$ factors through $\mathbf{G}(\mathbb{A}_f^p) \subset \mathbf{H}(\mathbb{A}_f^p)$, and hence through a compact open subgroup $\mathrm{K}^p \subset \mathbf{H}(\mathbb{A}_f^p)$. Write $\mathrm{K} := \mathrm{K}_p\mathrm{K}^p$.

We now define a point of $\mathrm{Sh}_{\mathrm{K}}(\mathbf{H}, X_{\mathbf{H}})$ using the Hodge embedding $\iota' : (\mathbf{H}, X_{\mathbf{H}}) \to \mathbf{GSp}(W), S^{\pm})$. Consider the abelian variety up to isogeny $A^F = A \otimes_{\mathbb{Q}} F$ given by the Serre tensor construction [Con04, §7], equipped with the isomorphism $\varepsilon : \widehat{V}(A^F) \simeq V \otimes_{\mathbb{Q}} \mathbb{A}_f \otimes_{\mathbb{Q}} F$ induced by the identity on V. Since $\rho_p^{\mathbf{G}}$ and ρ^p act via K, the K-orbit of ε is Γ_{E}-invariant. Thus, the triple $(A^F, \lambda \otimes F, \varepsilon)$, defines a point $\widetilde{x}_A \in \mathrm{Sh}_{\mathrm{K}}(\mathbf{H}, X_{\mathbf{H}})(\mathrm{E})$. (Note that, since ψ is $\mathbf{H}$-invariant, up to scalars, λ is defined over E as a weak polarization).

By our choice of F, the triple $(\mathbf{H}, X_{\mathbf{H}}, \mathcal{H})$ satisfies the assumptions of Theorem 6.1.4. Thus we may apply it to the reduction $x_A \in \mathscr{S}_{\mathrm{K}}(\mathbf{H}, X_{\mathbf{H}})(\mathbb{F}_q)$, where $\mathscr{S}_{\mathrm{K}}(\mathbf{H}, X_{\mathbf{H}})$ is the integral model constructed from a choice of auxiliary Hodge-type Shimura datum. This implies that there exists $\gamma \in \mathrm{Conj}_{\mathbf{H}}(\mathbb{Q})$ such that for all $\ell \neq p$, we have $\gamma = \gamma_\ell(v)$ in $\mathrm{Conj}_{\mathbf{H}}(\mathbb{Q}_\ell)$. By Lemma 7.1.8, it follows that $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ and $\gamma = \gamma_\ell(v)$ in $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$. □

7.3. **Refinements.**

7.3.1. We retain the notation introduced above. Write $\tilde{\gamma}_\ell(v) = \rho_\ell^{\mathbf{G}}(\mathrm{Fr}_v)$. There are a number of ways one might try to refine Theorem 7.2.5. For example one could ask if γ lifts to a point $\tilde{\gamma} \in \mathbf{G}(\mathbb{Q})$. When this is the case, one can also try to refine the relationship between $\tilde{\gamma}_\ell(v)$ and $\tilde{\gamma}$. We will prove such a result when $\mathbf{G}$ has simply connected derived group, and is quasi-split at p.

7.3.2. Let H, H' be connected reductive groups over a field K with algebraic closure $\bar{K}$. Suppose we are given an isomorphism $H \simeq H'$ over $\bar{K}$. Recall that an *inner twisting* between H and H' is an isomorphism $\psi : H \simeq H'$ over $\bar{K}$ such that for $\sigma \in \mathrm{Gal}(\bar{K}/K)$, there exists $g_\sigma \in H'(\bar{K})$ so that $\sigma(\psi(h)) = g_\sigma \psi(\sigma(h)) g_\sigma^{-1}$ for $h \in H(\bar{K})$. We say that a subgroup $M \subset H$ *transfers* to H' via ψ, if $\psi(M) \subset H'$ is defined over K, and ψ induces an isomorphism $M \simeq \psi(M)$ over K. We say that M *transfers* to H', if it transfers to H' via some ψ.

An element $h \subset H_{\mathbb{R}}$ is called *elliptic* if it is contained in an elliptic maximal torus, that is a maximal torus which is anisotropic modulo the center of H.

Lemma 7.3.3. $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ *lifts to an elliptic element* $\tilde{\gamma}_{\mathbb{R}} \in \mathbf{G}(\mathbb{R})$.

Proof. The composite $\mathbb{G}_m \overset{\mu}{\to} \mathbf{G} \overset{c_{\mathbf{G}}}{\to} \mathbb{G}_m$ is given by $x \mapsto x^i$ for some i. Here, as above, μ and $c_{\mathbf{G}}$ are the Hodge cocharacter and multiplier homomorphism respectively. For any lift $\tilde{\gamma} \in \mathbf{G}(\mathbb{C})$ of γ, we have $c_{\mathbf{G}}(\tilde{\gamma}) = q^i$ [Del79, 2.2.3]. In particular, $c_{\mathbf{G}}(\tilde{\gamma}) \in \mathbb{R}^{\times,+}$. Hence there exists $z \in Z_{\mathbf{G}}(\mathbb{R})$ with $c_{\mathbf{G}}(z) = c_{\mathbf{G}}(\tilde{\gamma})$. Set $\gamma_1 = \gamma z^{-1} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{R})$. It suffices to show γ_1 admits an elliptic lift in $\mathbf{G}(\mathbb{R})$.

Let $\tilde{\gamma}_1 \in \mathbf{G}(\mathbb{C})$ be any lift. Under any representation of $\mathbf{G}$ (for example its canonical symplectic one), the eigenvalues of the image of $\tilde{\gamma}_1$ have absolute value 1. Hence $\tilde{\gamma}_1$ is contained in a maximal compact subgroup of $\mathbf{G}(\mathbb{C})$. Let $\bar{\mathbf{G}} = \mathbf{G}/w(\mathbb{G}_m)$, and denote by $\tilde{\gamma}_2 \in \bar{\mathbf{G}}(\mathbb{C})$ the image of $\tilde{\gamma}_1$ Then $\tilde{\gamma}_2$ is contained in a maximal compact subgroup of $\bar{\mathbf{G}}(\mathbb{C})$. Such a subgroup has the form $\bar{\mathbf{G}}^c(\mathbb{R})$, where $\bar{\mathbf{G}}^c$ is a real form of $\bar{\mathbf{G}}$. Consider the canonical isomorphism $\psi : \bar{\mathbf{G}}^c_{\mathbb{C}} \simeq \bar{\mathbf{G}}_{\mathbb{C}}$. As the center of $\bar{\mathbf{G}}_{\mathbb{R}}$ is anisotropic, ψ induces an isomorphism between the centers of $\bar{\mathbf{G}}$ and $\bar{\mathbf{G}}^c$, over $\mathbb{R}$. Moreover, $\mathbf{G}^{\mathrm{der}}$ is an inner form of its compact form, so this implies that ψ is an inner twisting. Let $T \subset \bar{\mathbf{G}}^c_{\mathbb{R}}$ be a maximal torus containing $\psi^{-1}(\tilde{\gamma}_2)$. Then T transfers to $\bar{\mathbf{G}}$ [LR87, Lem. 5.6], and $\psi^{-1}(\tilde{\gamma}_2) \in T(\mathbb{R}) \subset \bar{\mathbf{G}}(\mathbb{R})$ is elliptic. Any lift of $\psi^{-1}(\tilde{\gamma}_2)$ to $\mathbf{G}(\mathbb{R})$ yields the required lift of γ. □

Corollary 7.3.4. *With the assumptions of Theorem 7.2.5, suppose that* $\mathbf{G}^{\mathrm{der}}$ *is simply connected and that* $\mathbf{G}_{\mathbb{Q}_p}$ *is quasi-split. Then* γ *lifts to an element* $\gamma_0 \in \mathbf{G}(\mathbb{Q})$ *such that*

- $\gamma_0 \in \mathbf{G}(\mathbb{R})$ *is elliptic*
- γ_0 *is conjugate to* $\tilde{\gamma}_\ell(v)$ *in* $\mathbf{G}(\mathbb{Q}_\ell)$ *for all but at most one prime* $\ell \neq p$.

Proof. Since γ lifts to an elliptic element by Lemma 7.3.3, this follows from the argument of [Kot90, p188]. □

Remarks 7.3.5.

(1) When $\mathbf{G}$ is *not* quasi-split at p, there does not seem to be any reason to believe that γ in the statement of Theorem 7.2.5 should lift to an element of $\mathbf{G}(\mathbb{Q})$.

(2) When $\mathbf{G}$ is quasi-split at p, one expects the conclusion of Corollary 7.3.4 to hold without assuming that $\mathbf{G}^{\mathrm{der}}$ is simply connected, and without excluding one prime $\ell \neq p$. Indeed this follows when one can show that the isogeny class on the corresponding Shimura variety contains a point which lifts to a special point. This is conjectured to hold in general [KPS22, Conj. 2.3.8]. One way to motivate this conjecture would be to prove the analogous statement for the admissible morphisms which appear in the Langlands–Rapoport conjecture [LR87]. This is done in *loc. cit* when the level at p is hyperspecial.
(3) It follows from the argument of [Kot90, p188] that the exceptional prime in the statement of the corollary can actually be chosen in a set of positive density. Of course the choice of this prime affects the choice of γ_0.
(4) It is possible to prove a version of Theorem 7.2.5 and Corollary 7.3.4 which includes $\ell = p$, using the crystalline Frobenius. We aim to return to this in a future work.
(5) In a paper in preparation [KZ], we extend our methods to prove a version of Theorem 7.2.5 at a place v of E where A has bad reduction. This involves an independence of ℓ statement for representations of the Weil–Deligne group.

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