

INDEPENDENCE OF ℓ FOR FROBENIUS CONJUGACY CLASSES ATTACHED TO ABELIAN VARIETIES

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ABSTRACT. Let A be an abelian variety over a number field $E \subset \mathbb{C}$ and let \mathbf{G} denote the Mumford–Tate group of A . After replacing E by a finite extension, the action of the absolute Galois group $\text{Gal}(\overline{E}/E)$ on the ℓ -adic cohomology $H_{\text{ét}}^1(A_{\overline{E}}, \mathbb{Q}_{\ell})$ factors through $\mathbf{G}(\mathbb{Q}_{\ell})$. We show that for v an odd prime of E where A has good reduction, the conjugacy class of Frobenius Frob_v in $\mathbf{G}(\mathbb{Q}_{\ell})$ is independent of ℓ . Along the way we prove that every point in the μ -ordinary locus of the special fiber of Shimura varieties has a special point lifting it.

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1. INTRODUCTION

Let A be an abelian variety over a number field $E \subset \mathbb{C}$ and \bar{E} an algebraic closure of E . For v a place of E dividing a prime p where A has good reduction and $\ell \neq p$ a prime, the action of $\text{Gal}(\bar{E}/E)$ on the ℓ -adic cohomology $H_{\text{et}}^1(A_{\bar{E}}, \mathbb{Q}_{\ell})$ is unramified, and the characteristic polynomial $P_{v,\ell}(t)$ of a geometric Frobenius $\text{Frob}_v \in \text{Gal}(\bar{E}/E)$ has coefficients in \mathbb{Z} , and is independent of ℓ . The aim of this paper is to prove a refinement of this statement for the image of Frob_v in the *Mumford–Tate* group of A .

Recall that the Mumford–Tate group \mathbf{G} of A is a reductive group over \mathbb{Q} , defined as the Tannakian group of the \mathbb{Q} -Hodge structure given by the Betti cohomology $V_B := H_B^1(A(\mathbb{C}), \mathbb{Q})$. It may also be defined as the stabilizer in $\mathbf{GL}(V_B)$ of all Hodge cycles on A . A fundamental result of Deligne [Del82] asserts that there exists a finite extension E'/E in \bar{E} such that for any prime ℓ , the action of $\text{Gal}(\bar{E}/E')$ on $H_{\text{et}}^1(A_{\bar{E}}, \mathbb{Q}_{\ell})$ is induced by a representation

$$\rho_{\ell}^{\mathbf{G}} : \text{Gal}(\bar{E}/E') \rightarrow \mathbf{G}(\mathbb{Q}_{\ell}).$$

It is not hard to see that for any finite extension E'/E , if $\rho_{\ell}^{\mathbf{G}}$ exists for one ℓ , then it exists for all ℓ . Moreover there is a minimal such extension E' . The existence of $\rho_{\ell}^{\mathbf{G}}$ is in fact predicted by the (in general still unproved) Hodge conjecture for A . Upon replacing E by E' , we assume there is a map $\rho_{\ell}^{\mathbf{G}} : \text{Gal}(\bar{E}/E) \rightarrow \mathbf{G}(\mathbb{Q}_{\ell})$.

For any reductive group \mathbf{H} over \mathbb{Q} we write $\text{Conj}_{\mathbf{H}}$ for the variety of semisimple conjugacy classes of \mathbf{H} and $\chi_{\mathbf{H}} : \mathbf{H} \rightarrow \text{Conj}_{\mathbf{H}}$ for the natural projection map. We thus obtain a well-defined element

$$\gamma_{\ell} = \gamma_{\ell}(v) := \chi_{\mathbf{G}}(\rho_{\ell}^{\mathbf{G}}(\text{Frob}_v)) \in \text{Conj}_{\mathbf{G}}(\mathbb{Q}_{\ell}),$$

the conjugacy class of ℓ -adic Frobenius at v . Our main theorem is the following.

Theorem 1.1. *Let $p > 2$ and $v|p$ a prime of E where A has good reduction. Then there exists $\gamma \in \text{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that*

$$\gamma = \gamma_{\ell} \in \text{Conj}_{\mathbf{G}}(\mathbb{Q}_{\ell}), \quad \forall \ell \neq p.$$

Since $P_{v,\ell}(t)$ is independent of ℓ , the image of γ_{ℓ} in $\text{Conj}_{\mathbf{GL}(V)}(\mathbb{Q}_{\ell})$ is defined over \mathbb{Q} and independent of ℓ . However, in general the map $\text{Conj}_{\mathbf{G}}(\mathbb{Q}) \rightarrow \text{Conj}_{\mathbf{GL}(V)}(\mathbb{Q})$ is not injective, so the theorem gives more information than the ℓ -independence of $P_{v,\ell}(t)$.

An analogue of the above theorem for any algebraic variety (or more generally motive) over a number field was conjectured by Serre in [Ser94, 12.6], but in general one does not even know the analogue of Deligne’s theorem on the existence of $\rho_{\ell}^{\mathbf{G}}$.

Previously proved cases of our theorem include a result of Noot who showed a version of this theorem where $\text{Conj}_{\mathbf{G}}$ is replaced by a certain quotient $\text{Conj}'_{\mathbf{G}_A}$ and under the additional assumption that the Frobenius element γ_{ℓ} is weakly neat [Noo09]. More recently, one of us [Kis17] proved the Theorem when \mathbf{G} is unramified at p , at least for some E' . Noot’s argument uses the independence of ℓ of $P_{v,\ell}(t)$, together with group theoretic arguments to analyze the map $\text{Conj}_{\mathbf{G}} \rightarrow \text{Conj}_{\mathbf{GL}(V)}$. The result of [Kis17] is proved by showing that, on the Shimura variety associated to \mathbf{G} , the isogeny class corresponding to A contains a point which admits a CM lift. It does not seem possible to extend either method to prove Theorem 1.1.

For the rest of the introduction we assume $p > 2$. Our proof of Theorem 1.1 makes use of families of abelian varieties with Mumford–Tate group contained in

\mathbf{G} , and especially the structure of their mod p reductions. These families are parameterized by a Shimura variety $\mathrm{Sh}_K(\mathbf{G}, X)$ associated to \mathbf{G} , and defined over a number field (its reflex field) $\mathbf{E} \subset \mathbb{C}$ which is contained in \mathbf{E} . We take $K = K_p K^p$ with $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ a parahoric subgroup and $K^p \subset \mathbf{G}(\mathbb{A}_f^p)$ a compact open subgroup. Let w be the restriction of v to \mathbf{E} . Write \mathbf{E}_w for the completion of \mathbf{E} at w , $\mathcal{O}_{\mathbf{E}_w}$ for the ring of integers of \mathbf{E}_w and $\kappa(w)$ for its residue field. Under some mild conditions we show that $\mathrm{Sh}_K(\mathbf{G}, X)$ has an integral model $\mathcal{S}_K(\mathbf{G}, X)$ over $\mathcal{O}_{\mathbf{E}_w}$, which is smoothly equivalent to a “local model”, defined as the closure of an orbit of \mathbf{G} acting on a certain Grassmannian. This extends the results of the first author and Pappas [KP18], which were restricted to the case when $\mathbf{G}_{\mathbb{Q}_p}$ was a tamely ramified group.

For each prime $\ell \neq p$, $\mathcal{S}_K(\mathbf{G}, X)$ is equipped with a $\mathbf{G}(\mathbb{Q}_\ell)$ -torsor \mathbb{L}_ℓ . In particular, for any finite extension $\kappa/\kappa(w)$ and $x \in \mathcal{S}_K(\mathbf{G}, X)(\kappa)$, the $q = |\kappa|$ -Frobenius acting on the geometric fiber of \mathbb{L}_ℓ at x , gives rise to an element $\gamma_{x,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$. We say x has the property (ℓ -ind), or the ℓ -independence property, if there exists an element $\gamma \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that

$$\gamma = \gamma_{x,\ell} \in \mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell), \forall \ell \neq p.$$

Now suppose that (\mathbf{G}, X) satisfies the conditions needed to guarantee the existence of $\mathcal{S}_K(\mathbf{G}, X)$ (cf. Theorem 5.2.13); the general case of Theorem 1.1 is eventually reduced to this one. Then for a suitable choice of K , our abelian variety A corresponds to a point $\tilde{x}_A \in \mathrm{Sh}_K(\mathbf{G}, X)(\mathbf{E})$ and its mod v reduction is a point x_A of the special fiber $\mathcal{S}_K := \mathcal{S}_K(\mathbf{G}, X) \otimes_{\mathcal{O}_{\mathbf{E}_w}} \kappa(w)$. Moreover there is an equality $\gamma_\ell(v) = \gamma_{x_A,\ell}$ as elements of $\mathrm{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$. Thus in order to show Theorem 1.1, it suffices to prove

(†) If $\kappa/\kappa(w)$ is finite, and $x \in \mathcal{S}_K(\kappa)$, then x satisfies (ℓ -ind).

By considering A as a point on a larger Shimura variety related to a group of the form $\mathrm{Res}_{F/\mathbb{Q}} \mathbf{G}$ where F is a suitably chosen totally real field, one can show that Theorem 1.1 follows from the following special case of (†).

Theorem 1.2. *Let (\mathbf{G}, X) be a Shimura datum of Hodge type and assume $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split, K_p is a very special parahoric and the triple (\mathbf{G}, X, K_p) is acceptable. Then for any $\kappa/\kappa(w)$ finite and $x \in \mathcal{S}_K(\kappa)$, x satisfies (ℓ -ind).*

The condition of acceptability of the triple (\mathbf{G}, X, K_p) is a technical one, and we refer the reader to §5.2.8 for the definition (cf. also Definition 5.2.6). We only mention here that this condition is always satisfied if $p > 3$ and the associated adjoint datum $(\mathbf{G}_{\mathrm{ad}}, X_{\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$.

As a first step towards Theorem 1.2, we prove the following analogue of Serre–Tate theory, which allows us to show that under the assumptions of Theorem 1.2, (ℓ -ind) holds on a dense, Zariski open subset of \mathcal{S}_K .

Theorem 1.3. *Assume (\mathbf{G}, X) is Hodge type and the triple (\mathbf{G}, X, K_p) is acceptable. Then*

- (1) *Any closed point x lying in the μ -ordinary locus $\mathcal{S}_{K,[b]_\mu} \subset \mathcal{S}_K$ admits a lifting to a special point $\tilde{x} \in \mathrm{Sh}_K(\mathbf{G}, X)$.*
- (2) *If in addition $\mathbf{G}_{\mathbb{Q}_p}$ is quasi-split and K_p is very special. Then $\mathcal{S}_{K,[b]_\mu}$ is Zariski open and dense in \mathcal{S}_K .*

The lifting constructed in (1) is the analogue in our setting of the canonical lift for ordinary abelian varieties and had been considered for Shimura varieties with good reduction in previous work of Moonen [Moo04] and Shankar and the second author [SZ21]. For these points, the Frobenius lifts to an automorphism of the associated CM abelian variety, and we obtain the desired element $\gamma \in \text{Conj}_{\mathbf{G}}(\mathbb{Q})$ by considering the induced action on Betti cohomology.

To prove Theorem 1.2, one considers a smooth curve \mathcal{C} with a map $\pi : \mathcal{C} \rightarrow \mathcal{S}_K$. Using a theorem of L. Lafforgue [Laf02, Théorème VII.6] on the existence of compatible local systems on smooth curves, we show that if the property (ℓ -ind) holds for a dense open subset of points on \mathcal{C} then it holds for all points of \mathcal{C} . Our results on the structure of the integral models $\mathcal{S}_K(\mathbf{G}, X)$ imply that \mathcal{S}_K is equipped with a certain combinatorially described stratification, the Kottwitz–Rapoport stratification. The stratum of maximal dimension is the smooth locus of \mathcal{S}_K . A theorem of Poonen [Poo04] shows that π can be chosen so that its image intersects $\mathcal{S}_{K,[b]_\mu}$ and any point x of the maximal stratum. The μ -ordinary case explained above then implies that any such x satisfies (ℓ -ind). We now argue by induction on the codimension of the strata; for a closed point x in some stratum of \mathcal{S}_K , we show that π can be chosen so that its image contains x , and also meets some higher dimensional stratum.

In fact, using general arguments with ampleness, it is not hard to construct a π whose image contains any closed point $x \in \mathcal{S}_K$, and meets the μ -ordinary locus. This would appear to avoid the induction on strata above. However, this argument would only allow us to prove the ℓ -independence result for some power of the Frobenius. To prove Theorem 1.2 in full, one needs the existence of a $y \in \mathcal{C}$, with $\pi(y) = x$, such that π induces an isomorphism of residue fields $\kappa(x) \simeq \kappa(y)$. To construct such curves, we first construct a sequence of smooth curves which are *subschemes* of the local model associated to $\mathcal{S}_K(\mathbf{G}, X)$, using the explicit group theoretic description of this local model. These are then pulled back to $\mathcal{S}_K(\mathbf{G}, X)$ via the local model diagram. The assumption that K_p is very special is key to our argument, as this not only guarantees the density of $\mathcal{S}_{K,[b]_\mu}$, but also that the Kottwitz–Rapoport stratification on the local model has a particularly simple description (cf. §6.2.2) which is used in the construction of π .

The induction argument would also be unnecessary if one could show a conjecture of Deligne [Del80, Conjecture 1.2.10] on the existence of compatible local systems on a normal variety. For smooth schemes Deligne’s conjecture has been proved by Drinfeld [Dri12], but the special fiber \mathcal{S}_K is not smooth, so Drinfeld’s theorem does not suffice for our purposes.

We now explain the organization of the paper. In §2–5 we construct the integral models of the Shimura varieties we will need. These are then used to prove Theorem 1.1 in §6,7. As explained above, there are two main results we need about these integral models: the local model diagram, which relates them to an orbit closure on a Grassmannian, and an analogue of Serre–Tate theory at μ -ordinary points. The properties of these local models are established in §3. In particular, we show that a suitable Hodge embedding induces a closed immersion on local models (cf. Proposition 3.2.6) which generalizes [KP18, Proposition 2.3.6]. In §4 we review the deformation theory of p -divisible groups equipped with a collection of crystalline tensors following [KP18], and show the existence of canonical deformations for μ -ordinary p -divisible groups. The latter uses a generalization to general parahorics

of a result of Wortmann on μ -ordinary σ -conjugacy classes, which is proved in §2.3. We combine the previous results to construct the required integral models in §5, first in some special Hodge type cases, then in general following [KP18, §4.4-6]. A key input for the general case is the notion of R -smoothness, introduced in §2, which allows us to extend the twisting construction of [KP18, §4.4] beyond the tamely ramified case.

In §6, we prove Theorem 1.2 following the strategy outlined above and in §7 we prove Theorem 1.1 using Theorem 1.2. Finally we remark that for technical reasons related to the level structure on A , we actually work with Shimura stacks (i.e. Shimura varieties where the level structure is not neat) in §5-7.

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2. GROUP THEORETIC RESULTS

In this section, we prove some group theoretic results which will be needed in §5. §2.1–§2.3 contains the main results needed for the construction of canonical liftings in §4.3 and §5.3. In §2.4 we study some properties of Néron models of tori which will be used in §5.2 to construct integral models for Shimura varieties of abelian type.

2.1. σ -straight elements.

2.1.1. Let F be a non-archimedean local field with ring of integers \mathcal{O}_F . We fix a uniformizer $\varpi_F \in \mathcal{O}_F$ and we let k_F denote the residue field of \mathcal{O}_F . We let \check{F} denote the completion of the maximal unramified extension of F and $\mathcal{O}_{\check{F}}$ its ring of integers, and we fix \overline{F} an algebraic closure of F . We let k be the residue field of $\mathcal{O}_{\check{F}}$ which is an algebraic closure of k_F . We write Γ for the absolute Galois group $\text{Gal}(\overline{F}/F)$ of F and I for the inertia subgroup, which is identified with $\text{Gal}(\overline{F}/\check{F})$. We let σ denote the Frobenius element of $\text{Aut}(\check{F}/F)$.

Let S be a scheme. If X is a scheme over S and $S' \rightarrow S$ is a morphism of schemes, then we write $X_{S'}$ for the base change of X along $S' \rightarrow S$.

2.1.2. Let G be a reductive over F . Let S be a maximal \check{F} -split torus of G defined over F and T its centralizer (cf. [Tit79, 1.10] for the existence of S). By Steinberg's Theorem, G is quasi-split over \check{F} and T is a maximal torus of G . We let $\mathcal{B}(G, F)$ (resp. $\mathcal{B}(G, \check{F})$) denote the (extended) Bruhat–Tits building of G over F (resp. \check{F}). Let \mathfrak{a} denote a σ -invariant alcove in the apartment $V := \mathcal{A}(G, S, \check{F})$ over \check{F} associated to S ; we write \mathcal{I} for the corresponding Iwahori group scheme over \mathcal{O}_F . The relative Weyl group W_0 and the Iwahori Weyl group are defined as

$$(2.1.2.1) \quad W_0 = N(\check{F})/T(\check{F}) \quad W = N(\check{F})/\mathcal{T}_0(\mathcal{O}_{\check{F}})$$

where N is the normalizer of T and \mathcal{T}_0 is the connected Néron model for T . These are related by an exact sequence

$$0 \longrightarrow X_*(T)_I \longrightarrow W \longrightarrow W_0 \longrightarrow 0.$$

For an element $\lambda \in X_*(T)_I$ we write t_λ for the corresponding element in W ; such elements will be called translation elements. We will sometimes write W_G or $W_{G_{\check{F}}}$ for W if we want to specify the group that we are working with.

2.1.3. We also fix a special vertex \mathfrak{s} lying in the closure of \mathfrak{a} . Such a vertex induces a splitting of the exact sequence (2.1.2.1) and gives an identification

$$(2.1.3.1) \quad V \cong X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}.$$

Let $\text{Aff}(V)$ denote the group of affine transformations of V . Then we have an identification $\text{Aff}(V) \cong V \rtimes \text{GL}(V)$. The Frobenius σ acts on V via affine transformations and we write $\varsigma \in \text{GL}(V)$ for the linear part of this action. The identification (2.1.3.1) also determines a dominant chamber $C_+ \subset X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$; namely by taking the one containing \mathfrak{a} , and we write B for the corresponding Borel subgroup defined over \check{F} . We write σ_0 for the automorphism of $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ defined by $\sigma_0 := w_0 \circ \varsigma$ where $w_0 \in W_0$ is the unique element such that $w_0 \circ \varsigma(C_+) = C_+$. We call this the L -action on $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$; by definition it preserves C_+ .

2.1.4. Let \mathbb{S} denote the set of simple reflections in the walls of \mathfrak{a} . We let W_a denote the affine Weyl group; it is the subgroup of W generated by the reflections in \mathbb{S} . Then (W_a, \mathbb{S}) has the structure of a Coxeter group and hence a notion of length and Bruhat order. The Iwahori Weyl group and affine Weyl group are related via the following exact sequence

$$(2.1.4.1) \quad 0 \longrightarrow W_a \longrightarrow W \longrightarrow \pi_1(G)_I \longrightarrow 0.$$

The choice of \mathfrak{a} induces a splitting of this exact sequence and $\pi_1(G)_I$ can be identified with the subgroup $\Omega \subset W$ consisting of elements which preserve \mathfrak{a} . The length function ℓ and Bruhat order \leq extend to W via this choice of splitting and Ω is identified with the set of length 0 elements.

We let $\tilde{\kappa}_G(w)$ denote the image of $w \in W$ in $\pi_1(G)_I$ and $\kappa_G(w)$ its projection to $\pi_1(G)_{\Gamma}$. For $w \in W$, there is an integer n such that σ^n acts trivially on W and $w\sigma(w) \dots \sigma^{n-1}(w) = t_\lambda$ for some $\lambda \in X_*(T)_I$. We define the (non-dominant) Newton cocharacter $\nu_w \in X_*(T)_{I, \mathbb{Q}} \cong X_*(T)_{\mathbb{Q}}^I$ to be $\frac{1}{n}\lambda$, which is easily seen to be independent of n . We let $\bar{\nu}_w \in X_*(T)_{\mathbb{Q}}^{I,+}$ be the dominant representative of ν_w .

2.1.5. Let T_{sc} , denote the preimage of T in the simply connected covering G_{sc} of the derived group of G . Then W_a is the Iwahori Weyl group for G_{sc} and we have the following exact sequence

$$0 \longrightarrow X_*(T_{\text{sc}})_I \longrightarrow W_a \longrightarrow W_0 \longrightarrow 0.$$

Since the action of I permutes the set of absolute coroots, $X_*(T_{\text{sc}})_I$ is torsion free and there is an inclusion $X_*(T_{\text{sc}})_I \hookrightarrow X_*(T)_I$. By [HR08], there exists a reduced root system Σ such that

$$W_a \simeq Q^\vee(\Sigma) \rtimes W_0$$

where $Q^\vee(\Sigma)$ and $W(\Sigma)$ denotes the coroot lattice and Weyl group of Σ respectively. The roots of Σ are proportional to the roots of the relative root system for $G_{\check{F}}$; however the root systems themselves may not be proportional.

As explained in [HR08, p7], we may consider elements of Σ as functions on $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$, and we write $\langle \cdot, \cdot \rangle$ for the induced pairing between $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ and

the root lattice associated to Σ . We let ρ denote the half sum of all positive roots in Σ . Then for any $\lambda \in X_*(T)_I$ we have the equality

$$(2.1.5.1) \quad \ell(t_\lambda) = \langle \bar{\lambda}, 2\rho \rangle,$$

where $\bar{\lambda} \in W_0 \cdot \lambda$ is the dominant representative of λ , i.e. the image of $\bar{\lambda}$ in $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ lies in C_+ .

2.1.6. We say that an element $w \in W$ is σ -straight if for any $n \in \mathbb{N}$,

$$\ell(w\sigma(w) \dots \sigma^{n-1}(w)) = n\ell(w).$$

It is straightforward to check that this is equivalent to the condition $\ell(w) = \langle \bar{\nu}_w, 2\rho \rangle$.

In this paper, we are particularly interested in translation elements $t_{\mu'}$ which are also σ -straight; the key property of these elements that we will need is that they are central for some Levi subgroup of G defined over F .

For any $v \in X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$, we let $\Phi_{v,0}$ be the set of relative roots α for $G_{\bar{F}}$ such that $\langle v, \alpha \rangle = 0$. We may then associate to v the semi-standard Levi subgroup $M_v \subset G_{\bar{F}}$ generated by T and the root subgroups U_α corresponding to $\alpha \in \Phi_{v,0}$. If in addition v is fixed by ς , then M_v is defined over F . We say $\lambda \in X_*(T)_I$ is central in G if it pairs with any relative root (equivalently any root in Σ) to give 0.

Lemma 2.1.7. *Let $\mu' \in X_*(T)_I$ such that $t_{\mu'}$ is a σ -straight element and let $M := M_{\nu_{t_{\mu'}}}$ be the semi-standard Levi subgroup of G associated to the Newton cocharacter $\nu_{t_{\mu'}}$. Then M is defined over F and μ' is central in M .*

Proof. For any $\lambda \in X_*(T)_I$, and for sufficiently divisible n we have

$$n\nu_{\sigma(t_\lambda)} = \sigma(t_\lambda) \dots \sigma^n(t_\lambda) = t_\lambda^{-1} n\nu_{t_\lambda} t_\lambda = n\nu_{t_\lambda}.$$

Note that $\sigma(t_\lambda) = t_{\varsigma(\lambda)}$; it follows that $\nu_{\sigma(t_\lambda)} = \varsigma(\nu_{t_\lambda})$ and hence ν_{t_λ} is fixed by ς . Therefore M is defined over F .

We let $u \in W_0$ be such that $u(\nu_{t_{\mu'}}) = \bar{\nu}_{t_{\mu'}}$. For a sufficiently divisible n , we have

$$\ell(t_{\mu'}) = \langle \bar{\nu}_{t_{\mu'}}, 2\rho \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle u\varsigma^i(\mu'), 2\rho \rangle$$

where the first equality follows from the σ -straightness of $t_{\mu'}$. Now $\langle u\varsigma^i(\mu'), 2\rho \rangle \leq \ell(t_{\mu'})$ with equality if and only if $u\varsigma^i(\mu')$ is dominant. Therefore $u\varsigma^i(\mu')$ is dominant for all i and hence $\varsigma^i(\mu')$ is contained in the translate C' of the dominant chamber C_+ by u^{-1} for all i .

Now M corresponds to a sub-root system Σ_M of Σ consisting of the roots $\alpha \in \Sigma$ such that $\langle \nu_{t_{\mu'}}, \alpha \rangle = 0$. Then Σ_M is also the reduced root system associated to the affine Weyl group for M as in §2.1.5. We must show for all $\alpha \in \Sigma_M$, we have $\langle \mu', \alpha \rangle = 0$. Let $\alpha \in \Sigma_M$ be a root, then since $\varsigma^i(\mu')$ is contained in a single Weyl chamber for all i , it follows that $\langle \varsigma^i(\mu'), \alpha \rangle$ have the same sign for all i .

Without loss of generality, assume $\langle \varsigma^i(\mu'), \alpha \rangle \geq 0, \forall i$. Then we have

$$(2.1.7.1) \quad 0 = \langle \nu_{t_{\mu'}}, \alpha \rangle = \frac{1}{n} \sum_{i=0}^{n-1} \langle \varsigma^i(\mu'), \alpha \rangle.$$

Since all the terms in the sum are non-negative, they must be 0. Hence μ' is central in M . □

2.1.8. Now let $\{\mu\}$ be a geometric conjugacy class of cocharacters of G . Let $\mu \in X_*(T)_I$ denote the image of a dominant (with respect to the choice of Borel B defined above) representative $\tilde{\mu} \in X_*(T)$ of $\{\mu\}$.

Lemma 2.1.9. *Let $w \in W_0$ such that for $\mu' := w(\mu)$, $t_{\mu'}$ is a σ -straight element. Let $\tilde{\lambda} := w(\tilde{\mu}) \in X_*(T)$. Then $\tilde{\lambda}$ is central in $M := M_{\nu_{\mu'}}$. Here, we consider W_0 as a subgroup of the absolute Weyl group for G .*

Proof. Let $w(C_+) \subset X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{R}$ be the translate of the dominant chamber by w . Then $w(C_+)$ determines a chamber C_M for M (it is the unique chamber for M such that $w(C_+) \subset C_M$) and $\mu' \in C_M$. The chamber C_M determines an ordering of the root system Σ_M . Let α be a positive root for Σ_M and $\tilde{\alpha} \in X^*(T)$ an (absolute) root lifting α ; such a lift exists by the construction of Σ , see eg. [Bou68, VI, 2.1]. We let $(\ , \) : X_*(T) \times X^*(T) \rightarrow \mathbb{Z}$ denote the natural pairing.

Let K/\check{F} be a finite Galois extension over which T splits. We have by definition of Σ_M

$$0 = \langle \mu', \alpha \rangle = c \sum_{\tau \in \text{Gal}(K/\check{F})} (\tilde{\lambda}, \tau(\tilde{\alpha}))$$

for some positive $c \in \mathbb{R}$, where the first equality follows since μ' is central in M . For any $\tau \in \text{Gal}(K/\check{F})$, C_M is preserved by τ and hence $\tau(\tilde{\alpha})$ is a positive root for M . Therefore $(\tilde{\lambda}, \tau(\tilde{\alpha})) \geq 0$, and hence $(\tilde{\lambda}, \tau(\tilde{\alpha})) = 0$ for all τ . Applying this to every relative root α for M , we see that $\tilde{\lambda}$ is central in M . \square

2.2. μ -ordinary σ -conjugacy classes.

2.2.1. Let $\{\mu\}$ be a geometric conjugacy class of cocharacters of G ; we let $\tilde{\mu} \in X_*(T)$ and $\mu \in X_*(T)_I$ as above. The μ -admissible set is defined to be

$$\text{Adm}(\{\mu\}) = \{w \in W \mid w \leq t_{x(\mu)} \text{ for some } x \in W_0\}.$$

It has a unique minimal element denoted $\tau_{\{\mu\}}$, which is also the unique element of $\text{Adm}(\{\mu\}) \cap \Omega$.

For $b \in G(\check{F})$, we let $[b]$ denote the set $\{g^{-1}b\sigma(g) \mid g \in G(\check{F})\}$, the σ -conjugacy class of b . The set of σ -conjugacy classes $B(G)$ has been classified by Kottwitz in [Kot92] and [Kot97]. For $b \in G(\check{F})$, we let $\nu_b : \mathbb{D} \rightarrow G_{\check{F}}$ denote its Newton cocharacter and

$$\bar{\nu}_b \in X_*(T)_{I, \mathbb{Q}}^+ \cong X_*(T)_{\mathbb{Q}}^{I, +}$$

the dominant representative for ν_b ; it is known that $\bar{\nu}_b$ is invariant under the action of σ_0 . We let $\tilde{\kappa}_G : G(\check{F}) \rightarrow \pi_1(G)_I$ denote the Kottwitz homomorphism and we write

$$\kappa_G : G(\check{F}) \rightarrow \pi_1(G)_{\Gamma}$$

for the composition of $\tilde{\kappa}_G$ and the projection map $\pi_1(G)_I \rightarrow \pi_1(G)_{\Gamma}$. This induces a well-defined map $B(G) \rightarrow \pi_1(G)_{\Gamma}$, also denoted κ_G . Then there is an injective map

$$(2.2.1.1) \quad B(G) \xrightarrow{(\kappa_G, b \mapsto \bar{\nu}_b)} \pi_1(G)_{\Gamma} \times (X_*(T)_{\mathbb{Q}}^{I, +})^{\sigma_0}.$$

2.2.2. There is a more explicit description of this map using W . For $w \in W$, its σ -conjugacy class is the set $\{u^{-1}w\sigma(w)|u \in W\}$. We let $B(W, \sigma)$ denote the set of σ -conjugacy classes in W . For $w \in W$, we let $\dot{w} \in N(\check{F})$ denote a lift of w . Then to $w \in W$, we associate the σ -conjugacy class of \dot{w} ; by Lang's theorem this does not depend on the choice of representative \dot{w} . We write

$$\Psi : B(W, \sigma) \rightarrow B(G)$$

for the map induced by $w \mapsto [\dot{w}]$.

By [He14, Theorem 3.7], Ψ is surjective and we have a commutative diagram

$$(2.2.2.1) \quad \begin{array}{ccc} B(W, \sigma) & \xrightarrow{\Psi} & B(G) \\ & \searrow (\bar{\nu}, \kappa_G) & \swarrow (\bar{\nu}, \kappa_G) \\ & (X_*(T)_{\mathbb{Q}}^{I,+}) \times \pi_1(G)_{\Gamma} & \end{array}$$

The map Ψ is not injective in general, however it is proved in [He14, Theorem 3.7] that its restriction to the set of σ -straight σ -conjugacy classes is a bijection. Here, a σ -conjugacy class in W is said to be σ -straight if it contains a σ -straight element.

2.2.3. Note that there is a partial order on the set $X_*(T)_{\mathbb{Q}}^+$; for $\lambda, \lambda' \in X_*(T)_{\mathbb{Q}}^+$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative rational linear combination of positive roots. For $\{\mu\}$ as above, we write μ^{\natural} for the common image of an element of $\{\mu\}$ in $\pi_1(G)_{\Gamma}$ and we define

$$\mu^{\diamond} = \frac{1}{N} \sum_{i=1}^N \sigma_0^i(\mu) \in X_*(T)_{I, \mathbb{Q}}^+ \cong X_*(T)_{\mathbb{Q}}^{I,+}.$$

where N is the order of the element σ_0 acting on $X_*(T)_I \otimes_{\mathbb{Z}} \mathbb{Q}$. We set

$$B(G, \{\mu\}) = \{[b] \in B(G) : \kappa_G(b) = \mu^{\natural}, \bar{\nu}_b \leq \mu^{\diamond}\}.$$

Note that for $[b_1], [b_2] \in B(G, \{\mu\})$ such that $\bar{\nu}_{[b_1]} = \bar{\nu}_{[b_2]}$, we have $[b_1] = [b_2]$ since $[b_1]$ and $[b_2]$ have common image μ^{\natural} under κ_G .

Definition 2.2.4. Suppose there exists a class $[b] \in B(G, \{\mu\})$ such that $\bar{\nu}_{[b]} = \mu^{\diamond}$ (such a class is necessarily unique if it exists by the above remark). We write $[b]_{\mu}$ for this class; it is called the μ -ordinary σ -conjugacy class.

Remark 2.2.5. [HN18, Theorem 1.1] have shown that $B(G, \{\mu\})$ always contains a maximal element with respect to the partial order \leq . When G is quasi-split, this class is just $[b]_{\mu}$. However if G is not quasi-split, there may be no $[b] \in B(G, \{\mu\})$ such that $\bar{\nu}_{[b]} = \mu^{\diamond}$.

Lemma 2.2.6. Assume there exists $[b]_{\mu} \in B(G, \{\mu\})$ with $\bar{\nu}_{[b]_{\mu}} = \mu^{\diamond}$. There exists $\mu' \in W_0 \cdot \mu$ with $t_{\mu'}$ σ -straight such that $t_{\mu'} \in [b]_{\mu}$.

Proof. Since $[b]_{\mu} \in B(G, \{\mu\})$, there exists a σ -straight element $w \in \text{Adm}(\{\mu\})$ such that $\dot{w} \in [b]_{\mu}$ by [He16, Theorem 4.1]. The commutativity of diagram (2.2.2.1) implies that $\bar{\nu}_w = \mu^{\diamond}$. Since w is σ -straight, we have

$$\ell(w) = \langle \bar{\nu}_w, 2\rho \rangle = \langle \mu^{\diamond}, 2\rho \rangle = \langle \mu, 2\rho \rangle = \ell(t_{\mu}),$$

where the final equality uses (2.1.5.1) and the fact that μ is dominant. Since $w \in \text{Adm}(\{\mu\})$, $\ell(w) \leq \ell(t_{\mu})$ with equality if and only if $w = t_{\mu'}$ for some $\mu' \in W_0 \cdot \mu$. \square

2.2.7. Now let G' be another reductive group over F and $f : G \rightarrow G'$ a group scheme morphism which induces an isogeny $G_{\text{der}} \rightarrow G'_{\text{der}}$. We write $\{\mu'\}$ for the G' -conjugacy class of cocharacters induced by $\{\mu\}$. We have the following relationship between μ -ordinary σ -conjugacy classes for G and G' .

Lemma 2.2.8. (1) *There exists $[b]_{\mu} \in B(G, \{\mu\})$ with $\bar{\nu}_{[b]_{\mu}} = \mu^{\diamond}$ if and only if there exists $[b']_{\mu'} \in B(G', \{\mu'\})$ with $\bar{\nu}_{[b']_{\mu'}} = \mu'^{\diamond}$.*
 (2) *Let $[b] \in B(G, \mu)$ and $[b'] := [f(b)] \in B(G', \{\mu'\})$. Then $[b] = [b]_{\mu}$ if and only if $[b'] = [b']_{\mu'}$.*

Proof. (1) Note that we have a commutative diagram

$$\begin{array}{ccc} B(G) & \longrightarrow & (X_*(T)_{\mathbb{Q}}^{I,+}) \times \pi_1(G)_{\Gamma} \\ \downarrow & & \downarrow \\ B(G') & \longrightarrow & (X_*(T')_{\mathbb{Q}}^{I,+}) \times \pi_1(G')_{\Gamma} \end{array}$$

where T' is the centralizer of a maximal \check{F} -split torus of G' containing $f(T)$. Thus one direction of (1) is clear.

For the converse, suppose there exists $[b']_{\mu'} \in B(G', \{\mu'\})$. Note that by assumption, there is an identification of relative Weyl groups for G and G' . Then by Lemma 2.2.6, there exists $w_0 \in W_0$ such that $t_{w_0(\mu')}$ is a σ -straight element of the Iwahori Weyl group for G' and $t_{w_0(\mu')} \in [b']_{\mu'}$. Then it is easy to check that $t_{w_0(\mu')}$ is a σ -straight element of the Iwahori Weyl group for G and that $\bar{\nu}_{t_{w_0(\mu')}} = \mu'^{\diamond}$. It follows that $[t_{w_0(\mu')}] = [b]_{\mu} \in B(G, \{\mu\})$.

(2) One direction is clear. Suppose then that $[b'] = [b']_{\mu'}$. It follows that $\bar{\nu}_{[b]} = \mu^{\diamond} + \alpha$ for some $\alpha \in X_*(\ker(G \rightarrow G'))^I$. But $[b] \in B(G, \{\mu\})$ and hence $\mu^{\diamond} - \bar{\nu}_{[b]}$ is a rational linear combination of positive coroots. Thus $\alpha = 0$ and $[b] = [b]_{\mu}$. \square

2.3. Parahoric group schemes.

2.3.1. Recall the extended Bruhat–Tits buildings $\mathcal{B}(G, F)$ and $\mathcal{B}(G, \check{F})$ associated to G . For a non-empty bounded subset $\Xi \subset \mathcal{B}(G, F)$ which is contained in an apartment, we let $G(F)_{\Xi}$ (resp. $G(\check{F})_{\Xi}$) denote the subgroup of $G(F)$ (resp. $G(\check{F})$) which fixes Ξ pointwise. By the main result of [BT84], there exists a smooth affine group scheme $\tilde{\mathcal{G}}_{\Xi}$ over \mathcal{O}_F with generic fiber G which is uniquely characterized by the property $\tilde{\mathcal{G}}_{\Xi}(\mathcal{O}_{\check{F}}) = G(\check{F})_{\Xi}$. As in [KP18, §1.1.2], we will call such a group scheme the Bruhat–Tits stabilizer scheme associated to Ξ . If $\Xi = \{x\}$ is a point we write $G(F)_x$ (resp. $\tilde{\mathcal{G}}_x$) for $G(F)_{\{x\}}$ (resp. $\tilde{\mathcal{G}}_{\{x\}}$).

For $\Xi \subset \mathcal{B}(G, F)$, we write \mathcal{G}_{Ξ} for the “connected stabilizer” Ξ (cf. [BT84, §4]). We are mainly interested in the cases where Ξ is a point x or an open facet \mathfrak{f} . In this case, \mathcal{G}_x (resp. $\mathcal{G}_{\mathfrak{f}}$) is the parahoric group scheme associated to x (resp. \mathfrak{f}). By [HR08], $\mathcal{G}_{\Xi}(\mathcal{O}_{\check{F}}) = \tilde{\mathcal{G}}_{\Xi}(\mathcal{O}_{\check{F}}) \cap \ker \tilde{\kappa}_G$. It follows that $\mathcal{G}_{\Xi}(\mathcal{O}_F) = \tilde{\mathcal{G}}_{\Xi}(\mathcal{O}_F) \cap \ker \tilde{\kappa}_G$. If \mathfrak{f} is a facet of $\mathcal{B}(G, F)$ we say $x \in \mathfrak{f}$ is generic if every element of $G(F)$ which fixes x also fixes \mathfrak{f} pointwise. The set of generic points in \mathfrak{f} is an open dense subset of \mathfrak{f} , and for any generic point $x \in \mathfrak{f}$, we have $\tilde{\mathcal{G}}_x = \tilde{\mathcal{G}}_{\mathfrak{f}}$ and $\mathcal{G}_x = \mathcal{G}_{\mathfrak{f}}$.

We may also consider the corresponding objects over \check{F} and we use the same notation in this case. When it is understood which point of $\mathcal{B}(G, F)$ or $\mathcal{B}(G, \check{F})$

we are referring to, we simply write $\tilde{\mathcal{G}}$ and \mathcal{G} for the corresponding group schemes. A parahoric group scheme \mathcal{G} is said to be a *connected* parahoric if there exists $x \in \mathcal{B}(G, F)$ such that $\tilde{\mathcal{G}}_x = \mathcal{G}_x = \mathcal{G}$; if such a point exists, it is necessarily a generic point in the facet containing it.

Let G' be another connected reductive group and assume there is an identification $G_{\text{ad}} \cong G'_{\text{ad}}$ between their respective adjoint groups. Then there are surjective maps of buildings $\mathcal{B}(G, F) \rightarrow \mathcal{B}(G_{\text{ad}}, F)$ and $\mathcal{B}(G', F) \rightarrow \mathcal{B}(G'_{\text{ad}}, F)$ which are equivariant for $G(F)$ and $G'(F)$ respectively. If $\mathcal{G} = \mathcal{G}_x$ is a parahoric group scheme for G corresponding to $x \in \mathcal{B}(G, F)$, then \mathcal{G} determines a parahoric group scheme $\mathcal{G}' = \mathcal{G}'_{x'}$ for G' where $x' \in \mathcal{B}(G', F)$ lies in the preimage of the image of x in $\mathcal{B}(G_{\text{ad}}, F)$.

2.3.2. Now let $J \subset \mathbb{S}$ be a subset and we write W_J for the subgroup of W generated by J . If W_J is finite, J corresponds to a parahoric group scheme \mathcal{G} over $\mathcal{O}_{\tilde{F}}$; such parahorics are called *standard* (with respect to \mathfrak{a}). We let W^J (resp. JW) denote the set of minimal length representatives of the cosets W/W_J (resp. $W_J \backslash W$).

We recall the Iwahori decomposition. For $w \in W$, the map $w \mapsto \dot{w}$ induces a bijection

$$W_J \backslash W / W_J \cong \mathcal{G}(\mathcal{O}_{\tilde{F}}) \backslash G(\tilde{F}) / \mathcal{G}(\mathcal{O}_{\tilde{F}}).$$

We now assume J is σ -stable; in this case \mathcal{G} is defined over \mathcal{O}_F and is a parahoric group scheme for G . For the rest of §2.3, we fix a geometric conjugacy class of cocharacters $\{\mu\}$ of G and assume the existence of $[b]_{\mu} \in B(G, \{\mu\})$. We define $\text{Adm}(\{\mu\})_J$ to be the image of $\text{Adm}(\{\mu\})$ in $W_J \backslash W / W_J$. We sometimes write $\text{Adm}_G(\{\mu\})_J$ if we want to specify the group G we are working with. The following is the key group theoretic result that we need in order to prove the existence of canonical liftings in §5.3.

Proposition 2.3.3. *Let $b \in \left(\bigcup_{w \in \text{Adm}(\{\mu\})_J} \mathcal{G}(\mathcal{O}_{\tilde{F}}) \dot{w} \mathcal{G}(\mathcal{O}_{\tilde{F}}) \right) \cap [b]_{\mu}$. Then*

- (1) $b \in \mathcal{G}(\mathcal{O}_{\tilde{F}}) \dot{t}_{\mu'} \mathcal{G}(\mathcal{O}_{\tilde{F}})$ for some σ -straight element $t_{\mu'}$.
- (2) There exists $g \in \mathcal{G}(\mathcal{O}_{\tilde{F}})$ such that $g^{-1} b \sigma(g) = \dot{t}_{\mu'}$.

Proof. By [HR17, Theorem 6.1 (b)], there exists $h \in \mathcal{G}(\mathcal{O}_{\tilde{F}})$ such that $h^{-1} b \sigma(h) \in \mathcal{I}(\mathcal{O}_{\tilde{F}}) \dot{w} \mathcal{I}(\mathcal{O}_{\tilde{F}})$ for some $w \in {}^JW$. Thus $w \in {}^JW \cap \text{Adm}(\{\mu\})_J$ and hence lies in ${}^JW \cap \text{Adm}(\{\mu\})$ by [He16, Theorem 6.1]. Thus upon replacing b by $h^{-1} b \sigma(h)$, we may assume $b \in \mathcal{I}(\mathcal{O}_{\tilde{F}}) \dot{w} \mathcal{I}(\mathcal{O}_{\tilde{F}})$. By [HZ20, Theorem 4.1], there exists a σ -straight element $x \leq w$ such that $[b]_{\mu} \cap \mathcal{I}(\mathcal{O}_{\tilde{F}}) \dot{x} \mathcal{I}(\mathcal{O}_{\tilde{F}}) \neq \emptyset$ (the Theorem in *loc. cit.* proves the non-emptiness of the affine Deligne–Lusztig variety $X_x(b)$, which is equivalent to this statement). By [He14, Theorem 3.5], $\dot{x} \in [b]_{\mu}$ and by the same argument as in Lemma 2.2.6 we have $x = t_{\mu'}$ for some $\mu' \in W_0 \cdot \mu$. Since $x \leq w$ and $w \in \text{Adm}(\{\mu\})$, we have $w = t_{\mu'}$. This proves (1).

For (2), the above argument shows that we may assume $b \in \mathcal{I}(\mathcal{O}_{\tilde{F}}) \dot{t}_{\mu'} \mathcal{I}(\mathcal{O}_{\tilde{F}})$ for $t_{\mu'}$ a σ -straight element. By [He14, Proposition 4.5], there exists $i \in \mathcal{I}(\mathcal{O}_{\tilde{F}})$ such that $i^{-1} b \sigma(i) = \dot{t}_{\mu'}$; the result follows. \square

Remark 2.3.4. This result is a generalization to general parahorics of [SZ21, Proposition 2.5] which is due to Wortmann. In the case when \mathcal{G} is a hyperspecial parahoric, this result is the group theoretic analogue of the fact that there is exactly one isomorphism class of ordinary F -crystal over $\mathcal{O}_{\tilde{F}}$.

2.4. Néron models of tori.

2.4.1. In this subsection, we introduce the notion of R -smooth tori and discuss some consequences for Bruhat–Tits group schemes.

Let T be a torus over a local field F ; recall we have defined \mathcal{T}_0 to be the connected Néron model of T . We let \mathcal{T} (resp. \mathcal{T}_{ft}) denote the lft Néron model (resp. finite type Néron model) for T . Then we have $\mathcal{T}(\mathcal{O}_{\tilde{F}}) = T(\tilde{F})$ and \mathcal{T}_{ft} is characterized by the condition $\mathcal{T}_{\text{ft}}(\mathcal{O}_{\tilde{F}}) = \{t \in T(\tilde{F}) \mid \tilde{\kappa}_T(t) \in X_*(T)_{I, \text{tors}}\}$ where $X_*(T)_{I, \text{tors}}$ is the torsion subgroup of $X_*(T)_I$. Alternatively, by [Rap05, n°1] the connected components of the special fiber of \mathcal{T} are parameterized by $X_*(T)_I$ and \mathcal{T}_{ft} is the unique smooth subgroup scheme of \mathcal{T} whose special fiber is given by the set of connected components corresponding to the torsion subgroup $X_*(T)_{I, \text{tors}}$ of $X_*(T)_I$.

2.4.2. Let \tilde{F}/F be a finite Galois extension over which T splits and we let $\mathcal{T}_{\mathcal{O}_{\tilde{F}}}$ denote the lft Néron model of $T_{\tilde{F}}$.¹ By [BLR90, §7.6, Proposition 6], $\text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{\mathcal{O}_{\tilde{F}}}$ is the lft Néron model over \mathcal{O}_F for $\text{Res}_{\tilde{F}/F} T_{\tilde{F}}$. There is a natural map $T \rightarrow \text{Res}_{\tilde{F}/F} T_{\tilde{F}}$ and we define \mathcal{T}^c to be the Zariski closure of T inside $\text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{\mathcal{O}_{\tilde{F}}}$. As in [BT84, §4.4.8], \mathcal{T}^c does not depend on the choice of Galois splitting field of T .

Definition 2.4.3. We say a torus T is R -smooth if \mathcal{T}^c is smooth.

Since \mathcal{T}^c satisfies the Néron mapping property (cf. [Edi92, Proof of Theorem 4.2]), we have $\mathcal{T} \cong \mathcal{T}^c$ if T is R -smooth.

We can similarly define a notion of R -smoothness for tori over \tilde{F} . It is easy to see using compatibility of Néron models with base change along $\mathcal{O}_F \rightarrow \mathcal{O}_{\tilde{F}}$ that a torus over F is R -smooth if and only if $T_{\tilde{F}}$ is R -smooth.

The main property concerning R -smooth tori that we need is the following.

Lemma 2.4.4. *Suppose we have a closed immersion $f : T_1 \rightarrow T_2$ between tori where T_1 is R -smooth, then f extends to a closed immersion $\mathcal{T}_1 \rightarrow \mathcal{T}_2$ of lft Néron models.*

Proof. Let \tilde{F} be a finite Galois splitting field for T_1 and T_2 . Then since $T_{1, \tilde{F}}$ and $T_{2, \tilde{F}}$ are just products of multiplicative group schemes, the map $T_{1, \tilde{F}} \rightarrow T_{2, \tilde{F}}$ extends to a closed immersion of lft Néron models $\mathcal{T}_{\mathcal{O}_{\tilde{F}}} \rightarrow \mathcal{T}_{2, \mathcal{O}_{\tilde{F}}}$ over $\mathcal{O}_{\tilde{F}}$. We obtain a diagram

$$\begin{array}{ccc} \mathcal{T}_1 & \xrightarrow{f} & \mathcal{T}_2 \\ g \downarrow & & \downarrow h \\ \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{1, \mathcal{O}_{\tilde{F}}} & \xrightarrow{i} & \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{2, \mathcal{O}_{\tilde{F}}} \end{array}$$

where i is a closed immersion since it is given by applying restriction of scalars to a closed immersion and g is a closed immersion since T_1 is R -smooth. It follows that $h \circ f = i \circ g$ is a closed immersion, and hence f is a closed immersion since h is separated. \square

¹We are abusing notation here since $\mathcal{T}_{\mathcal{O}_{\tilde{F}}}$ is not necessarily the base change to $\mathcal{O}_{\tilde{F}}$ of the Néron model \mathcal{T} of T over \mathcal{O}_F .

2.4.5. The proof of [Edi92, Theorem 4.2] shows that if T splits over a tamely ramified extension of F , then T is R -smooth. In addition, the main examples of R -smooth tori that we will consider are given by the following Proposition.

Proposition 2.4.6. (1) *Let K/F be a finite extension and S an R -smooth torus over K . Then $T := \text{Res}_{K/F} S$ is R -smooth.*

(2) *Suppose we have tori T_1, T_2 and T_3 with T_1 and T_2 R -smooth, together with group scheme morphisms $f : T_1 \rightarrow T_3$ and $g : T_2 \rightarrow T_3$ satisfying the following properties*

(i) *f is surjective and induces a smooth map $f : \mathcal{T}_1 \rightarrow \mathcal{T}_3$ on lft Néron models.*

(ii) *g is a closed immersion.*

Then the connected component T of the identity of the fiber product $T_1 \times_{T_3} T_2$ is an R -smooth torus.

Proof. (1) Let \tilde{F} be a finite Galois splitting field of T which necessarily contains K . For any F -morphism $\tau : K \rightarrow \tilde{F}$, the base change of S along τ is split. Since S is R -smooth, it follows that we have a closed immersion of \mathcal{O}_K -group schemes

$$\mathcal{S} \rightarrow \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_K} \mathcal{S}_{\mathcal{O}_{\tilde{F}}},$$

where \mathcal{S} (resp. $\mathcal{S}_{\mathcal{O}_{\tilde{F}}}$) is the lft Néron model for S (resp. $S_{\tilde{F}}$).

Applying $\text{Res}_{\mathcal{O}_K/\mathcal{O}_F}$ we obtain a closed immersion

$$\text{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathcal{S} \rightarrow \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{S}_{\mathcal{O}_{\tilde{F}}}.$$

Taking the product over all $\tau : K \rightarrow \tilde{F}$ we obtain a closed immersion

$$\text{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathcal{S} \rightarrow \prod_{\tau: K \rightarrow \tilde{F}} \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{S}_{\mathcal{O}_{\tilde{F}}} \cong \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{\mathcal{O}_{\tilde{F}}}.$$

Since $\text{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathcal{S}$ is the lft Néron model \mathcal{T} for T , it follows that \mathcal{T} is the closure of its generic fiber inside $\text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{\mathcal{O}_{\tilde{F}}}$ and hence T is R -smooth.

(2) We may assume $F = \tilde{F}$. We let \mathcal{T}'' denote the fiber product $\mathcal{T}_1 \times_{\mathcal{T}_3} \mathcal{T}_2$, where the \mathcal{T}_i are the lft Néron models for T_i . Then condition (i) implies that the map $\mathcal{T}'' \rightarrow \mathcal{T}_2$ is smooth, and hence \mathcal{T}'' is smooth over \mathcal{O}_F . We let $\mathcal{T}' \subset \mathcal{T}''$ denote the connected component of the identity; then \mathcal{T}' is a smooth scheme over \mathcal{O}_F . Moreover \mathcal{T}' satisfies the Néron mapping property for T ; it follows that \mathcal{T}' is isomorphic to the lft Néron model \mathcal{T} for T .

Let \tilde{F} denote a finite Galois splitting field for T_1 (and hence also for T); we obtain a commutative diagram:

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\quad} & \mathcal{T}_1 \\ \downarrow & & \downarrow \\ \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{\mathcal{O}_{\tilde{F}}} & \longrightarrow & \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{1, \mathcal{O}_{\tilde{F}}} \end{array}$$

Condition (ii) and the R -smoothness of T_2 implies that the natural map $\mathcal{T} \rightarrow \mathcal{T}_1$ is a closed immersion. By R -smoothness of \mathcal{T}_1 , the map $\mathcal{T}_1 \rightarrow \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{1, \mathcal{O}_{\tilde{F}}}$ is a closed immersion. It follows that $\mathcal{T} \rightarrow \text{Res}_{\mathcal{O}_{\tilde{F}}/\mathcal{O}_F} \mathcal{T}_{\mathcal{O}_{\tilde{F}}}$ is a closed immersion and hence T is R -smooth. \square

Corollary 2.4.7. *Let $T_1 = \prod_{i=1}^s \text{Res}_{K_i/F} S_{1,i}$ and $T_3 = \prod_{i=1}^s \text{Res}_{K_i/F} S_{3,i}$ respectively where K_i is a finite extension of F and $S_{1,i}, S_{3,i}$ are K_i -tori which split over a tamely ramified extension of F_i , and let T_2 be an F -torus which splits over a tamely ramified extension of F .*

Suppose we are given a group scheme morphism $f : T_1 \rightarrow T_3$ which arises from a product of surjective maps $S_{1,i} \rightarrow S_{3,i}$ over K_i , and $g : T_2 \rightarrow T_3$ a group scheme morphism which is a closed immersion. Then the connected component T of the identity of the fiber product $T_1 \times_{T_3} T_2$ is an R -smooth torus.

Proof. By Proposition 2.4.6 (1) and [Edi92, Theorem 4.2], T_1 and T_2 are R -smooth tori. By part (2) of Proposition 2.4.6, it suffices to show that $f : T_1 \rightarrow T_3$ induces a smooth map $\mathcal{T}_1 \rightarrow \mathcal{T}_3$ on lft Néron models over F . For this it suffices to consider the case $s = 1$; we thus drop the index i from the notation.

We first reduce to the case $\ker(f)$ is connected. Let $D := \ker(f)$ and let D° denote the connected component of the identity of D . We assume $f = \text{Res}_{K/F} h$ where $h : S_1 \rightarrow S_3$; then $D = \text{Res}_{K/F} \ker h$ and $D^\circ = \text{Res}_{K/F} (\ker h)^\circ$, where $(\ker h)^\circ$ is the connected component of the identity of $\ker h$. The quotient $S'_3 := S_1/(\ker h)^\circ$ is a torus equipped with an isogeny $S'_3 \rightarrow S_3$ and we have an exact sequence

$$0 \longrightarrow (\ker h)^\circ \longrightarrow S_1 \longrightarrow S'_3 \longrightarrow 0.$$

Setting $T'_3 := \text{Res}_{K/F} S'_3$, we obtain an exact sequence

$$0 \longrightarrow D^\circ \longrightarrow T_1 \xrightarrow{f'} T'_3 \longrightarrow 0.$$

We define T'_2 to be the connected component of the identity of $T_2 \times_{T_3} T'_3$. Then we may identify T with the connected component of the identity of $T_1 \times_{T'_3} T'_2$. Since $T'_2 \rightarrow T'_3$ is a closed immersion, we may replace T_2 and T_3 , by T'_2 and T'_3 respectively and hence assume that $\ker f$ is connected.

By properties of Weil restriction, it is enough to show that the map $S_1 \rightarrow S_3$ on lft Néron models over \mathcal{O}_K , obtained from $S_1 \rightarrow S_2$ over K , is smooth. We reduce to showing that a surjective map $T \rightarrow T'$ between F -tori which split over tamely ramified extensions of F and whose kernel is connected induces a smooth map $\mathcal{T} \rightarrow \mathcal{T}'$ between lft Néron models. This now follows from the same argument as [Edi92, Theorem 6.1 (5) \Rightarrow (6)] using the fact that $\ker(T \rightarrow T')$ is a torus. \square

2.4.8. The previous results have the following consequences for Bruhat–Tits group schemes. Let G be a reductive group over F and \tilde{G} a Bruhat–Tits stabilizer scheme corresponding to $x \in \mathcal{B}(G, F)$ which is generic in the facet containing it. Let $\beta : G \hookrightarrow G'$ be a closed immersion of reductive groups over F , which induces an isomorphism on derived groups. As in [KP18, §1.1.3], x determines a point $x' \in \mathcal{B}(G', F)$ and we write \tilde{G}' for the corresponding Bruhat–Tits stabilizer scheme of G' ; then β extends to a group scheme homomorphism $\beta : \tilde{G} \rightarrow \tilde{G}'$.

Proposition 2.4.9. *Assume that the centralizer of any maximal \check{F} -split torus in G is an R -smooth torus. Then $\beta : \tilde{G} \rightarrow \tilde{G}'$ is a closed immersion.*

Proof. Since the construction of Bruhat–Tits stabilizer schemes is compatible with unramified base extensions, it is enough to prove the result in the case $F = \check{F}$.

We let S be a maximal \check{F} -split torus in G such that x lies in $\mathcal{A}(G, S, \check{F})$. Let T be the centralizer of S which by assumption is an R -smooth torus. Let S' be a

maximal split torus of G' such that $S' \cap G = S$ and we let T' denote the centralizer of S' . By the construction of Bruhat–Tits stabilizer schemes in [BT84, §4.6], the Zariski closure of T (resp. T') inside $\tilde{\mathcal{G}}$ (resp. $\tilde{\mathcal{G}}'$) can be identified with the finite type Néron model \mathcal{T}_{ft} (resp. \mathcal{T}'_{ft}). We claim that the natural map $T \rightarrow T'$ extends to a closed immersion

$$(2.4.9.1) \quad \mathcal{T}_{\text{ft}} \rightarrow \mathcal{T}'_{\text{ft}}$$

between finite type Néron models.

Assuming this, we can prove the proposition. For any relative root α , the map $G \rightarrow G'$ induces an isomorphism between the corresponding root subgroups U_α and U'_α . If we let \mathcal{U}_α and \mathcal{U}'_α denote the corresponding schematic closures, then by the construction of $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}'$ in [BT84, §4.6], the map $G \rightarrow G'$ also induces an isomorphism $\mathcal{U}_\alpha \rightarrow \mathcal{U}'_\alpha$. Thus by [BT84, Theorem 2.2.3] the schematic closure $\hat{\mathcal{G}}$ of G in $\tilde{\mathcal{G}}'$ contains the smooth big open cell

$$\prod_{\alpha} \mathcal{U}_{-\alpha} \times \mathcal{T}_{\text{ft}} \times \prod_{\alpha} \mathcal{U}_{\alpha};$$

hence by [BT84, Corollary 2.2.5], $\hat{\mathcal{G}}$ is smooth. Since $\hat{\mathcal{G}}(\mathcal{O}_{\check{F}}) = G(\check{F}) \cap \tilde{\mathcal{G}}'(\mathcal{O}_{\check{F}})$, it follows that $\hat{\mathcal{G}} \cong \tilde{\mathcal{G}}$, and hence we obtain a closed immersion $\tilde{\mathcal{G}} \hookrightarrow \tilde{\mathcal{G}}'$ as desired.

It remains to show the existence of the closed immersion (2.4.9.1).

By Lemma 2.4.4, we have a closed immersion $\mathcal{T} \rightarrow \mathcal{T}'$ of lft Néron models. We let $\phi : X_*(T)_I \rightarrow X_*(T')_I$ denote the morphism on the targets of the Kottwitz homomorphism. Then it is easy to see that

$$\phi^{-1}(X_*(T')_{I, \text{tors}}) = X_*(T)_{I, \text{tors}}.$$

As the finite type Néron models \mathcal{T}_{ft} and \mathcal{T}'_{ft} correspond to the subschemes of \mathcal{T} and \mathcal{T}' whose special fibers are given by the connected components parameterized by $X_*(T)_{I, \text{tors}}$ and $X_*(T')_{I, \text{tors}}$ respectively, it follows that $\mathcal{T} \rightarrow \mathcal{T}'$ induces a closed immersion $\mathcal{T}_{\text{ft}} \rightarrow \mathcal{T}'_{\text{ft}}$ as desired. \square

Remark 2.4.10. As all maximal \check{F} -split tori are \check{F} -conjugate, the centralizer of any maximal \check{F} -split torus is R -smooth if there exists one such centralizer which is R -smooth.

2.4.11. Now let $\beta : G \rightarrow G'$ be a central extension between reductive groups with kernel Z and \mathcal{G} the parahoric group scheme associated to some $x \in \mathcal{B}(G, F)$. We let \mathcal{G}' denote the parahoric of G' corresponding to \mathcal{G} ; then as above, β extends to a group scheme homomorphism $\mathcal{G} \rightarrow \mathcal{G}'$.

Proposition 2.4.12. *Assume Z is an R -smooth torus. Then the Zariski closure \tilde{Z} of Z inside \mathcal{G} is smooth and there is an (fppf) exact sequence*

$$(2.4.12.1) \quad 0 \longrightarrow \tilde{Z} \longrightarrow \mathcal{G} \xrightarrow{\beta} \mathcal{G}' \longrightarrow 0$$

of group schemes over \mathcal{O}_F .

Proof. As in Proposition 2.4.9, it suffices to prove the Proposition when $F = \check{F}$. Let S be a maximal \check{F} -split torus of G such that x lies in $\mathcal{A}(G, S, \check{F})$. Let T be the centralizer of S and we let T' be the corresponding maximal torus of G' .

Assume there exists an fppf exact sequence

$$(2.4.12.2) \quad 1 \longrightarrow \tilde{\mathcal{Z}} \longrightarrow \mathcal{T}_0 \longrightarrow \mathcal{T}'_0 \longrightarrow 1$$

where \mathcal{T}_0 and \mathcal{T}'_0 are the connected Néron models of T and T' respectively. Then we may argue as in [KP18, Proposition 1.1.4] to obtain the desired exact sequence (2.4.12.1).

It remains to exhibit the exact sequence (2.4.12.2); we follow the argument of [PR08, Lemma 6.7].

By assumption we obtain a closed immersion between lft Néron models $\mathcal{Z} \rightarrow \mathcal{T}$. We let $\tilde{\mathcal{Z}}'$ denote the subgroup scheme of \mathcal{Z} with generic fiber Z , and special fiber corresponding to the connected components of the special fiber of \mathcal{Z} parameterized by $\ker(X_*(Z)_I \rightarrow X_*(T)_I)$. Then $\tilde{\mathcal{Z}}'$ is smooth and we have a closed immersion $\tilde{\mathcal{Z}}' \rightarrow \mathcal{T}_0$. It follows that $\tilde{\mathcal{Z}}'$ coincides with $\tilde{\mathcal{Z}}$ and we obtain a closed immersion $\tilde{\mathcal{Z}} \rightarrow \mathcal{T}_0$. As in [PR08, Lemma 6.7] we have an exact sequence:

$$1 \longrightarrow \tilde{\mathcal{Z}}(\mathcal{O}_{\tilde{F}}) \longrightarrow \mathcal{T}_0(\mathcal{O}_{\tilde{F}}) \longrightarrow \mathcal{T}'_0(\mathcal{O}_{\tilde{F}}) \longrightarrow 1$$

The quotient $\mathcal{T}_0/\tilde{\mathcal{Z}}$ is a smooth affine commutative group scheme with the same generic fiber as \mathcal{T}'_0 and with the same $\mathcal{O}_{\tilde{F}}$ -points; hence by [BT84, Proposition 1.7.6] we have $\mathcal{T}'_0 \cong \mathcal{T}_0/\tilde{\mathcal{Z}}$. The result follows. \square

3. LOCAL MODELS OF SHIMURA VARIETIES

In this section we assume F is a finite extension of \mathbb{Q}_p with residue field k_F . We recall the construction of local models for Weil-restricted groups following [Lev16] and prove some properties of these models which will be needed in §5.

3.1. Local models for Weil-restricted groups.

3.1.1. Let K_0/F be a finite unramified extension. Let $P(u) \in \mathcal{O}_{K_0}[u]$ be a monic polynomial and $\underline{\mathcal{G}}$ a smooth affine group scheme over $\mathcal{O}_{K_0}[u]$. We consider the functor $\mathrm{Fl}_{\underline{\mathcal{G}},0}^{P(u)}$ on \mathcal{O}_{K_0} -algebras R given by

$$\mathrm{Fl}_{\underline{\mathcal{G}},0}^{P(u)}(R) = \{\text{iso. classes of pairs } (\mathcal{E}, \beta)\},$$

where \mathcal{E} is a $\underline{\mathcal{G}}$ -torsor over $R[u]$ and $\beta : \mathcal{E}|_{R[u][1/P(u)]} \xrightarrow{\sim} \mathcal{E}^0$ is an isomorphism of $\underline{\mathcal{G}}$ -torsors, where \mathcal{E}^0 denotes the trivial $\underline{\mathcal{G}}$ -torsor. We then define the mixed characteristic affine Grassmannian

$$\mathrm{Fl}_{\underline{\mathcal{G}}}^{P(u)} := \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F} \mathrm{Fl}_{\underline{\mathcal{G}},0}^{P(u)}.$$

By embedding $\underline{\mathcal{G}}$ into a general linear group, one deduces as in [Lev16, Proposition 4.1.4], that $\mathrm{Fl}_{\underline{\mathcal{G}}}^{P(u)}$ is representable by an ind-scheme over \mathcal{O}_F .

3.1.2. Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple over F in the sense of [HPR20, §2.1]. Thus

- G is a reductive group scheme over F .
 - $\{\mu\}$ is a geometric conjugacy class of minuscule cocharacters of G .
 - $\mathcal{G} = \mathcal{G}_x$ for some $x \in \mathcal{B}(G, F)$ which is generic in the facet containing it.
- In addition, we will often make the following assumption.

(*) G is isomorphic to $\prod_{i=1}^r \text{Res}_{K_i/F} H_i$ where K_i/F is a finite extension and H_i is a reductive group over K_i which splits over a tamely ramified extension of K_i . When $r = 1$, we simply write $G = \text{Res}_{K/F} H$.

If $p \geq 5$, then any adjoint group satisfies (*). Using this fact, one can define local models for any group G when $p \geq 5$ (see [Lev16, Remark 4.2.3]) although we will not need the construction in this level of generality.

3.1.3. Let $(G, \{\mu\}, \mathcal{G})$ be a triple with $G \cong \text{Res}_{K/F} H$ as above. By [Pra01, p 172], there is an identification of buildings $\mathcal{B}(G, F) \cong \mathcal{B}(H, K)$. Therefore we may identify the set of parahoric subgroups of $G(F)$ with the set of parahoric subgroups of $H(K)$; see [HR20, §4.2] for example. Thus, there is a parahoric group scheme \mathcal{H} over \mathcal{O}_K such that $\mathcal{G}(\mathcal{O}_F)$ is identified with $\mathcal{H}(\mathcal{O}_K)$ as subgroups of $G(F) \cong H(K)$. By [HR20, Proposition 4.7], we have $\mathcal{G} \cong \text{Res}_{\mathcal{O}_K/\mathcal{O}_F} \mathcal{H}$. If we consider x as a point in $\mathcal{B}(H, K)$, then \mathcal{H} is the parahoric group scheme of H associated to x .

Let K_0 denote the maximal unramified extension of F contained in K and write \mathcal{O}_{K_0} (resp. k_0) for its ring of integers (resp. residue field). We let $\mathcal{O}_{K_0}[u^\pm]$ denote the ring $\mathcal{O}_{K_0}[u, u^{-1}]$. We fix a uniformizer ϖ of K and we write $Q(u) \in \mathcal{O}_{K_0}[u]$ for the Eisenstein polynomial which is the minimal polynomial for ϖ over K_0 . Then [Lev16, §3.4] constructs a smooth affine group scheme $\underline{\mathcal{H}}$ over $\mathcal{O}_{K_0}[u]$ which specializes to \mathcal{H} under the map $\mathcal{O}_{K_0}[u] \rightarrow \mathcal{O}_K$, $u \mapsto \varpi$ and such that

$$\underline{H} := \underline{\mathcal{H}}|_{\mathcal{O}_{K_0}[u^\pm]}$$

is a reductive group scheme. Applying the construction of §3.1.1 we obtain the ind-scheme $\text{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ over \mathcal{O}_F which is ind-projective by [Lev16, Theorem 4.2.11].

3.1.4. For a K_0 -algebra R , the completion $\widehat{R[u]}$ of $R[u]$ along $Q(u)$, contains the completion of $K_0[u]$ along $Q(u)$. The latter ring may be identified with $K[[t]]$, by a map sending t to $Q(u)$ and inducing the identity on residue fields. Then $\widehat{R[u]}$ may be identified with $(K \otimes_{K_0} R)[[t]]$ by sending t to $Q(u)$. This induces an isomorphism from the generic fiber of $\text{Fl}_{\underline{\mathcal{H}}, 0}^{Q(u)}$ with the affine Grassmannian $\text{Gr}_{\text{Res}_{K/K_0} H}$ (cf. [HR20, Corollary 3.5]), and hence an isomorphism between the generic fiber of $\text{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ with $\text{Gr}_{\text{Res}_{K/F} H} \cong \text{Gr}_G$ (recall that this is the fpqc sheaf associated to the functor on F -algebras R given by $R \mapsto G(R((t)))/G(R[[t]])$).

The special fiber of $\text{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ can be identified with the partial affine flag variety $\text{Res}_{k_0/k_F} \mathcal{FL}_{\underline{\mathcal{H}}_{k_0[[t]]}}$; here $\mathcal{FL}_{\underline{\mathcal{H}}_{k_0[[t]]}}$ is the fpqc sheaf associated to the functor

$$R \mapsto \underline{\mathcal{H}}_{k_0[[t]]}(R((t)))/\underline{\mathcal{H}}_{k_0[[t]]}(R[[t]])$$

on k_0 -algebras. A representative μ of $\{\mu\}$ over \overline{F} determines an element of $G(\overline{F}((t)))$ and hence a point $e_\mu := \mu(t) \in \text{Gr}_G(\overline{F})$. The Schubert variety S_μ is then defined to be the closure of the $G(\overline{F}[[t]])$ -orbit of e_μ in Gr_G . The conjugacy class $\{\mu\}$ has a minimal field of definition E known as the (local) reflex field, and the Schubert variety $S_\mu \subset \text{Gr}_G$ is defined over E . The local model $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}$ is defined to be the Zariski closure of S_μ in $\text{Fl}_{\underline{\mathcal{H}}}^{Q(u)} \otimes_{\mathcal{O}_F} \mathcal{O}_E$.

3.1.5. In general, if $G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ as in (*), we define $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}$ to be the product $M_{\mathcal{G}, \{\mu\}}^{\text{loc}} := \prod_{i=1}^r M_{\mathcal{G}_i, \{\mu_i\}}^{\text{loc}} \otimes_{\mathcal{O}_{E_i}} \mathcal{O}_E$. Here the parahoric \mathcal{G}_i of $\text{Res}_{K_i/F} H_i$ is determined by $\mathcal{G} \cong \prod_{i=1}^r \mathcal{G}_i$, $\{\mu_i\}$ is the $\text{Res}_{K_i/F} H_i$ factor of the G -conjugacy class

$\{\mu\}$, and E_i (resp. E) is the field of definition of $\{\mu_i\}$ (resp. $\{\mu\}$). The following theorem follows immediately from [Lev16, Theorem 4.2.7].

Theorem 3.1.6. *Suppose G satisfies $(*)$ and that p does not divide the order of the algebraic fundamental group $\pi_1(G_{\text{der}})$ of the derived group G_{der} of G . Then the scheme $M_{\mathcal{G},\{\mu\}}^{\text{loc}}$ is normal with reduced special fiber. Moreover each geometric irreducible component of $M_{\mathcal{G},\{\mu\}}^{\text{loc}} \otimes_{\mathcal{O}_E} k$ is normal and Cohen–Macaulay.*

□

Remark 3.1.7. (1) Note that the input for the constructions in this subsection is a parahoric group scheme \mathcal{H} over \mathcal{O}_K and a finite extension K/F . When $K = F$, the group scheme $\underline{\mathcal{H}}$ and the mixed characteristic affine Grassmannian $\text{Fl}_{\underline{\mathcal{H}}}^{u-\varpi}$ agrees with those constructed by Pappas–Zhu [PZ13].

(2) Using the argument in [Lev16, Proposition 4.2.4, Remark 4.2.5], one can show that the local model $M_{\mathcal{G},\{\mu\}}^{\text{loc}}$ depends only on \mathcal{G} and $\{\mu\}$ and not on the choice of extension K or the uniformizer ϖ .

3.1.8. We may identify the geometric special fiber of $M_{\mathcal{G},\{\mu\}}^{\text{loc}}$ with a certain union of Schubert varieties corresponding to the μ -admissible set $\text{Adm}(\{\mu\})_J$ defined in 2.2.1; we explain this in the remainder of §3.1. To do this, we first explain the relationship between the Iwahori Weyl group of G and a certain reductive group over $k_F[[u]]$.

Let S denote a maximal \check{K} -split torus of H defined over K such that x lies in a σ_K -invariant facet in the apartment $\mathcal{A}(H, S, \check{K})$ corresponding to S (here σ_K denotes the Frobenius element of $\text{Aut}(\check{K}/K)$). Then the construction in [Lev16, Proposition 3.1.2] provides us with a maximal $\mathcal{O}_{\check{K}_0}[u^\pm]$ -split torus \underline{S} of \underline{H} defined over $\mathcal{O}_{K_0}[u^\pm]$ which extends S . The choice of \underline{S} gives us an identification of apartments

$$(3.1.8.1) \quad \mathcal{A}(\underline{H}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u))) \cong \mathcal{A}(H, S, \check{K})$$

for $\kappa = \check{K}_0, k$. Moreover there is an identification of Iwahori Weyl groups

$$(3.1.8.2) \quad W_{\underline{H}_{\kappa((u))}} \cong W_{H_{\check{K}}}$$

for $\underline{H}_{\kappa((u))}$ and $H_{\check{K}}$ such that the identification (3.1.8.1) is equivariant for the actions of these groups on the respective apartments. We let

$$x_{\kappa((u))} \in \mathcal{A}(\underline{H}_{\kappa((u))}, \underline{S}_{\kappa((u))}, \kappa((u)))$$

be the point corresponding to x under the identification (3.1.8.1). Then the group scheme $\underline{H}/\mathcal{O}_{K_0}[u]$ has the property that its specialization to $\kappa[[u]]$ is isomorphic to the parahoric group scheme corresponding to $x_{\kappa((u))}$.

3.1.9. Let $\underline{\mathcal{G}}_{k_F[[u]]}$ denote the group scheme $\underline{\mathcal{G}}_{k_F[[u]]} := \text{Res}_{k_0[[u]]/k_F[[u]]} \underline{\mathcal{H}}_{k_0[[u]]}$ and we write $\underline{\mathcal{G}}_{k_F((u))}$ for its generic fiber. We let $\underline{\mathcal{G}}_{k[[u]]}$ (resp. $\underline{\mathcal{G}}_{k((u))}$) denote the base change of $\underline{\mathcal{G}}_{k_F[[u]]}$ (resp. $\underline{\mathcal{G}}_{k_F((u))}$) to $k[[u]]$ (resp. $k((u))$). Then by construction, the special fiber of $\text{Fl}_{\underline{\mathcal{H}}}^{Q(u)}$ is identified with the usual partial affine flag variety associated to $\underline{\mathcal{G}}_{k_F[[u]]}$; here we use [HR20, Corollary 3.6 and Lemma 3.7] for the identification $\text{Res}_{k_0/k_F} \mathcal{FL}_{\underline{\mathcal{H}}_{k_0[[u]]}} \cong \mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}}$. The isomorphism (3.1.8.2) induces an isomorphism of Iwahori Weyl groups

$$(3.1.9.1) \quad W_G \cong W_{\underline{\mathcal{G}}_{k_F((u))}}.$$

Indeed we have identifications

$$W_G \cong \prod_{\psi: K_0 \rightarrow \check{F}} W_H, \quad W_{\underline{G}_{k_F((u))}} \cong \prod_{\psi: k_0 \rightarrow k} W_{\underline{H}_{k_F((u))}},$$

where ψ runs over F (resp. k_F)-embeddings. Identifying $k_0 \rightarrow k$ with the unique lift $K_0 \rightarrow \check{F}$ and using (3.1.8.2), we obtain the identification (3.1.9.1).

Similarly, we obtain an identification of apartments

$$(3.1.9.2) \quad \mathcal{A}(\underline{G}_{k_F((u))}, \underline{S}'_{k((u))}, k((u))) \cong \mathcal{A}(G, S', \check{F}).$$

Here S' is the maximal \check{F} -split torus of G determined by the maximal \check{K} -split torus of H as in [HR20, §4.2], and $\underline{S}'_{k((u))}$ is the maximal $k((u))$ -split torus of $\underline{G}_{k_F((u))}$ obtained from the maximal $\mathcal{O}_{\check{K}_0}[u^\pm]$ -split torus \underline{S} of \underline{H} . Moreover the identification (3.1.9.2) is compatible with the action of Iwahori Weyl groups under the identification (3.1.9.1).

3.1.10. We fix a σ -invariant alcove $\mathfrak{a} \subset \mathcal{A}(G, S', \check{F})$ whose closure contains x . This determines a set of simple reflections \mathbb{S} for W_G and the parahoric \mathcal{G} is a standard parahoric for this choice of alcove; hence it corresponds to a σ -stable subset $J \subset \mathbb{S}$. We let $\underline{\mathfrak{a}}$ denote the alcove in $\mathcal{A}(\underline{G}_{k_F((u))}, \underline{S}'_{k((u))}, k((u)))$ corresponding to \mathfrak{a} and $\underline{\mathbb{S}}$ the set of simple reflections in the walls of $\underline{\mathfrak{a}}$. There is an identification $\mathbb{S} \cong \underline{\mathbb{S}}$ and we let $\underline{J} \subset \underline{\mathbb{S}}$ be the subset corresponding to $J \subset \mathbb{S}$; then $\underline{\mathcal{G}}_{k_F[[u]]}$ is the standard parahoric group scheme for $\underline{G}_{k_F((u))}$ associated to \underline{J} . Writing W_J (resp. $W_{\underline{J}}$) for the finite group generated by the reflections in J (resp. \underline{J}), we obtain an identification $W_J \cong W_{\underline{J}}$, and an identification

$$(3.1.10.1) \quad W_J \backslash W_G / W_J \cong W_{\underline{J}} \backslash W_{\underline{G}_{k_F((u))}} / W_{\underline{J}}.$$

In particular we may consider $\text{Adm}(\{\mu\})_J$ as a subset of $W_{\underline{J}} \backslash W_{\underline{G}_{k_F((u))}} / W_{\underline{J}}$.

For an element $w \in W_G$, we write $\underline{w} \in W_{\underline{G}_{k_F((u))}}$ for the corresponding element and $\dot{w} \in \underline{G}_{k_F((u))}(k((u)))$ a lift of w . We let \underline{S}_w denote the closure of the $\underline{\mathcal{G}}_{k_F[[u]]}(k[[u]])$ -orbit of \dot{w} considered as a point of the partial affine flag variety $\mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}} \otimes_{k_F} k$ for $\underline{\mathcal{G}}_{k_F[[u]]}$.

3.1.11. If $G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$, we may define $\underline{\mathcal{G}}_{k_F[[u]]} := \prod_{i=1}^r \underline{\mathcal{G}}_{i, k_F[[u]]}$, where the $\underline{\mathcal{G}}_{i, k_F[[u]]}$ are the $k_F[[u]]$ -group schemes constructed in the previous paragraphs using the groups $\text{Res}_{K_i/F} H_i$. We let $\underline{\mathcal{G}}_{i, k_F((u))}$ denote the generic fiber of $\underline{\mathcal{G}}_{i, k_F[[u]]}$ and we define $\underline{G}_{k_F((u))} := \prod_{i=1}^r \underline{G}_{i, k_F((u))}$. Since the construction of Iwahori Weyl groups and apartments are compatible with products, the above discussion extends to this case. In particular, we have an identification of double cosets for the Iwahori Weyl group (3.1.10.1), and for $w \in W_J \backslash W_G / W_J$ we have the associated Schubert variety \underline{S}_w in $\mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}} \otimes_{k_F} k$. Applying [Lev16, Proposition 4.3.2] to each of the factors $\text{Res}_{K_i/F} H_i$, we obtain the following theorem.

Theorem 3.1.12. *Let $G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ and assume that $p \nmid |\pi_1(G_{\text{der}})|$. We have an identification*

$$M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \otimes_{\mathcal{O}_E} k \cong \bigcup_{w \in \text{Adm}(\{\mu\})_J} \underline{S}_w$$

as closed subschemes of $\mathcal{FL}_{\underline{\mathcal{G}}_{k_F[[u]]}} \otimes_{k_F} k$.

□

3.2. Embedding local models.

3.2.1. We now show that a suitable embedding of local model triples induces a closed immersion of local models (Proposition 3.2.6). This generalizes [KP18, Proposition 2.3.6] and is proved using a similar method. This result and its consequences in §3.4 are the main technical results needed to establish the local model diagram in §5.1.

We begin by recalling the construction of certain lattice chains of $\mathcal{O}_{K_0}[u]$ -modules from [PZ13, §5.2.1]. Let $\underline{W} = \mathcal{O}_{K_0}[u]^n$ and $W = \underline{W} \otimes_{\mathcal{O}_{K_0}[u], u \mapsto 0} \mathcal{O}_{K_0} \cong \mathcal{O}_{K_0}^n$. Write $W = \bigoplus_{i=0}^r V_i$ for some r and direct summands V_i of W , and let $U_i = \bigoplus_{j \geq i} V_j$ which forms a flag of subspaces of W ; we write $P \subset \mathrm{GL}(W)$ for the corresponding parabolic. For $i = 0, \dots, r-1$ we let $\underline{W}_i \subset \underline{W}$ denote the inverse image of U_i under $\underline{W} \rightarrow W$; the sequence W_i satisfies

$$uW \subset \underline{W}_{r-1} \subset \dots \subset \underline{W}_0 = \underline{W}.$$

We extend the sequence to \mathbb{Z} by letting $\underline{W}_{i+kr} = u^k \underline{W}_i$ and we write \underline{W}_\bullet for the resulting chain indexed by \mathbb{Z} . As in [PZ13, §5.2.1], the dilatation $\mathrm{GL}(\underline{W}_\bullet)$ of $\mathrm{GL}(\underline{W})$ along P can be identified with the closed subscheme of $\prod_{i=0}^{r-1} \mathrm{GL}(\underline{W}_i)$ which respect the maps $\underline{W}_i \rightarrow \underline{W}_{i+1}$. Let \mathcal{GL} be the parahoric group scheme over \mathcal{O}_K of $\mathrm{GL}_n(K)$ corresponding to the stabilizer of the lattice chain $\underline{W}_i \otimes_{\mathcal{O}_{K_0}[u], u \mapsto \varpi} \mathcal{O}_K$ in K^n . Then $\mathrm{GL}(\underline{W}_\bullet)$ is isomorphic to the $\mathcal{O}_{K_0}[u]$ -group scheme $\underline{\mathcal{GL}}$ associated to \mathcal{GL} and the extension K/F in §3.1.3. Since every parahoric of $\mathrm{GL}_n(K)$ arises in this way, this gives an explicit description of the associated $\mathcal{O}_{K_0}[u]$ -group scheme $\underline{\mathcal{GL}}$ attached to any parahoric of $\mathrm{GL}_n(K)$.

3.2.2. Let $(G, \{\mu\}, \mathcal{G})$ be a local model triple as in §3.1.2 with $G \cong \mathrm{Res}_{K/F} H$. Let $\rho : G \rightarrow \mathrm{GL}(V)$ be a faithful minuscule representation, where V is a finite dimensional vector space over F , such that $\rho \circ \mu$ is conjugate to a standard (i.e. having weights $0, -1$) minuscule coweight and such that G contains the scalars. We will show that we may replace ρ by a different faithful minuscule representation $\rho' : G \rightarrow \mathrm{GL}(W)$ such that ρ' induces a closed immersion of local models

$$\mathrm{M}_{\mathcal{G}, \{\mu\}}^{\mathrm{loc}} \hookrightarrow \mathrm{M}_{\mathcal{GL}_W, \{\rho' \circ \mu\}}^{\mathrm{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

where \mathcal{GL}_W is a certain parahoric group scheme of $\mathrm{GL}(W)$.

Base changing ρ to K , we obtain a map $H \rightarrow \mathrm{GL}(V_K)$ given by composing

$$\rho_K : G_K \rightarrow \mathrm{GL}(V_K)$$

with the diagonal map $H \rightarrow G_K$. Let W denote the underlying F -vector space corresponding to V_K . We consider the composition

$$\rho' : G = \mathrm{Res}_{K/F} H \xrightarrow{\rho_1} \mathrm{Res}_{K/F} \mathrm{GL}(V_K) \xrightarrow{\rho_2} \mathrm{GL}(W)$$

where ρ_1 is obtained by applying restriction of scalars to the map $H \rightarrow \mathrm{GL}(V_K)$, and ρ_2 is induced by the restriction of structure functor from K -vector spaces to F -vector spaces.

3.2.3. Since H splits over a tame extension of K and $H \rightarrow \mathrm{GL}(V_K)$ is a minuscule representation, it follows from [KP18, §1.2] that there exists a $H(K)$ -equivariant toral embedding of buildings

$$(3.2.3.1) \quad \mathcal{B}(H, K) \rightarrow \mathcal{B}(\mathrm{GL}(V_K), K).$$

There are canonical identifications of $\mathcal{B}(G, F)$ (resp. $\mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V_K), F)$) with $\mathcal{B}(H, K)$ (resp. $\mathcal{B}(\mathrm{GL}(V_K), K)$); we thus obtain a $G(F)$ -equivariant toral embedding of buildings

$$(3.2.3.2) \quad \mathcal{B}(G, F) \rightarrow \mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V_K), F).$$

Similarly, restriction of structure induces a $\mathrm{GL}(V_K)$ -equivariant map of buildings

$$\mathcal{B}(\mathrm{GL}(V_K), K) \cong \mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V_K), F) \rightarrow \mathcal{B}(\mathrm{GL}(W), F).$$

Let y (resp. z) denote the image of x in $\mathcal{B}(\mathrm{Res}_{K/F}\mathrm{GL}(V_K), F)$ (resp. $\mathcal{B}(\mathrm{GL}(W), F)$). We write $\mathcal{GL}_{K/F}$ (resp. \mathcal{GL}_W) for the parahoric group schemes over \mathcal{O}_F for $\mathrm{Res}_{K/F}\mathrm{GL}(V_K)$ (resp. $\mathrm{GL}(W)$) corresponding to y (resp. z), and we write $\{\mu_{K/F}\}$ and $\{\mu_W\}$ for the respective conjugacy class of cocharacters of $\mathrm{Res}_{K/F}\mathrm{GL}(V_K)$ and $\mathrm{GL}(W)$ induced by $\{\mu\}$. If we write \mathcal{GL} for the parahoric \mathcal{O}_K -group scheme of $\mathrm{GL}(V_K)$ associated to y ; then $\mathcal{GL}_{K/F} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}\mathcal{GL}$.

The natural map of group schemes $\tilde{\mathcal{G}} \rightarrow \mathcal{GL}_{K/F}$ is a closed immersion since this map is obtained by Weil restriction of a closed immersion between \mathcal{O}_K -group schemes as in [KP18, Proposition 1.3.3]. We will need the following lemma.

Lemma 3.2.4. *Let K be a non-archimedean local field (in possibly equal characteristic) and K'/K a finite (not necessarily separable) extension. Let V be a vector space over K' and let W denote V considered as a vector space over K . Let \mathcal{GL} be a parahoric group scheme of $\mathrm{GL}(V)$ corresponding to the stabilizer of an $\mathcal{O}_{K'}$ -lattice chain $\{\Lambda_i\}_{i=1,\dots,r}$ in V . We write $\{\Lambda_{W,i}\}_{i=1,\dots,r}$ for the associated \mathcal{O}_K -lattice chain of W and we let \mathcal{GL}_W denote the parahoric group scheme of $\mathrm{GL}(W)$ stabilizing $\{\Lambda_{W,i}\}_{i=1,\dots,r}$. Then the natural closed immersion $\mathrm{Res}_{K'/K}\mathrm{GL}(V) \rightarrow \mathrm{GL}(W)$ extends to a closed immersion of \mathcal{O}_K -group schemes*

$$\mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_K}\mathcal{GL} \hookrightarrow \mathcal{GL}_W.$$

Proof. The group scheme \mathcal{GL} is the schematic closure of $\mathrm{GL}(V) \rightarrow \prod_{i=1}^r \mathrm{GL}(V)$ (under the diagonal embedding) in $\prod_{i=1}^r \mathrm{GL}(\Lambda_i)$. Similarly \mathcal{GL}_W is the schematic closure of $\mathrm{GL}(W) \rightarrow \prod_{i=1}^r \mathrm{GL}(W)$ in $\prod_{i=1}^r \mathrm{GL}(\Lambda_{W,i})$. Thus we have a commutative diagram of \mathcal{O}_K -schemes

$$\begin{array}{ccc} \mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_K}\mathcal{GL} & \longrightarrow & \mathcal{GL}_W \\ \downarrow & & \downarrow \\ \prod_{i=1}^r \mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_K}\mathrm{GL}(\Lambda_i) & \longrightarrow & \prod_{i=1}^r \mathrm{GL}(\Lambda_{W,i}) \end{array}$$

where the vertical arrows are closed immersions. It therefore suffices to show the bottom arrow is a closed immersion, and hence we reduce to proving the lemma when $r = 1$, i.e. when \mathcal{GL} is the stabilizer $\mathrm{GL}(\Lambda)$ of a single \mathcal{O}_K -lattice $\Lambda \subset V$. This case can be proved, for example, by explicitly writing down the equations for the morphism. \square

By Lemma 3.2.4, the map $\mathcal{GL}_{K/F} \rightarrow \mathcal{GL}_W$ is a closed immersion. Composing with $\tilde{\mathcal{G}} \rightarrow \mathcal{GL}_{K/F}$ we obtain a closed immersion of \mathcal{O}_F -group schemes $\tilde{\mathcal{G}} \rightarrow \mathcal{GL}_W$ extending ρ' .

3.2.5. By our assumption on $\rho \circ \mu$, μ_W is conjugate to a standard minuscule coweight

$$a \mapsto \text{diag}(1^{(n-d)}, (a^{-1})^{(d)})$$

of $\text{GL}(W)$, where $n = \dim_F W$. The generic fiber of $M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}}$ is the Grassmannian $\text{Gr}(d, n)$ of d -dimensional subspaces of W . We let X_μ denote the generic fiber of $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}$; it can be identified with the E -variety G/P_μ , where P_μ is the parabolic subgroup of G corresponding to μ . Then the representation $\rho' : G \rightarrow \text{GL}(W)$ induces a closed immersion

$$(3.2.5.1) \quad X_\mu \rightarrow \text{Gr}(d, n) \otimes_{\mathcal{O}_F} E.$$

Proposition 3.2.6. *The map (3.2.5.1) extends to a closed immersion of local models*

$$(3.2.6.1) \quad \rho^{\text{loc}} : M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \rightarrow M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E.$$

Proof. Recall ρ' factors as $\rho_2 \circ \rho_1$; it suffices to show there are closed immersions

$$M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \hookrightarrow M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\text{loc}} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E \hookrightarrow M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

where the first map is induced by ρ_1 and the second map is induced by ρ_2 . Here, $M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\text{loc}}$ is the local model attached to the \mathcal{O}_K -group scheme \mathcal{GL} and the extension K/F as in §3.1.3, and E' is the local reflex field for the $\text{Res}_{K/F} \text{GL}(V_K)$ -conjugacy class of cocharacters $\{\mu_{K/F}\}$.

Step (1): $M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \hookrightarrow M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\text{loc}} \otimes_{\mathcal{O}_{E'}} \mathcal{O}_E$.

As in [KP18, Proposition 2.3.7], it follows from descent that it suffices to show that such a closed immersion exists upon base change to \check{E} . Thus we need to show that there exists a closed immersion

$$M_{\mathcal{G}_{\mathcal{O}_{\check{F}}}, \{\mu\}}^{\text{loc}} \hookrightarrow M_{\mathcal{GL}_{K/F, \mathcal{O}_{\check{F}}}, \{\mu_{K/F}\}}^{\text{loc}} \otimes_{\mathcal{O}_{\check{E}'}} \mathcal{O}_{\check{E}}$$

where $\mathcal{G}_{\mathcal{O}_{\check{F}}}$ (resp. $\mathcal{GL}_{K/F, \mathcal{O}_{\check{F}}}$) denotes the corresponding parahoric group schemes for $G_{\check{F}}$ (resp. $\text{Res}_{K/F} \text{GL}(V_K) \otimes_F \check{F}$) and these are the analogues of the local models defined over \check{F} .

We have isomorphisms

$$G_{\check{F}} \cong \prod_{\tau: K_0 \rightarrow \check{F}} \text{Res}_{\check{K}/\check{K}_0} H_{\check{K}}, \quad \text{Res}_{K/F} \text{GL}(V_K) \otimes_F \check{F} \cong \prod_{\tau: K_0 \rightarrow \check{F}} \text{Res}_{\check{K}/\check{K}_0} \text{GL}(V_{\check{K}})$$

and the embedding $\rho_{1, \check{F}}$ is given by the product embedding; it suffices to consider each factor separately. Thus upon relabeling we may assume $G_{\check{F}} \cong \text{Res}_{\check{K}/\check{K}_0} H_{\check{K}}$ and that ρ_1 is induced by restriction of scalars from an embedding

$$\phi : H_{\check{K}} \rightarrow \text{GL}(V_{\check{K}}).$$

For notational simplicity, we write $\underline{\mathcal{H}}$ for the $\mathcal{O}_{\check{K}_0}[u]$ -group scheme associated to $\mathcal{H}_{\check{K}}$.

The same proof as [PZ13, Proposition 8.1] shows that it suffices to show that there exists a lattice chain \underline{V}_\bullet in $\mathcal{O}_{\check{K}_0}[u]^n$ such that ϕ extends to a homomorphism of $\mathcal{O}_{\check{K}_0}[u]$ -group schemes

$$\phi_{\mathcal{O}_{\check{K}_0}[u]} : \underline{\mathcal{H}} \rightarrow \text{GL}(\underline{V}_\bullet)$$

satisfying the following two conditions

- ρ extends to a group scheme morphism $\underline{H} \rightarrow \mathrm{GL}(V_\bullet)$ over $\mathcal{O}_{\check{K}_0}[u]$.
- The homomorphism

$$\underline{H}_{k[[u]]} := \underline{H} \otimes_{\mathcal{O}_{\check{K}_0}[u]} k[[u]] \rightarrow \mathrm{GL}(V_\bullet \otimes_{\mathcal{O}_{\check{K}_0}[u]} k[[u]])$$

is a locally closed immersion, and the Zariski closure of $\underline{H}_{k((u))} := \underline{H} \otimes_{\mathcal{O}_{\check{K}_0}[u]} k((u))$ in $\mathrm{GL}(V_\bullet \otimes_{\mathcal{O}_{\check{K}_0}[u]} k[[u]])$ is a smooth group scheme \mathcal{P}' whose connected component may be identified with $\underline{H}_{k[[u]]}$.

Indeed, under these assumptions, the proof in [PZ13, Proposition 8.1] shows that extending torsors along $\phi_{\mathcal{O}_{\check{K}_0}[u]}$ gives a morphism $\mathrm{Fl}_{\underline{H}}^{Q(u)} \rightarrow \mathrm{Fl}_{\mathrm{GL}(V_\bullet)}^{Q(u)}$ which restricts to a closed immersion $M_{\mathcal{O}_{\check{F}}, \{\mu\}}^{\mathrm{loc}} \hookrightarrow M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\mathrm{loc}} \otimes_{\mathcal{O}_{\check{E}'}} \mathcal{O}_{\check{E}'}$.

The construction of the map $\phi_{\mathcal{O}_{\check{K}_0}[u]}$ follows, with some minor modifications, from the same argument as [KP18, Proposition 2.3.7]; as in the construction of the group scheme \underline{H} in [Lev16], the key point is to realize the tame descent over $\mathcal{O}_{\check{K}_0}[u^\pm]$ as opposed to $\mathcal{O}_{\check{K}}[u^\pm]$ in [KP18]. We briefly sketch their argument, pointing out what modifications are needed in our situation.

Let \tilde{K}/\check{K} be a splitting field for $H_{\check{K}}$ which we may assume is finite, tamely ramified and Galois. We let $\tilde{e} := [\tilde{K} : \check{K}]$ and fix a uniformizer $\tilde{\varpi}$ of \tilde{K} . The action of $\mathrm{Gal}(\tilde{K}/\check{K})$ extends to an action on $\mathcal{O}_{\check{K}_0}[w^\pm]/\mathcal{O}_{\check{K}_0}[u^\pm]$, where $w^{\tilde{e}} = u$. Using the argument in [KP18, Proposition 2.3.7, Step 1], we obtain a representation

$$\phi_{\mathcal{O}_{\check{K}_0}[u^\pm]} : \underline{H}_{\mathcal{O}_{\check{K}_0}[u^\pm]} \rightarrow \mathrm{GL}_n(\mathcal{O}_{\check{K}_0}[u^\pm])$$

which extends ϕ under the map $u \mapsto \varpi$; this is constructed by descending along the cover $\mathcal{O}_{\check{K}_0}[w^\pm]/\mathcal{O}_{\check{K}_0}[u^\pm]$. (In *loc. cit.*, they apply the argument to the cover $\mathcal{O}_{\check{K}}[w^\pm]/\mathcal{O}_{\check{K}}[u^\pm]$ to obtain a representation over $\mathcal{O}_{\check{K}}[u^\pm]$). Here, the specialization of $\mathrm{GL}_n(\mathcal{O}_{\check{K}_0}[u^\pm])$ along $u \mapsto \varpi$ is identified with $\mathrm{GL}(V_{\check{K}})$ via a suitable choice of basis for $V_{\check{K}}$.

The construction of V_\bullet then proceeds in the same way as [KP18, Proposition 2.3.7, Step 1]. We write T for the diagonal torus of GL_n ; then the basis of $V_{\check{K}}$ is chosen so that $y \in \mathcal{A}(\mathrm{GL}_n, T, \check{K})$. Using the identification of apartments

$$(3.2.6.2) \quad \mathcal{A}(\mathrm{GL}_n, T, \check{K}) \cong \mathcal{A}(\underline{GL}_{n, \check{K}_0((u))}, \underline{T}_{\check{K}_0((u))}, \check{K}_0((u))).$$

we obtain a lattice chain \underline{N}_\bullet of $\check{K}_0[[u]]$ -modules in $\check{K}_0((u))^n$ corresponding to the image of y in $\mathcal{A}(\underline{GL}_{n, \check{K}_0((u))}, \underline{T}_{\check{K}_0((u))}, \check{K}_0((u)))$. Then if we define $\underline{V}_\bullet := \underline{N}_\bullet \cap \mathcal{O}_{\check{K}_0}[u^\pm]^n$, $\phi_{\mathcal{O}_{\check{K}_0}[u^\pm]}$ extends to a map $\phi_{\mathcal{O}_{\check{K}_0}[u]} : \underline{H} \rightarrow \mathrm{GL}(V_\bullet)$ satisfying the required conditions.

Step (2): $M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\mathrm{loc}} \hookrightarrow M_{\mathcal{GL}_W, \{\mu_W\}}^{\mathrm{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_{E'}$.

Since $\mathrm{GL}(W)$ is a split F -group, the local model $M_{\mathcal{GL}_W, \{\mu_W\}}^{\mathrm{loc}}$ is naturally a subscheme of $\mathrm{Fl}_{\mathcal{GL}_W}^{v-\varpi_F}$. Here, $\underline{\mathcal{GL}}_W$ is an $\mathcal{O}_F[v]$ -group scheme and $\mathrm{Fl}_{\underline{\mathcal{GL}}_W}^{v-\varpi_F}$ is defined by applying §3.1.3 with $K = F$. We first show there exists a map $\mathrm{Fl}_{\underline{\mathcal{GL}}}^{Q(u)} \rightarrow \mathrm{Fl}_{\underline{\mathcal{GL}}_W}^{v-\varpi_F}$; here $\underline{\mathcal{GL}}$ is the $\mathcal{O}_{K_0}[u]$ -group scheme associated to the \mathcal{O}_K -group scheme \mathcal{GL} and the extension K/F as in §3.1.3.

Let W_0 denote the underlying K_0 -vector space of V . Denote by \mathcal{GL}_{W_0} the parahoric group scheme over \mathcal{O}_{K_0} corresponding to the image of y under the map

of buildings

$$\mathcal{B}(\mathrm{GL}(V_K), K) = \mathcal{B}(\mathrm{Res}_{K/K_0} \mathrm{GL}(V_K), K_0) \rightarrow \mathcal{B}(\mathrm{GL}(W_0), K_0)$$

We first define a map $\mathrm{Fl}_{\underline{\mathcal{GL}}, 0}^{Q(u)} \rightarrow \mathrm{Fl}_{\underline{\mathcal{GL}}_{W_0}, 0}^{v-\varpi_F}$. (This amounts to constructing the map above in the special case when $F = K_0$).

Define the map

$$r : \mathcal{O}_{K_0}[v] \rightarrow \mathcal{O}_{K_0}[u], \quad v \mapsto Q(u) + \varpi_F,$$

which lifts the inclusion $\mathcal{O}_{K_0} \rightarrow \mathcal{O}_K$, via $v \mapsto \varpi_F$, and $u \mapsto \varpi$. Let $\underline{\mathcal{GL}}_{K/K_0}$ denote the group scheme given by Weil restriction of $\underline{\mathcal{GL}}$ along r ; then the base change of $\underline{\mathcal{GL}}_{K/K_0}$ along $\mathcal{O}_{K_0}[v] \rightarrow \mathcal{O}_{K_0}$, $v \mapsto \varpi_F$ is identified with $\mathcal{GL}_{K/K_0} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_{K_0}} \mathcal{GL}$. We begin by constructing a map

$$i : \underline{\mathcal{GL}}_{K/K_0} \rightarrow \underline{\mathcal{GL}}_{W_0}$$

extending the map of \mathcal{O}_{K_0} -schemes $\mathcal{GL}_{K/K_0} \rightarrow \mathcal{GL}_{W_0}$ under the specialization $v \mapsto \varpi_F$, such that the base change to $k[[v]]$

$$i_{k[[v]]} : \underline{\mathcal{GL}}_{K/K_0, k[[v]]} \rightarrow \underline{\mathcal{GL}}_{W_0, k[[v]]}$$

is a closed immersion.

To construct i , let W_\bullet denote the lattice chain of $\mathcal{O}_{K_0}[u]$ -modules associated to \mathcal{GL} via the construction in §3.2.1; then $\underline{\mathcal{GL}}$ may be identified with the automorphism group of W_\bullet . We may view W_\bullet , via r , as a lattice chain of $\mathcal{O}_{K_0}[v]$ -modules $W_{0,\bullet}$. Then we may identify $\underline{\mathcal{GL}}_{W_0}$ with the automorphism group of $W_{0,\bullet}$. Since any $\mathcal{O}_{K_0}[u]$ -automorphism of W_\bullet gives an $\mathcal{O}_{K_0}[v]$ -automorphism of $W_{0,\bullet}$, we obtain a natural map of $\mathcal{O}_{K_0}[v]$ -group schemes $i : \underline{\mathcal{GL}}_{K/K_0} \rightarrow \underline{\mathcal{GL}}_{W_0}$ as desired. The base change $i_{k[[v]]} : \underline{\mathcal{GL}}_{K/K_0, k[[v]]} \rightarrow \underline{\mathcal{GL}}_{W_0, k[[v]]}$ is induced by restriction of structure from $k[[u]]$ -lattices to $k[[v]]$ -lattices under the map $v \mapsto u^e$, where $e = [K : K_0]$. Therefore it is a closed immersion by Lemma 3.2.4.

By [HR20, Corollary 3.6], the Weil restriction of torsors along r induces an isomorphism

$$\mathrm{Fl}_{\underline{\mathcal{GL}}, 0}^{Q(u)} \xrightarrow{\sim} \mathrm{Fl}_{\underline{\mathcal{GL}}_{K/K_0}, 0}^{v-\varpi_F}.$$

Combining this isomorphism with the map given by extending torsors along i , we obtain the required map

$$\iota_0 : \mathrm{Fl}_{\underline{\mathcal{GL}}, 0}^{Q(u)} \cong \mathrm{Fl}_{\underline{\mathcal{GL}}_{K/K_0}, 0}^{v-\varpi_F} \rightarrow \mathrm{Fl}_{\underline{\mathcal{GL}}_{W_0}, 0}^{v-\varpi_F}.$$

Now applying $\mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F}$ we obtain a map

$$\iota : \mathrm{Fl}_{\underline{\mathcal{GL}}}^{Q(u)} \rightarrow \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F} \mathrm{Fl}_{\underline{\mathcal{GL}}_{W_0}}^{v-\varpi_F}.$$

A standard argument (cf. [PR08, Theorem 1.4]) shows that $\iota \otimes_{\mathcal{O}_F} k$ is a locally closed immersion. Since the domain of this map is ind-projective it follows that $\iota \otimes_{\mathcal{O}_F} k$ is a closed immersion.

We compose ι with the map

$$\iota' : \mathrm{Res}_{\mathcal{O}_{K_0}/\mathcal{O}_F} \mathrm{Fl}_{\underline{\mathcal{GL}}_{W_0}}^{v-\varpi_F} \rightarrow \mathrm{Fl}_{\underline{\mathcal{GL}}_W}^{v-\varpi_F}$$

induced by the embedding $\text{Res}_{K_0/F} \text{GL}(W_0) \rightarrow \text{GL}(W)$. As in [PZ13, Proof of Proposition 8.1], $\iota' \otimes_{\mathcal{O}_F} k$ is a closed immersion, since $\text{Res}_{K_0/F} \text{GL}(W_0)$ is an unramified group and the embedding $\text{Res}_{K_0/F} \text{GL}(W_0) \rightarrow \text{GL}(W)$ is minuscule. It follows that the composite map $\iota' \circ \iota$ is a closed immersion on special fibers.

Restricting to the local models we obtain a map

$$(3.2.6.3) \quad M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\text{loc}} \rightarrow M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_{E'}$$

which is a closed immersion on special fibers. An argument involving Nakayama's Lemma as in [PZ13, Proposition 8.1] shows that (3.2.6.3) itself is a closed immersion.

It remains to check the statement regarding the generic fiber. This follows from the definition of local models in §3.1.4, and the fact that the map r takes $v - \varpi_F$ to $Q(u)$. \square

3.2.7. More generally if $G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ as in (*) and $\rho : G \rightarrow \text{GL}(V)$ is a faithful representation such that $\rho \circ \mu$ is a conjugate to a standard minuscule coweight and G contains the scalars, we let W_i denote the underlying F -vector space of $V \otimes_{\mathcal{O}_F} K_i$. Then as before we obtain a new faithful minuscule representation given by the composition

$$\rho' : G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i \rightarrow \prod_{i=1}^r \text{GL}(W_i) \rightarrow \text{GL}(W).$$

where the first map is induced from a product of maps $\rho'_i : \text{Res}_{K_i/F} H_i \rightarrow \text{GL}(W_i)$ and $W := \prod_{i=1}^r W_i$. We let \mathcal{GL}_{W_i} denote the parahoric for $\text{GL}(W_i)$ as constructed in §3.2.3; this determines a parahoric \mathcal{GL}_W of $\text{GL}(W)$ given by the stabilizer of the lattice chain in W formed by all possible products of the lattice chains in W_i corresponding to \mathcal{GL}_{W_i} . We let μ_{W_i} denote the i^{th} -component of the $\prod_{i=1}^r \text{GL}(W_i)$ -conjugacy class of cocharacters induced by $\{\mu\}$. By [KP18, Proposition 2.3.7], there is a closed immersion

$$(3.2.7.1) \quad \prod_{i=1}^r M_{\mathcal{GL}_{W_i}, \{\mu_{W_i}\}}^{\text{loc}} \hookrightarrow M_{\mathcal{GL}_W, \{\rho' \circ \mu\}}^{\text{loc}}$$

Applying Proposition 3.2.6 to each factor and composing with (3.2.7.1), we obtain the following.

Proposition 3.2.8. *There is a closed immersion*

$$(3.2.8.1) \quad \rho^{\text{loc}} : M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \rightarrow M_{\mathcal{GL}_W, \{\rho' \circ \mu\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E.$$

extending the natural map on generic fibers. \square

3.3. Local models and the admissible set.

3.3.1. We keep the notation of the previous subsection. We now give a more explicit description of the closed immersion

$$\rho^{\text{loc}} \otimes_{\mathcal{O}_E} k : M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \otimes_{\mathcal{O}_E} k \hookrightarrow M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \otimes_{\mathcal{O}_F} k$$

constructed in Proposition 3.2.8 on the level of k -points. This description will be needed in §4.3 in order to relate the relative position of Frobenius to the Kottwitz–Rapoport stratification on local models.

We first consider the case $G \cong \text{Res}_{K/F} H$ with K, H as above. Let $\underline{\mathcal{G}}_{k_F[[u]]}$ denote the $k_F[[u]]$ -group scheme defined in §3.1.9 and $\underline{\mathcal{G}}_{k[[u]]}$ its base change to $k[[u]]$. We may identify $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k)$ with the union

$$(3.3.1.1) \quad M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k) = \bigcup_{w \in \text{Adm}(\{\mu\})_J} S_w(k) \subset \underline{\mathcal{G}}_{k_F[[u]]}(k((u))) / \underline{\mathcal{G}}_{k_F[[u]]}(k[[u]]).$$

For notational convenience we write GL_W for the group scheme $\text{GL}(W)$. We also write $\underline{\mathcal{GL}}_W$ for the $\mathcal{O}_F[v]$ group scheme associated to \mathcal{GL}_W in [PZ13], and we let $\underline{\text{GL}}_W$ denote its base change to $\mathcal{O}_F[v^\pm]$. Then similarly to (3.3.1.1), we may identify

$$M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}}(k) \subset \underline{\text{GL}}_W(k((v))) / \underline{\mathcal{GL}}_W(k[[v]])$$

with a union of Schubert varieties for $\text{Adm}_{\text{GL}_W}(\{\mu_W\})_{J'}$. Here J' is a subset of the set of simple reflections for the Iwahori Weyl group of GL_W corresponding to the parahoric \mathcal{GL}_W . On the other hand, the discussion in [Zho20, §3.4] shows that there is an embedding

$$M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}}(k) \subset \text{GL}_W(\check{F}) / \mathcal{GL}_W(\mathcal{O}_{\check{F}}).$$

Note that the convention in *loc. cit.* is that $g \in \text{GL}_W(\check{F}) / \mathcal{GL}_W(\mathcal{O}_{\check{F}})$ corresponds to the filtration induced $\{g\varpi\Lambda_i\}_{i \in \mathbb{Z}}$, where $\{\Lambda_i\}_{i \in \mathbb{Z}}$ are the constituent lattices of the lattice chain corresponding to \mathcal{GL}_W . Thus we may consider $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k)$ as a subset of $\text{GL}_W(\check{F}) / \mathcal{GL}_W(\mathcal{O}_{\check{F}})$.

Now the embedding $\rho' : G \rightarrow \text{GL}_W$ may be extended to a morphism $\rho' : \mathcal{G} \rightarrow \mathcal{GL}_W$; hence we obtain a map

$$(3.3.1.2) \quad H(\check{K}) / \mathcal{H}(\mathcal{O}_{\check{K}}) \cong G(\check{F}) / \mathcal{G}(\mathcal{O}_{\check{F}}) \rightarrow \text{GL}_W(\check{F}) / \mathcal{GL}_W(\mathcal{O}_{\check{F}}).$$

If \mathcal{G} is a connected parahoric, i.e. $\tilde{\mathcal{G}} = \mathcal{G}$, this map is an injection. The following proposition is the analogue of [Zho20, Proposition 3.4] in our setting.

Proposition 3.3.2. *Assume \mathcal{G} is a connected parahoric. Let $g \in G(\check{F})$ with*

$$g \in \mathcal{G}(\mathcal{O}_{\check{F}}) \dot{w} \mathcal{G}(\mathcal{O}_{\check{F}})$$

for some $w \in W_J \backslash W / W_J$. Then the image of $\rho'(g)$ in $\text{GL}_W(\check{F}) / \mathcal{GL}_W(\mathcal{O}_{\check{F}})$ lies in $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k)$ if and only if $w \in \text{Adm}(\{\mu\})_J$.

Proof. By the construction of the map

$$\rho'^{\text{loc}} : M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \otimes_{\mathcal{O}_E} k \hookrightarrow M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \otimes_{\mathcal{O}_F} k$$

in Proposition 3.2.6, the map on the special fiber of local models is given by the composition

$$M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \otimes_{\mathcal{O}_E} k \xrightarrow{\rho_1^{\text{loc}}} M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\text{loc}} \otimes_{\mathcal{O}_{E'}} k \xrightarrow{\rho_2^{\text{loc}}} M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \otimes_{\mathcal{O}_F} k.$$

We let W_\bullet denote the $\mathcal{O}_{K_0}[u]$ -lattice chain constructed in the proof of Proposition 3.2.6 Step (2), and we write $\underline{W}_{\bullet, k[[u]]}$ for the lattice chain given by base change to $k[[u]]$. We let $\underline{\mathcal{GL}}_{k[[u]]}$ denote the stabilizer of the $k[[u]]$ -lattice chain $\prod_{\psi: k_0 \rightarrow k} \underline{W}_{\bullet, k[[u]]}$, and $\underline{\text{GL}}_{k((u))}$ its generic fiber. Then we may identify $M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \otimes_{\mathcal{O}_E} k$ (resp. $M_{\mathcal{GL}_{K/F}, \{\mu_{K/F}\}}^{\text{loc}} \otimes_{\mathcal{O}_{E'}} k$) with a closed subscheme of

$$\mathcal{FL}_{\underline{\mathcal{GL}}_{k[[u]]}} \quad (\text{resp. } \mathcal{FL}_{\underline{\mathcal{GL}}_{k((u))}})$$

and the map ρ_1^{loc} is induced by extending torsors along a morphism

$$\underline{\mathcal{G}}_{k[[u]]} \rightarrow \underline{\mathcal{G}}_{\mathcal{L}_{k[[u]]}}.$$

Recall $e := [K : K_0]$ and we let $k[[v]] \rightarrow k[[u]]$ denote the map sending v to u^e . We write $\underline{\mathcal{G}}_{\mathcal{L}_{W,k[[v]]}}$ for the base change to $k[[v]]$ of the $\mathcal{O}_{K_0}[v]$ -group $\underline{\mathcal{G}}_{\mathcal{L}_W}$. Then $\underline{\mathcal{G}}_{\mathcal{L}_{W,k[[v]]}}$ is identified with the stabilizer of $\prod_{\psi: k_0 \rightarrow k} \underline{W}_{\bullet, k[[u]]}$ as $k[[v]]$ -modules; here we take all possible products of lattices in the lattice chain. There is a natural map of $k[[v]]$ -group schemes

$$(3.3.2.1) \quad \text{Res}_{k[[u]]/k[[v]]} \underline{\mathcal{G}}_{\mathcal{L}_{k[[u]]}} \rightarrow \underline{\mathcal{G}}_{\mathcal{L}_{W,k[[v]]}}$$

induced by the forgetful functor from $k[[u]]$ -modules to $k[[v]]$ -modules. Then we may identify $M_{\mathcal{G}_{\mathcal{L}_W, \{\mu_W\}}}^{\text{loc}} \otimes_{\mathcal{O}_F} k$ with a closed subscheme of $\mathcal{FL}_{\mathcal{G}_{\mathcal{L}_{W,k[[v]]}}}$ and the map ρ_2^{loc} is given by extending torsors along (3.3.2.1). The map ρ^{loc} on k -points is then given by the injection

$$(3.3.2.2) \quad \underline{\mathcal{G}}_{k((u))}(k((u))) / \underline{\mathcal{G}}_{k[[u]]}(k[[u]]) \hookrightarrow \underline{\text{GL}}_{W,k((v))}(k((v))) / \underline{\mathcal{G}}_{\mathcal{L}_{W,k[[v]]}}(k[[v]]).$$

We have a commutative diagram of maps of apartments

$$(3.3.2.3) \quad \begin{array}{ccccc} \mathcal{A}(G, S', \check{F}) & \hookrightarrow & \mathcal{A}(\text{Res}_{K/F} \text{GL}(V_K), T', \check{F}) & \hookrightarrow & \mathcal{A}(\text{GL}_W, T'_W, \check{F}) \\ \downarrow \parallel & & \downarrow \parallel & & \downarrow \parallel \\ \mathcal{A}(\underline{G}_{k((u))}, \underline{S}', k((u))) & \hookrightarrow & \mathcal{A}(\underline{\text{GL}}_{k((u))}, \underline{T}'_{k((u))}, k((u))) & \hookrightarrow & \mathcal{A}(\underline{\text{GL}}_{W,k((v))}, \underline{T}'_{k((v))}, k((v))) \end{array}.$$

Here the tori T' and $\underline{T}'_{k((u))}$ are defined as follows. Let $\underline{\Lambda}$ denote the $\mathcal{O}_{\check{K}_0}[u^{\pm}]$ -module corresponding to the base change to $\mathcal{O}_{\check{K}_0}$ of the common generic fiber of \underline{W}_{\bullet} . The torus $\underline{T}' \subset \underline{\text{GL}}_{k((u))}$ is the maximal split torus determined by a suitable choice of basis \underline{b} of $\underline{\Lambda}$; cf. [KP18, Proof of Proposition 2.3.7]. Then T' (resp. $\underline{T}'_{k((u))}$) is the base change of \underline{T}' to \check{K} (resp. $k((u))$). The existence of the left square follows from the construction of the basis \underline{b} ; cf. [Zho20, §3.3].

The tori T'_W and $\underline{T}'_{k((v))}$ are determined by T' , $\underline{T}'_{k((u))}$ and the choice of uniformizers ϖ, u of \check{K} and $k((u))$ respectively. The commutativity of the right square then follows from the explicit description of the apartments in terms of lattice chains. We may also identify Iwahori Weyl groups for the groups in the top row with the respective Iwahori Weyl group in the bottom row, and the vertical isomorphisms are compatible with the action of the Iwahori Weyl groups. Moreover the horizontal maps induce morphisms of Iwahori Weyl groups and they are equivariant for the actions of these groups on the apartment.

We now argue as in [Zho20, Proposition 3.4]. Since $\mathcal{G}(\mathcal{O}_{\check{F}})$ maps to $\mathcal{GL}_W(\mathcal{O}_{\check{F}})$, we may assume $g = g_1 \dot{w}$. There is a $\mathcal{G} \otimes_{\mathcal{O}_F} \mathcal{O}_E$ -action on $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}$. Over the special fiber this action coincides with the one given by left multiplication by $\mathcal{G}(\mathcal{O}_{\check{F}})$ on

$$M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k) \subset M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}} \subset \text{GL}_W(\check{F}) / \mathcal{GL}_W(\mathcal{O}_{\check{F}});$$

note that the action of $\mathcal{G}(\mathcal{O}_{\check{F}})$ necessarily factors through $\mathcal{G}(k)$ since $\rho' \circ \mu$ is minuscule. Thus upon modifying g by g_1 on the left, we may assume that $g = \dot{w}$.

Using the commutativity of the diagram (3.3.2.3) and the fact that this diagram is equivariant for the action of Iwahori Weyl groups, it follows that the image of g in $M_{\mathcal{GL}_W, \{\mu_W\}}^{\text{loc}}(k) \subset \underline{\text{GL}}_{W,k((v))}(k((v))) / \underline{\mathcal{G}}_{\mathcal{L}_{k[[v]]}}(k[[v]])$ is given by the image of

$\underline{w} \in \underline{G}_{k((u))}(k((u)))$, where \underline{w} is a lift of the element $w \in W_{\underline{G}_{k((u))}}$ corresponding to w under the isomorphism (3.1.9.1). It follows from Theorem 3.1.12 that g gives a point in $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k)$ if and only if $w \in \text{Adm}(\{\mu\})_J$. \square

3.3.3. We now let $G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ as in (*), and $\rho : G \rightarrow \text{GL}(V)$ a faithful representation as in §3.2.7. As before we write W for the F -vector space underlying $\prod_{i=1}^r V_{K_i}$ and $\rho'^{\text{loc}} : M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \hookrightarrow M_{\mathcal{GL}_W, \{\rho' \circ \mu\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E$ the closed immersion of local models constructed in Proposition 3.2.8. This factors as

$$M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \hookrightarrow \prod_{i=1}^r M_{\mathcal{GL}_{W_i}, \{\mu_{W_i}\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E \hookrightarrow M_{\mathcal{GL}_W, \{\rho' \circ \mu\}}^{\text{loc}} \otimes_{\mathcal{O}_F} \mathcal{O}_E.$$

As before, we may identify $M_{\mathcal{GL}_W, \{\rho' \circ \mu\}}^{\text{loc}}(k)$ with a subset of $\text{GL}_W(\check{F})/\mathcal{GL}(\mathcal{O}_{\check{F}})$. Using the fact that the embedding $G(\check{F}) \hookrightarrow \text{GL}_W(\check{F})$ factors through $\prod_{i=1}^r \text{GL}_{W_i}(\check{F})$ and applying Proposition 3.3.2 we obtain the following.

Proposition 3.3.4. *Let $G \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ and assume \mathcal{G} is a connected parahoric. Let $g \in G(\check{F})$ with*

$$g \in \mathcal{G}(\mathcal{O}_{\check{F}}) \dot{w} \mathcal{G}(\mathcal{O}_{\check{F}})$$

for some $w \in W_J \backslash W/W_J$. Then the image of $\rho'(g)$ in $\text{GL}_W(\check{F})/\mathcal{GL}_W(\mathcal{O}_{\check{F}})$ lies in $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k)$ if and only if $w \in \text{Adm}(\{\mu\})_J$, where $J \subset \mathbb{S}$ is the set of simple reflections corresponding to \mathcal{G} .

\square

3.4. More general local models.

3.4.1. In this subsection we extend the construction of local models to certain triples $(G, \mathcal{G}, \{\mu\})$ with the condition (*) relaxed. This is necessary for the later applications to Shimura varieties because groups of the form $\text{Res}_{K/F} H$ rarely arise as the group at p of a Shimura datum of Hodge type.

Let G be a reductive group over F and $\{\mu\}$ a conjugacy class of minuscule cocharacters for G . Let $\rho : G \rightarrow \text{GSp}(V)$ be a faithful symplectic representation, where V is a $2n$ -dimensional vector space over F equipped with a perfect alternating bilinear form Ψ . We assume that $\rho \circ \mu$ is conjugate to the standard minuscule coweight $a \mapsto \text{diag}(1^{(n)}, (a^{-1})^{(n)})$ and that G contains the scalars. We call such an embedding a local Hodge embedding.

Definition 3.4.2. The pair $(G, \{\mu\})$ is said to be *regular* if the following three conditions are satisfied.

- (1) G is a subgroup of a reductive group $G' \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ as in (*) such that the inclusion $G \subset G'$ induces an isomorphism $G_{\text{der}} \cong G'_{\text{der}}$.
- (2) There exists a local Hodge embedding $\rho : G \rightarrow \text{GSp}(V)$ such that ρ extends to a closed immersion $\rho : G' \rightarrow \text{GL}(V)$.
- (3) The centralizer T of a maximal \check{F} -split torus of G is R -smooth.

We say a local model triple $(G, \{\mu\}, \mathcal{G})$ is regular if the associated pair $(G, \{\mu\})$ is regular.

Remark 3.4.3. (1) For later applications, all Shimura varieties that we work with can be related to one whose associated local model triple is regular. Therefore, this assumption will not appear in our final result.

- (2) By Proposition 2.4.9, condition (3) implies the inclusion $G \subset G'$ induces a closed immersion $\tilde{\mathcal{G}} \rightarrow \tilde{\mathcal{G}}'$, where $\tilde{\mathcal{G}}'$ is the Bruhat–Tits stabilizer scheme for G' corresponding to $\tilde{\mathcal{G}}$.

3.4.4. Let $(G, \{\mu\}, \mathcal{G})$ be a regular triple and $G' \cong \prod_{i=1}^r \text{Res}_{K_i/F} H_i$ as in Definition 3.4.2. Since G and G' have the same derived group, the parahoric \mathcal{G} determines a parahoric group scheme \mathcal{G}' of G' . We define a local model for G by setting $M_{\mathcal{G}, \{\mu\}}^{\text{loc}} := M_{\mathcal{G}', \{\mu'\}}^{\text{loc}}$, where $\{\mu'\}$ is the G' -conjugacy class of cocharacters induced by $\{\mu\}$. If we let $P_\mu \subset G$ denote the parabolic subgroup corresponding to some representative μ of $\{\mu\}$, and $P'_{\mu'} \subset G'$ the corresponding parabolic of G' , then there is a canonical identification

$$X_\mu := G/P_\mu \cong G'/P'_{\mu'}$$

so such a definition is justified. It is possible to prove that the definition of $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}$ does not depend on the choice of G' , but we will not need this, and will always consider the definition via a choice of auxiliary group G' .

We choose a σ -invariant alcove $\mathfrak{a} \subset \mathcal{B}(G, \check{F})$ as in §3.1.10; this determines a set of simple reflections \mathbb{S} for the Iwahori Weyl group W and we let $J \subset \mathbb{S}$ be the subset corresponding to the parahoric \mathcal{G} . There is a natural $G(\check{F})$ -equivariant map of buildings $\mathcal{B}(G, \check{F}) \rightarrow \mathcal{B}(G', \check{F})$ and the alcove \mathfrak{a} determines an alcove $\mathfrak{a}' \subset \mathcal{B}(G', \check{F})$. We let W', \mathbb{S}' denote the corresponding objects for G' .

By construction, there is a canonical identification $\mathbb{S} \cong \mathbb{S}'$ and we let $J' \subset \mathbb{S}'$ denote the subset corresponding to J . Then J' corresponds to the parahoric \mathcal{G}' of G . The stratification of the special fiber of the local model has a stratification naturally indexed by the μ' -admissible $\text{Adm}(\{\mu'\})_{J'}$ set of G' . However the natural map $G \rightarrow G'$ induces a map $W \rightarrow W'$ between Iwahori Weyl groups and by [HR, Lemma 3.6], this induces a bijection

$$\text{Adm}_G(\{\mu\})_J \cong \text{Adm}_{G'}(\{\mu'\})_{J'}.$$

We may thus consider the strata as being indexed by $\text{Adm}(\{\mu\})_J$.

3.4.5. Let $\rho : G \rightarrow \text{GSp}(V)$ be a local Hodge embedding as in Definition 3.4.2 (2) and $\rho : G' \rightarrow \text{GL}(V)$ its extension to G' . Let $\rho' : G' \rightarrow \text{GL}(W)$ be the embedding obtained from ρ via the construction in §3.2.7; we write $2n' := \dim_F W$. Recall, that $W = \prod_{i=1}^r W_i$, with $W_i = V \otimes_F K_i$, viewed as an F -vector space. We may equip W_i with the alternating bilinear form given by

$$\Psi_i : W_i \times W_i \xrightarrow{\Psi \otimes_F K_i} K \xrightarrow{\text{tr}} F,$$

where $\text{tr} : K \rightarrow F$ is the trace map. We then define an alternating bilinear form Ψ' on W by setting $\Psi' := \sum \Psi_i$. It is easy to check that the induced map $G \rightarrow \text{GL}(W)$ factors through $\text{GSp}(W)$ and we write ρ^H for the induced map $G \rightarrow \text{GSp}(W)$.

There is a canonical equivariant toral embedding of buildings

$$\mathcal{B}(\text{GSp}(W), F) \rightarrow \mathcal{B}(\text{GL}(W), F);$$

see eg. [KP18, §2.3.2]. Arguing as in [KP18, Lemma 2.3.3], we may choose the embedding (3.2.3.2) such that the composition $\mathcal{B}(G, F) \rightarrow \mathcal{B}(\text{GL}(W), F)$ factors through $\mathcal{B}(\text{GSp}(W), F)$. We write \mathcal{GSP} (resp. \mathcal{GL}_W) for the parahoric group scheme of $\text{GSp}(W)$ (resp. $\text{GL}(W)$) corresponding to the image of x .

The local model $M_{\mathcal{GSP}, \{\rho^H \circ \mu\}}^{\text{loc}}$ agrees with the one studied by Görtz in [Gör03]; its generic fiber is the Lagrangian Grassmannian $\text{LGr}(W)$, which parameterizes n' -dimensional isotropic subspaces of W . The natural map

$$X_\mu \rightarrow \text{Gr}(n', 2n') \otimes_F E$$

factors through $\text{LGr}(W) \otimes_F E$. The following corollary follows immediately from Proposition 3.2.6, using the existence of the closed immersion

$$M_{\mathcal{GSP}, \{\rho^H \circ \mu\}}^{\text{loc}} \rightarrow M_{\mathcal{GL}_W, \{\rho' \circ \mu'\}}^{\text{loc}};$$

cf. [KP18, §2.3.4].

3.4.6. Arguing as in [KP18, §2.3.15], one can further modify ρ^H so that $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}$ maps into a smooth Grassmannian.

Corollary 3.4.7. *Let $(G, \mathcal{G}, \{\mu\})$ be a regular triple. Then there exists a good local Hodge embedding $G \rightarrow \text{GSp}(W')$.* \square

Definition 3.4.8. Let $(G, \{\mu\}, \mathcal{G})$ be a regular triple and we let $G \subset G'$ as in Definition 3.4.2 (1). Let W be an F -vector space and $\Lambda \subset W$ an \mathcal{O}_F -lattice. We say that a faithful representation $\varrho : G \rightarrow \text{GL}(W)$ is *good* with respect to Λ if the following two conditions are satisfied.

- (1) ϱ extends to a closed immersion $\tilde{\mathcal{G}}' \hookrightarrow \mathcal{GL}_W := \text{GL}(\Lambda)$.
- (2) There is a closed immersion of local models

$$M_{\mathcal{G}, \{\mu\}}^{\text{loc}} \hookrightarrow \text{Gr}(\Lambda) \otimes_{\mathcal{O}_F} \mathcal{O}_E$$

which extends the natural map on the generic fiber, where $\text{Gr}(\Lambda)$ is the Grassmannian of subspaces $\mathcal{F} \subset \Lambda$ of rank d . Here d is such that $\varrho \circ \mu$ is conjugate to the standard minuscule coweight $a \mapsto (1^{(n-d)}, (a^{-1})^{(d)})$.

A representation $\varrho : G \rightarrow \text{GL}(W)$ is said to be *good* if there exists an \mathcal{O}_F -lattice $\Lambda \subset W$ with respect to which ϱ is good, and we say that a local Hodge embedding $\rho : G \rightarrow \text{GSp}(W)$ is good if the induced representation $G \rightarrow \text{GL}(W)$ is good.

Corollary 3.4.9. *Let $(G, \mathcal{G}, \{\mu\})$ be a regular triple and $\rho^H : G \rightarrow \text{GSp}(W)$ a Hodge embedding as constructed in §3.4.5. Then we may find a new Hodge embedding $\rho'' : G \rightarrow \text{GSp}(W')$ such that ρ'' is good.* \square

3.4.10. Let $(G, \{\mu\}, \mathcal{G})$ be a regular local model triple and $\rho'' : G \rightarrow \text{GSp}(W')$ a good Hodge embedding. We let $\Lambda \subset W'$ be a lattice with respect to which ρ'' is good. As explained in [Zho20, §3.6], we may identify the k -points of $\text{Gr}(\Lambda)$ with a subset of $\text{GL}_{W'}(\check{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\check{F}})$, where $\mathcal{GL}_{W'} := \text{GL}(\Lambda)$. The following Corollary can be deduced easily from Proposition 3.3.2.

Corollary 3.4.11. *Assume the parahoric \mathcal{G} is connected. Let $g \in G(\check{F})$ with*

$$g \in \mathcal{G}(\mathcal{O}_{\check{F}}) \dot{w} \mathcal{G}(\mathcal{O}_{\check{F}})$$

for some $w \in W_J \backslash W/W_J$. Then the image of $\rho''(g)$ in $\text{GL}_{W'}(\check{F})/\mathcal{GL}_{W'}(\mathcal{O}_{\check{F}})$ lies in $M_{\mathcal{G}, \{\mu\}}^{\text{loc}}(k)$ if and only if $w \in \text{Adm}(\{\mu\})_J$. \square

4. DEFORMATION THEORY OF p -DIVISIBLE GROUPS

In this section we prove the deformation theoretic results needed to study integral models of Shimura varieties in §5. §4.1–§4.2 contains the results needed to establish the local model diagram in §5.1. In §4.3 we study the deformation theory of μ -ordinary p -divisible groups and construct an analogue of the Serre–Tate canonical lifting.

4.1. The versal deformation space with tensors.

4.1.1. We recall the deformation theory of p -divisible groups equipped with a collection of crystalline tensors following [KP18, §3]. As most of the arguments of *loc. cit.* go through unchanged in our setting, we discuss in detail only those points which do not.

In this section, we assume $p > 2$ and we work over the base field \mathbb{Q}_p so that $\check{\mathbb{Q}}_p = W(k)[\frac{1}{p}]$, where $W(k)$ denotes the Witt vectors of k . For any ring R and an R -module M , we let M^\otimes denote the direct sum of all R -modules obtained from M by taking duals, tensor products, symmetric and exterior products. If R is a complete local ring with residue field of positive characteristic and \mathcal{G} is a p -divisible group over R , we write $\mathbb{D}(\mathcal{G})$ for its (contravariant) Dieudonné crystal.

4.1.2. Let \mathcal{G}_0 be a p -divisible group over k and set $\mathbb{D} := \mathbb{D}(\mathcal{G}_0)(\check{\mathbb{Z}}_p)$. We write φ for the Frobenius on \mathbb{D} . Let $(s_{\alpha,0}) \subset \mathbb{D}^\otimes$ be a collection of φ -invariant tensors whose image in $\mathbb{D}(\mathcal{G}_0)(k)^\otimes$ lie in Fil^0 . We assume that there exists a \mathbb{Z}_p -module U and an isomorphism

$$(4.1.2.1) \quad U \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \cong \mathbb{D}$$

such that $s_{\alpha,0} \in U^\otimes$. Write $\tilde{\mathcal{G}} \subset \text{GL}(U)$ for the pointwise stabilizer of $\{s_{\alpha,0}\}_\alpha$ so that $\tilde{\mathcal{G}}_{\check{\mathbb{Z}}_p}$ can be identified with the stabilizer of $s_{\alpha,0}$ in $\text{GL}(\mathbb{D})$.

We assume that the generic fiber $G := \tilde{\mathcal{G}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a reductive group, and that $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_x$ for some $x \in \mathcal{B}(G, \mathbb{Q}_p)$ which is generic in its facet. We write \mathcal{G} for the parahoric group scheme corresponding to x .

Let $P \subset \text{GL}(\mathbb{D})$ be a parabolic subgroup lifting the parabolic P_0 corresponding to the filtration on $\mathbb{D}(\mathcal{G}_0)(k)$. Write $M^{\text{loc}} = \text{GL}(\mathbb{D})/P$ and $\text{Spf} A = \widehat{M}^{\text{loc}}$ the completion of M^{loc} at the identity; then A is isomorphic to a power series ring over $\check{\mathbb{Z}}_p$. Let $K'/\check{\mathbb{Q}}_p$ be a finite extensions and $y : A \rightarrow K'$ a continuous map such that $s_{\alpha,0} \in \text{Fil}^0 \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K'$ for the filtration induced by y on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K'$. By [Kis10, Lemma 1.4.5], the filtration corresponding to y is induced by a G -valued cocharacter μ_y . Let $G.y$ be the orbit of y in $M^{\text{loc}} \otimes_{\check{\mathbb{Z}}_p} K'$ which is defined over a finite extension $\check{E}/\check{\mathbb{Q}}_p$, and we write $M_{\mathcal{G}}^{\text{loc}}$ for the closure of this orbit in M^{loc} .

4.1.3. Let R be a complete local ring with maximal ideal \mathfrak{m} and residue field k . We let $W(R)$ denote the Witt vectors of R . Recall [Zin01] we have a subring

$$\widehat{W}(R) = W(k) \oplus \mathbb{W}(\mathfrak{m}) \subset W(R),$$

where $\mathbb{W}(\mathfrak{m}) \subset W(R)$ consists of Witt vectors $(w_i)_{i \geq 1}$ with $w_i \in \mathfrak{m}$ and $w_i \rightarrow 0$ in the \mathfrak{m} -adic topology. The Frobenius of $W(R)$ induces a map $\varphi : \widehat{W}(R) \rightarrow \widehat{W}(R)$, and we write I_R for the kernel of the projection $\widehat{W}(R) \rightarrow R$. We recall the following

definition, which is [Zho20, Definition 4.6] in the case that G splits over a tamely ramified extension of \mathbb{Q}_p .

Definition 4.1.4. Let $K/\check{\mathbb{Q}}_p$ be a finite extension. Let \mathcal{G} be a p -divisible group over \mathcal{O}_K whose special fiber is isomorphic to \mathcal{G}_0 . We say \mathcal{G} is $(\tilde{\mathcal{G}}, \mu_y)$ -adapted if the tensors $s_{\alpha,0}$ extend to Frobenius invariant tensors $\tilde{s}_\alpha \in \mathbb{D}(\mathcal{G})(\widehat{W}(\mathcal{O}_K))^\otimes$ such that the following two conditions hold:

- (1) There is an isomorphism $\mathbb{D}(\mathcal{G})(\widehat{W}(\mathcal{O}_K)) \cong \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \widehat{W}(\mathcal{O}_K)$ taking \tilde{s}_α to $s_{\alpha,0}$.
- (2) Under the canonical identification

$$\mathbb{D}(\mathcal{G})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \cong \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K$$

given by [KP18, Lemma 3.1.17], the filtration on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K$ is induced by a G -valued cocharacter conjugate to μ_y .

4.1.5. Consider the local model triple $(G, \{\mu_y^{-1}\}, \mathcal{G})$. We assume in addition that the following conditions are satisfied:

(4.1.5.1) The pair $(G, \{\mu_y^{-1}\})$ is regular and $p \nmid |\pi_1(G_{\text{der}})|$.

(4.1.5.2) The embedding $G \subset \text{GL}(U_{\mathbb{Q}_p})$ is good with respect to U .

(4.1.5.3) $G \subset \text{GL}(U_{\mathbb{Q}_p})$ contains the scalars.

Under these assumptions, Corollary 3.4.9 implies that the definition of $M_{\mathcal{G}}^{\text{loc}}$ above agrees with the local model $M_{\mathcal{G}, \{\mu_y^{-1}\}}^{\text{loc}} \otimes_{\mathcal{O}_E} \mathcal{O}_{\check{E}}$, cf. [Zho20, Example 3.3] regarding the sign convention for cocharacters defining local models. We write $\widehat{M}_{\mathcal{G}}^{\text{loc}} \cong \text{Spf } A_{\tilde{\mathcal{G}}}$ for the completion of $M_{\mathcal{G}}^{\text{loc}}$ at the identity element. By Theorem 3.1.6, $A_{\tilde{\mathcal{G}}}$ is normal and we have a natural surjective map $A \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_{\check{E}} \rightarrow A_{\tilde{\mathcal{G}}}$ corresponding to the closed immersion $\widehat{M}_{\mathcal{G}}^{\text{loc}} \subset \widehat{M}^{\text{loc}} \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_{\check{E}}$.

4.1.6. We now apply the construction in [KP18, 3.2]; the following is essentially [KP18, Proposition 3.2.17].

Proposition 4.1.7. *There exists a versal p -divisible group \mathcal{G}_A over $\text{Spf } A \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_{\check{E}}$ deforming \mathcal{G}_0 such that for any $K/\check{\mathbb{Q}}_p$ finite, a map $\varpi : A \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_{\check{E}} \rightarrow K$ factors through $A_{\tilde{\mathcal{G}}}$ if and only if the p -divisible group \mathcal{G}_{ϖ} given by the base change of \mathcal{G}_A along ϖ is $(\tilde{\mathcal{G}}, \mu_y)$ -adapted.*

Proof. Under our assumptions and using [Ans, Proposition 10.3] in place of [KP18, Proposition 1.4.3], we find that the conditions (3.2.2)-(3.2.4) of [KP18] are satisfied; we may thus apply the construction in [KP18, §3.2] to obtain \mathcal{G}_A .

By construction, the base change $\mathcal{G}_{A_{\tilde{\mathcal{G}}}} := \mathcal{G}_A \otimes_{A \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_{\check{E}}} A_{\tilde{\mathcal{G}}}$ is equipped with Frobenius invariant tensors $s_{\alpha,0,A_{\tilde{\mathcal{G}}}} \in \mathbb{D}(\mathcal{G}_{A_{\tilde{\mathcal{G}}}})(\widehat{W}(A_{\tilde{\mathcal{G}}}))^\otimes$. It is then clear that for $\varpi : A_{\tilde{\mathcal{G}}} \rightarrow K$, the tensors $s_{\alpha,0}$ extend to

$$\tilde{s}_\alpha \in \mathbb{D}(\mathcal{G}_{\varpi})(\widehat{W}(\mathcal{O}_K))^\otimes$$

so that Definition 4.1.4 (1) is satisfied. Indeed the tensors \tilde{s}_α are obtained from $s_{\alpha,0,A_{\tilde{\mathcal{G}}}}$ via base change. The argument in [Zho20, Proposition 4.7] shows that condition (2) is also satisfied, so that \mathcal{G}_{ϖ} is $(\tilde{\mathcal{G}}, \mu_y)$ -adapted.

The converse is [KP18, Proposition 3.2.17] □

4.2. Deformations with étale tensors.

4.2.1. Let $K/\check{\mathbb{Q}}_p$ be a finite extension and \mathcal{G} a p -divisible group over \mathcal{O}_K with special fiber \mathcal{G}_0 . We write $T_p\mathcal{G}$ for the p -adic Tate-module of \mathcal{G} and $T_p\mathcal{G}^\vee$ its linear dual. We let $s_{\alpha,\text{ét}} \in T_p\mathcal{G}^{\vee\otimes}$ be a collection of tensors whose stabilizer $\tilde{\mathcal{G}}$ has reductive generic G and $\tilde{\mathcal{G}} = \tilde{\mathcal{G}}_x$ for some $x \in \mathcal{B}(G, \mathbb{Q}_p)$ which is generic in the facet containing it. We write $\mathbb{D} := \mathbb{D}(\mathcal{G}_0)(\check{\mathbb{Z}}_p)$ and we let

$$s_{\alpha,0} \in D_{\text{cris}}(T_p\mathcal{G}^\vee \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^{\otimes} \simeq \mathbb{D}^{\otimes} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$$

denote the φ -invariant tensors corresponding to the image of $s_{\alpha,\text{ét}}$ under the p -adic comparison isomorphism.

Proposition 4.2.2. (1) We have $s_{\alpha,0} \in \mathbb{D}^{\otimes} \subset \mathbb{D}^{\otimes} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$. Moreover the $s_{\alpha,0}$ extend canonically to tensors $\tilde{s}_\alpha \in \mathbb{D}(\mathcal{G})(\widehat{W}(\mathcal{O}_K))^{\otimes}$ and there exists an isomorphism

$$(4.2.2.1) \quad T_p\mathcal{G}^\vee \otimes_{\check{\mathbb{Z}}_p} \widehat{W}(\mathcal{O}_K) \cong \mathbb{D}(\mathcal{G})(\widehat{W}(\mathcal{O}_K))$$

taking $s_{\alpha,0}$ to \tilde{s}_α .

- (2) There exists a G -valued cocharacter μ_y such that
(i) Under the canonical isomorphism

$$\gamma : \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K \cong \mathbb{D}(\mathcal{G})(\mathcal{O}_K) \otimes_{\mathcal{O}_K} K,$$

the filtration is induced by a G -valued cocharacter conjugate to μ_y .

- (ii) The filtration on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K$ induced by μ_y lifts the filtration on $\mathbb{D}(\mathcal{G}_0) \otimes_{\check{\mathbb{Z}}_p} k$.

Here we consider $G_{\check{\mathbb{Q}}_p} \subset \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$ via the base change of (4.2.2.1) to $\check{\mathbb{Q}}_p$.

Proof. The argument is the same as [KP18, Proposition 3.3.8, Corollary 3.3.10], where again we are using [Ans, Proposition 10.3] in place of [KP18, Proposition 1.4.3]. \square

4.2.3. The isomorphism (4.2.2.1) induces an isomorphism

$$T_p\mathcal{G}^\vee \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \cong \mathbb{D}$$

taking $s_{\alpha,\text{ét}}$ to $s_{\alpha,0}$ which we now fix. Taking $T_p\mathcal{G}^\vee$ to be U , we place ourselves in the setting of §4.1.2. Therefore we have a notion of $(\tilde{\mathcal{G}}, \mu_y)$ -adapted lifting where μ_y is as in Proposition 4.2.2. Moreover it follows from the same proposition that \mathcal{G} itself is a $(\tilde{\mathcal{G}}, \mu_y)$ -adapted lifting. The next proposition then follows immediately from Proposition 4.2.2 and the definition of $(\tilde{\mathcal{G}}, \mu_y)$ -adapted liftings (cf. [KP18, Proposition 3.3.13]).

Proposition 4.2.4. Let $K'/\check{\mathbb{Q}}_p$ be a finite extension and let \mathcal{G}' be a deformation of \mathcal{G}_0 to $\mathcal{O}_{K'}$ such that

- (1) The filtration on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K'$ corresponding to \mathcal{G}' is induced by a G -valued cocharacter conjugate to μ_y .
(2) The tensors $s_{\alpha,0} \in \mathbb{D}^{\otimes}$ correspond to tensors $s_{\alpha,\text{ét}} \in T_p\mathcal{G}'^{\vee\otimes}$ under the p -adic comparison isomorphism.

Then \mathcal{G}' is $(\tilde{\mathcal{G}}, \mu_y)$ -adapted lifting.

\square

4.3. Canonical liftings for μ -ordinary p -divisible groups.

4.3.1. We now study the deformation theory of μ -ordinary p -divisible groups. The results in this subsection will be used in §5.3 to prove our main result on CM (special) liftings for Shimura varieties.

We return to the setting of §4.1. Thus \mathcal{G}_0 is a p -divisible group over k equipped with $s_{\alpha,0} \in \mathbb{D}^\otimes$. We fix a $\check{\mathbb{Z}}_p$ -linear isomorphism

$$(4.3.1.1) \quad U \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \cong \mathbb{D}(\mathcal{G}_0)$$

as in (4.1.2.1) so that $s_{\alpha,0} \in U^\otimes$. In §4.3, we will assume in addition to (4.1.5.1)–(4.1.5.3), that \mathcal{G} is a connected parahoric so that $\mathcal{G} = \tilde{\mathcal{G}}$. Since the $s_{\alpha,0}$ are φ -invariant, the Frobenius is given by $b\sigma$ for an element $b \in G(\check{\mathbb{Q}}_p)$, and modifying (4.3.1.1) by an element $h \in \mathcal{G}(\check{\mathbb{Z}}_p)$ modifies b by $b \mapsto h^{-1}b\sigma(h)$. Therefore b is well-defined up to σ -conjugation by an element of $\mathcal{G}(\check{\mathbb{Z}}_p)$ and in particular we obtain a well-defined class $[b] \in B(G)$.

We choose a maximal $\check{\mathbb{Q}}_p$ -split torus S of G defined over $\check{\mathbb{Q}}_p$ such that $x \in \mathcal{A}(G, S, \check{\mathbb{Q}}_p)$ and we let T denote its centralizer. We fix a σ -stable alcove $\mathfrak{a} \subset \mathcal{A}(G, S, \check{\mathbb{Q}}_p)$ such that x lies in the closure of \mathfrak{a} ; thus \mathcal{G} corresponds to a subset $J \subset \mathbb{S}$ of the set of simple reflections of W determined by \mathfrak{a} . We follow the notation of §2 and let $\tilde{\mu} \in X_*(T)$ denote the dominant (with respect to a choice of Borel defined over $\check{\mathbb{Q}}_p$) representative of the conjugacy class $\{\mu_y\}$; we write μ for its image in $X_*(T)_I$. We have a closed immersion of local models

$$M_{\mathcal{G}, \{\mu_y^{-1}\}}^{\text{loc}} \hookrightarrow \text{Gr}(U) \otimes_{\mathbb{Z}_p} \mathcal{O}_E,$$

where $\text{Gr}(U)$ classifies submodules of U of rank $\dim_k \text{Fil}^0 \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} k$. By definition, the filtration on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} k$ corresponds to an element of $\text{Gr}(U)(k)$ which lies in $M_{\mathcal{G}, \{\mu_y^{-1}\}}^{\text{loc}}(k)$. This filtration is by definition the kernel of φ ; thus its preimage in \mathbb{D} is given by

$$\{v \in \mathbb{D} \mid b\sigma(v) \in p\mathbb{D}\}.$$

This is just the $\check{\mathbb{Z}}_p$ -lattice $\sigma^{-1}(b^{-1})p\mathbb{D}$. It follows from Corollary 3.4.11 that $\sigma^{-1}(b^{-1}) \in \mathcal{G}(\check{\mathbb{Z}}_p)w\mathcal{G}(\check{\mathbb{Z}}_p)$ for some element $w \in \text{Adm}(\{\mu_y^{-1}\})_J$, and hence that $b \in \mathcal{G}(\check{\mathbb{Z}}_p)\sigma(u)\mathcal{G}(\check{\mathbb{Z}}_p)$ for some $u \in \text{Adm}(\{\mu_y\})_J$. In particular we have $[\sigma^{-1}(b)] \in B(G, \{\mu_y\})$ by [Hel16, Theorem 1.1].

4.3.2. Now assume the existence of $[b]_\mu \in B(G, \{\mu_y\})$ as in Definition 2.2.4, and that $\sigma^{-1}(b) \in [b]_\mu$. We will construct a (\mathcal{G}, μ_y) -adapted (recall $\tilde{\mathcal{G}} = \mathcal{G}$) deformation of \mathcal{G}_0 which will be the analogue of the Serre–Tate canonical lifting in this context.

By Proposition 2.3.3 applied to $\sigma^{-1}(b)$, there exists an element $h \in \mathcal{G}(\check{\mathbb{Z}}_p)$ such that $h^{-1}b\sigma(h) = \sigma(t_{\mu'})$ for some $\mu' \in W_0 \cdot \mu$ with $t_{\mu'}$ σ -straight. Upon modifying the isomorphism (4.3.1.1), we may assume $b = \sigma(t_{\mu'})$; we fix this choice of (4.3.1.1) from now on. Let M be the semistandard Levi subgroup of G corresponding to $\nu_{t_{\mu'}} = \nu_{\sigma(t_{\mu'})}$; then $t_{\mu'}$ is central in W_M by Lemma 2.1.7. Let $w \in W_0$ such that $w \cdot \mu = \mu'$ and write $\tilde{\lambda} := w \cdot \tilde{\mu}$; then by Lemma 2.1.7, $\tilde{\lambda}$ is central in M . Let

$$\mathcal{M}(\check{\mathbb{Z}}_p) := M(\check{\mathbb{Q}}_p) \cap \mathcal{G}(\check{\mathbb{Z}}_p);$$

it is the $\check{\mathbb{Z}}_p$ -points of a parahoric group scheme \mathcal{M} of M defined over \mathbb{Z}_p . Since \mathcal{G} is a connected parahoric and $\pi_1(M)_I \rightarrow \pi_1(G)_I$ has torsion-free kernel, it follows that \mathcal{M} is a connected parahoric.

Lemma 4.3.3. *Let K be the field of definition of $\tilde{\lambda}$. The filtration induced by $\tilde{\lambda}$ on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K$ specializes to $\text{Fil}^0 \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} k$.*

Proof. Let $G \subset G'$ where G' is as in Definition 3.4.2 and let \mathcal{G}' be the corresponding parahoric. The cocharacter $\tilde{\lambda}$ determines a K -point $s_{\tilde{\lambda}^{-1}}$ of $\text{Gr}_{G'}$ which lies in $M_{\mathcal{G}, \{\mu_y^{-1}\}}^{\text{loc}}$ (cf. [Zho20, Example 3.3] for the sign convention) and whose image in $M^{\text{loc}} = \text{Gr}(U) \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p$ corresponds to the filtration induced by $\tilde{\lambda}$.

The geometric special fiber of $M_{\mathcal{G}, \{\mu_y^{-1}\}}^{\text{loc}}$ is a closed subscheme of $\mathcal{FL}_{\mathcal{G}'_{k[[u]]}}$ where $\mathcal{G}'_{k[[u]]}$ is a $k[[u]]$ -group scheme associated to \mathcal{G}' as in §3.1. By [Lev16, Proposition 4.2.8], $s_{\tilde{\lambda}^{-1}}$ extends to an \mathcal{O}_K -point of $M_{\mathcal{G}, \{\mu_y^{-1}\}}^{\text{loc}}$ whose special fiber is the point $t_{\mu'}^{-1}$. Here $t_{\mu'}^{-1}$ is the element of the Iwahori Weyl group for $\mathcal{G}'_{k((u))} := \mathcal{G}'_{k[[u]]} \otimes_{k[[u]]} k((u))$ corresponding to $t_{\mu'}^{-1}$ under the identification of Iwahori Weyl groups (3.1.9.1). By construction of the embedding $M_{\mathcal{G}}^{\text{loc}}(k) \hookrightarrow \text{GL}_U(\check{\mathbb{Q}}_p)/\text{GL}_U(\check{\mathbb{Z}}_p)$ in §3.4 (cf. Proof of Proposition 3.3.2), the filtration on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} k$ corresponding to the image of $t_{\mu'}^{-1}$ in $M^{\text{loc}}(k)$ is given by the reduction mod p of $t_{\mu'}^{-1} p \mathbb{D} = \sigma^{-1}(b^{-1}) p \mathbb{D}$. The proposition follows. \square

4.3.4. We extend the tensors $s_{\alpha,0} \in U^{\otimes}$ to tensors $t_{\beta,0} \in U^{\otimes}$ whose stabilizer is \mathcal{M} . Viewed in $\mathbb{D} \simeq U \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p$, the $t_{\beta,0}$ are φ -invariant as $b = \sigma(t_{\mu'}) \in M(\check{\mathbb{Q}}_p)$. Since $\tilde{\lambda}$ is an M -valued cocharacter, we may apply the construction in §4.1 to M and the tensors $t_{\beta,0}$. In particular we have a notion of $(\mathcal{M}, \tilde{\lambda})$ -adapted liftings of \mathcal{G}_0 . It is clear from the definition that any $(\mathcal{M}, \tilde{\lambda})$ -adapted lifting is also a (\mathcal{G}, μ_y) -adapted lifting.

4.3.5. Let J_b denote the σ -centralizer group for b . It is a reductive group over \mathbb{Q}_p such that

$$J_b(R) := \{g \in G(\check{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} R) \mid g^{-1} b \sigma(g) = b\}$$

for any \mathbb{Q}_p -algebra R . There is an action of $J_b(\mathbb{Q}_p)$ on \mathcal{G}_0 in the isogeny category. Since $\nu_{g^{-1} b \sigma(g)} = g^{-1} \nu_b g$ for any $g \in G(\check{\mathbb{Q}}_p)$, it follows that for $b = \sigma(t_{\mu'})$, we have $J_b(\mathbb{Q}_p) \subset M(\check{\mathbb{Q}}_p)$.

Theorem 4.3.6. *Let $K/\check{\mathbb{Q}}_p$ be an extension over which $\tilde{\lambda}$ is defined and suppose $\hat{\mathcal{G}} = \mathcal{G}$. There exists a (\mathcal{G}, μ_y) -adapted lifting \mathcal{G} to \mathcal{O}_K such that the action of $J_b(\mathbb{Q}_p)$ on \mathcal{G}_0 lifts to \mathcal{G} in the isogeny category.*

Proof. Suppose there exists an $(\mathcal{M}, \tilde{\lambda})$ -adapted lifting \mathcal{G} of \mathcal{G}_0 ; from the above discussion, we have that \mathcal{G} is also a (\mathcal{G}, μ_y) -adapted lifting. By Definition 4.1.4 (2), the filtration on the weakly admissible filtered φ -module associated to $T_p \mathcal{G}^{\vee}$ is induced by an M -valued cocharacter conjugate to $\tilde{\lambda}$, hence by $\tilde{\lambda}$ itself since it is central in M . Since $J_b(\mathbb{Q}_p) \subset M(\check{\mathbb{Q}}_p)$, the action of $J_b(\mathbb{Q}_p)$ respects the filtration and hence lifts to an action on \mathcal{G} in the isogeny category.

It suffices to show the existence of an $(\mathcal{M}, \tilde{\lambda})$ -adapted lifting. This follows from the same argument as [Zho20, Proposition 4.9]; we briefly recall the argument for the convenience of the reader.

We set $\mathfrak{S} := \check{\mathbb{Z}}_p[[u]]$ and we let $\sigma : \mathfrak{S} \rightarrow \mathfrak{S}$ be the map given by the usual Frobenius on $\check{\mathbb{Z}}_p$ and $u \mapsto u^p$. We define $\mathfrak{M} := \mathbb{D} \otimes_{\sigma^{-1}, \check{\mathbb{Z}}_p} \mathfrak{S}$, so that $\sigma^*(\mathfrak{M}) \cong \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \mathfrak{S}$, and we let $\mathcal{F} \subset \sigma^*(\mathfrak{M})$ denote the preimage of the filtration induced by $\tilde{\lambda}$ on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_K$; here the map $\mathfrak{S} \rightarrow \mathcal{O}_K$ is induced by sending u to a uniformizer ϖ in \mathcal{O}_K . By [KP18, Lemma 3.2.6], which holds by [Ans, Proposition 10.3], \mathcal{F} is a free \mathfrak{S} -module and $t_{\beta,0} \in \mathcal{F}$. Moreover the scheme of \mathfrak{S} -linear isomorphisms $\underline{\text{Isom}}_{t_{\beta,0}}(\mathcal{F}, \sigma^*(\mathfrak{M}))$ taking $t_{\beta,0}$ to $t_{\beta,0}$ is a trivial \mathcal{M} -torsor. Then arguing as in [Zho20, Proposition 4.9], we may construct a morphism $\varphi : \sigma^*(\mathfrak{M}) \rightarrow \mathfrak{M}$ which gives \mathfrak{M} the structure of an element of BT^φ (cf. [Zho20, §4.1]). By [Kis10, Theorem 1.4.2], \mathfrak{M} corresponds to a p -divisible group \mathcal{G} over \mathcal{O}_K , and \mathcal{G} is an $(\mathcal{M}, \tilde{\lambda})$ -adapted lifting for \mathcal{G}_0 . \square

5. INTEGRAL MODELS OF SHIMURA VARIETIES AND CANONICAL LIFTINGS

In this section we construct integral models for Shimura varieties with parahoric level structure and establish the main geometric properties needed for later applications. These include a version of the local model diagram, which is first proved in some Hodge type cases in §5.1 and then under some mild assumptions in the abelian type case in §5.2, and the existence of special liftings over the μ -ordinary locus which is proved in §5.3.

5.1. Integral models.

5.1.1. For the rest of this paper we fix an algebraic closure $\overline{\mathbb{Q}}$, and for each place v of \mathbb{Q} (including $v = \infty$) an algebraic closure $\overline{\mathbb{Q}}_v$ together with an embedding $i_v : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_v$ (here $\overline{\mathbb{Q}}_\infty \cong \mathbb{C}$).

Let \mathbf{G} be a reductive group over \mathbb{Q} and X a $\mathbf{G}_{\mathbb{R}}$ -conjugacy class of homomorphisms

$$h : \mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_m \rightarrow \mathbf{G}_{\mathbb{R}}$$

such that (\mathbf{G}, X) is a Shimura datum in the sense of [Del71].

Let c be complex conjugation. Then $\mathbb{S}(\mathbb{C}) = (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \cong \mathbb{C}^\times \times c^*(\mathbb{C}^\times)$ and we write μ_h for the cocharacter given by

$$\mathbb{C}^\times \rightarrow \mathbb{C}^\times \times c^*(\mathbb{C}^\times) \xrightarrow{h} \mathbf{G}(\mathbb{C}).$$

We set $w_h := \mu_h^{-1} \mu_h^{c-1}$.

Let \mathbb{A}_f denote the ring of finite adeles and \mathbb{A}_f^p the subring of \mathbb{A}_f with trivial p -component. Let $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ and $K^p \subset \mathbf{G}(\mathbb{A}_f)$ be compact open subgroups and write $K := K_p K^p$. Then

$$(5.1.1.1) \quad \text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}} = \mathbf{G}(\mathbb{Q}) \backslash X \times \mathbf{G}(\mathbb{A}_f) / K$$

can be identified with the complex points of a smooth algebraic stack over \mathbb{C} . The theory of canonical models implies that $\text{Sh}_K(\mathbf{G}, X)_{\mathbb{C}}$ has a model $\text{Sh}_K(\mathbf{G}, X)$ over the reflex field $\mathbf{E} \subset \mathbb{C}$, which is defined to be the field of definition of the conjugacy class $\{\mu_h\}$. We may consider \mathbf{E} as a subfield of \mathbb{Q} via the embedding $i_\infty : \mathbb{Q} \hookrightarrow \mathbb{C}$ and we write $\mathcal{O}_{\mathbf{E}}$ for the ring of integers of \mathbf{E} . If K^p is sufficiently small (indeed if K^p is neat), then $\text{Sh}_K(\mathbf{G}, X)$ is an algebraic variety.

We also define

$$\begin{aligned}\mathrm{Sh}_{K_p}(\mathbf{G}, X) &:= \lim_{\leftarrow K^p} \mathrm{Sh}_{K_p K^p}(\mathbf{G}, X) \\ \mathrm{Sh}_K(\mathbf{G}, X) &:= \lim_{\leftarrow K} \mathrm{Sh}_K(\mathbf{G}, X);\end{aligned}$$

these are pro-varieties equipped with actions of $\mathbf{G}(\mathbb{A}_f^p)$ and $\mathbf{G}(\mathbb{A}_f)$ respectively.

5.1.2. We now assume that there is an embedding of Shimura data

$$\iota : (\mathbf{G}, X) \rightarrow (\mathbf{GSp}(V), S^\pm).$$

Here $\mathbf{GSp}(V)$ is the group of symplectic similitudes of a \mathbb{Q} -vector space V equipped with a perfect alternating bilinear form Ψ , and S^\pm is the Siegel double space.

Fix a prime $p > 2$ and let v be the prime of \mathbf{E} above p induced by the embedding $i_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}_p}$. We let $\mathcal{O}_{\mathbf{E}}$ denote the ring of integers of \mathbf{E} and $\mathcal{O}_{\mathbf{E}(v)}$ the localization v , and we write E for the completion of \mathbf{E} at v . We let k_E denote the residue field at v and we fix an algebraic closure k of k_E . Set $G := \mathbf{G}_{\mathbb{Q}_p}$. We let $\tilde{\mathcal{G}} := \tilde{\mathcal{G}}_x$ for some $x \in \mathcal{B}(G, \mathbb{Q}_p)$ which is generic in the facet containing it and we write \mathcal{G} for the associated parahoric group scheme. For the rest of §5.1, we make following assumption.

$$(5.1.2.1) \quad (G, \{\mu_h\}) \text{ is regular and } p \nmid |\pi_1(G_{\mathrm{der}})|.$$

Then arguing as in [KP18, 2.3.15] (cf. Corollary 3.4.9), upon replacing ι by another Hodge embedding, we may assume that the local Hodge embedding $\iota_{\mathbb{Q}_p} : G \rightarrow \mathrm{GSp}(V_{\mathbb{Q}_p})$ is a good embedding. In this case, we say that ι itself is a good Hodge embedding.

5.1.3. We set $\tilde{K}_p := \tilde{\mathcal{G}}(\mathbb{Z}_p)$ and $K_p := \mathcal{G}(\mathbb{Z}_p)$, and we let $\tilde{K} := \tilde{K}_p K^p$ and $K := K_p K^p$. Let $\iota : (\mathbf{G}, X) \rightarrow (\mathbf{GSp}(V), S^\pm)$ be a good embedding and let $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ be a \mathbb{Z}_p -lattice with $V_{\mathbb{Z}_p} \subset V_{\mathbb{Z}_p}^\vee$ and such that $G \rightarrow \mathrm{GL}(V_{\mathbb{Q}_p})$ is good with respect to $V_{\mathbb{Z}_p}$.

Let $V_{\mathbb{Z}(p)} = V_{\mathbb{Z}_p} \cap V$. We write $G_{\mathbb{Z}(p)}$ for the Zariski closure of \mathbf{G} in $\mathrm{GL}(V_{\mathbb{Z}(p)})$; then $G_{\mathbb{Z}(p)} \otimes_{\mathbb{Z}(p)} \mathbb{Z}_p \cong \tilde{\mathcal{G}}$. Let $K' = K'_p K'^p$ where K'_p is the stabilizer in $\mathrm{GSp}(V_{\mathbb{Q}_p})$ of the lattice $V_{\mathbb{Z}_p}$ and $K'^p \subset \mathbf{GSp}(\mathbb{A}_f^p)$ is a compact open subgroup. The choice of $V_{\mathbb{Z}(p)}$ gives rise to an interpretation of $\mathrm{Sh}_{K'}(\mathbf{GSp}, S^\pm)$ as a moduli stack of abelian varieties up to prime to p isogeny and hence an integral model $\mathcal{S}_{K'}(\mathbf{GSp}, S^\pm)$ over $\mathbb{Z}_{(p)}$, see [KP18, §4] and [Zho20, §6].

Assume that K^p is a neat compact open subgroup. By [Kis10, Lemma 2.1.2], we can choose K'^p such that ι induces a closed immersion:

$$\mathrm{Sh}_{\tilde{K}}(\mathbf{G}, X) \hookrightarrow \mathrm{Sh}_{K'}(\mathbf{GSp}, S^\pm) \otimes_{\mathbb{Q}} \mathbf{E}.$$

Let $\mathcal{S}_{\tilde{K}}(\mathbf{G}, X)^-$ be the Zariski closure of $\mathrm{Sh}_{\tilde{K}}(\mathbf{G}, X)$ inside $\mathcal{S}_{K'}(\mathbf{GSp}, S^\pm) \otimes_{\mathbb{Z}(p)} \mathcal{O}_{\mathbf{E}(v)}$, and $\tilde{\mathcal{S}}_{\tilde{K}}(\mathbf{G}, X)$ to be the normalization of $\mathcal{S}_{\tilde{K}}(\mathbf{G}, X)^-$. We also define the pro-scheme

$$\mathcal{S}_{\tilde{K}_p}(\mathbf{G}, X) := \lim_{\leftarrow K^p} \mathcal{S}_{\tilde{K}_p K^p}(\mathbf{G}, X).$$

The $\mathbf{G}(\mathbb{A}_f^p)$ -action on $\mathrm{Sh}_{\tilde{K}_p}(\mathbf{G}, X)$ extends to $\mathcal{S}_{\tilde{K}_p}(\mathbf{G}, X)$. Hence we may define $\mathcal{S}_{\tilde{K}_p K^p}(\mathbf{G}, X)$ for a general (not necessarily neat) compact open subgroup $K^p \subset \mathbf{G}(\mathbb{A}_f)$ as the quotient stack $\mathcal{S}_{\tilde{K}_p}(\mathbf{G}, X)/K^p$. Alternatively, we may take a compact open subgroup $K_1^p \subset K^p$ which is neat and normal in K^p , and define $\mathcal{S}_{\tilde{K}}(\mathbf{G}, X)$ as the quotient of $\mathcal{S}_{\tilde{K}_p K_1^p}(\mathbf{G}, X)$ under the action of the finite group K^p/K_1^p .

5.1.4. In order to understand the local structure of $\mathcal{S}_{\bar{K}}(\mathbf{G}, X)$, we need to introduce Hodge cycles. By [Kis10, Proposition 1.3.2], the subgroup $G_{\mathbb{Z}_{(p)}}$ is the stabilizer of a collection of tensors $s_\alpha \in V_{\mathbb{Z}_{(p)}}^\otimes$. Let $h : \mathcal{A} \rightarrow \mathcal{S}_{\bar{K}}(\mathbf{G}, X)$ denote the pullback of the universal abelian scheme on $\mathcal{S}_{K'}(\mathbf{GSp}, S^\pm)$ and let $V_B := R^1 h_{\text{an},*} \mathbb{Z}_{(p)}$, where h_{an} is the map of complex analytic spaces associated to h . Since the tensors s_α are \mathbf{G} -invariant, they give rise to sections $s_{\alpha,B} \in V_B^\otimes$. We also let $\mathcal{V} = R^1 h_* \Omega^\bullet$ be the relative de Rham cohomology of \mathcal{A} . Using the de Rham isomorphism, the $s_{\alpha,B}$ give rise to a collection of Hodge cycles $s_{\alpha,\text{dR}} \in \mathcal{V}_\mathbb{C}^\otimes$, where $\mathcal{V}_\mathbb{C}$ is the complex analytic vector bundle associated to \mathcal{V} . By [Kis10, Corollary 2.2.2], these tensors are defined over \mathbf{E} .

Similarly for a finite prime $\ell \neq p$, we let $\mathcal{V}_\ell = \mathcal{V}_\ell(\mathcal{A}) = R^1 h_{\text{ét}*} \mathbb{Q}_\ell$ and $\mathcal{V}_p = \mathcal{V}_p(\mathcal{A}) = R^1 h_{\eta,\text{ét}*} \mathbb{Z}_p$ where h_η is the generic fibre of h . Using the étale-Betti comparison isomorphism, we obtain tensors $s_{\alpha,\ell} \in \mathcal{V}_\ell^\otimes$ and $s_{\alpha,p} \in \mathcal{V}_p^\otimes$.

For T an $\mathcal{O}_{\mathbf{E}(v)}$ -scheme (resp \mathbf{E} -scheme, resp. \mathbb{C} -scheme), $* = \ell$ or dR (resp. ét, resp. B) and $x \in \mathcal{S}_{\bar{K}}(\mathbf{G}, X)(T)$, we write \mathcal{A}_x for the pullback of \mathcal{A} to x and $s_{\alpha,*,x}$ for the pullback of $s_{\alpha,*}$ to x .

For T an $\mathcal{O}_{\mathbf{E}(v)}$ -scheme, an element $x \in \mathcal{S}_{\bar{K}}(\mathbf{G}, X)(T)$ corresponds to a triple $(\mathcal{A}_x, \lambda, \epsilon_{K'}^p)$, where λ is a weak polarization (cf. [Zho20, §6.3]) and $\epsilon_{K'}^p$ is a section of the étale sheaf $\underline{\text{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}_x), V_{\mathbb{A}_f^p})/K^p$; here

$$\widehat{V}(\mathcal{A}_x) = \varprojlim_{p \nmid n} \mathcal{A}_x[n]$$

is the adelic prime to p Tate module of \mathcal{A}_x . As in [Kis10, §3.4.2], $\epsilon_{K'}^p$ can be promoted to a section

$$\epsilon_K^p \in \Gamma(T, \underline{\text{Isom}}_{\lambda,\psi}(\widehat{V}(\mathcal{A}_x), V_{\mathbb{A}_f^p})/K^p)$$

which takes $s_{\alpha,\ell,x}$ to s_α for $\ell \neq p$.

5.1.5. Recall that k is an algebraic closure of k_E and $\check{\mathbb{Q}}_p = W(k)[1/p]$. Let $\bar{x} \in \mathcal{S}_{\bar{K}}(\mathbf{G}, X)(k)$ and $\tilde{x} \in \mathcal{S}_{\bar{K}}(\mathbf{G}, X)(\mathcal{O}_K)$ a point lifting \bar{x} , where $K/\check{\mathbb{Q}}_p$ is a finite extension.

Let $\mathcal{G}_{\tilde{x}}$ denote the p -divisible group associated to $\mathcal{A}_{\tilde{x}}$ and $\mathcal{G}_{\tilde{x}}$ its special fiber; we let $\mathbb{D} := \mathbb{D}(\mathcal{G}_{\tilde{x}})(\check{\mathbb{Z}}_p)$. Then $T_p \mathcal{G}_{\tilde{x}}^\vee$ is identified with $H_{\text{ét}}^1(\mathcal{A}_{\tilde{x},\bar{K}}, \mathbb{Z}_p)$ and we obtain $\text{Gal}(\bar{K}/K)$ -invariant tensors $s_{\alpha,p,\tilde{x}} \in T_p \mathcal{G}_{\tilde{x}}^{\vee \otimes}$ whose stabilizer can be identified with $\check{\mathcal{G}}$. Let $s_{\alpha,0,\tilde{x}} \in \mathbb{D}[\frac{1}{p}]^\otimes$ denote the tensors corresponding to $s_{\alpha,p,\tilde{x}}$ via the p -adic comparison isomorphism. By [KPS, Proposition 1.3.7], $s_{\alpha,0,\tilde{x}}$ are independent of the choice of lifting $\tilde{x} \in \mathcal{S}_K(\mathbf{G}, X)(\mathcal{O}_K)$. We may therefore denote them by $s_{\alpha,0,\bar{x}}$.

By Proposition 4.2.2, we have $s_{\alpha,0,\bar{x}} \in \mathbb{D}^\otimes$ and there is a $\check{\mathbb{Z}}_p$ -linear bijection

$$(5.1.5.1) \quad V_{\mathbb{Z}_p}^\vee \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \cong T_p \mathcal{G}_{\tilde{x}}^\vee \otimes_{\mathbb{Z}_p} \check{\mathbb{Z}}_p \cong \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Z}}_p$$

taking s_α to $s_{\alpha,0,\bar{x}}$. The filtration on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} K$ corresponding to $\mathcal{G}_{\tilde{x}}$ is induced by a G -valued cocharacter conjugate to μ_h^{-1} . By a result of Blasius and Wintenberger [Bla91], $s_{\alpha,\text{dR},\tilde{x}} \in \tilde{x}^*(\mathcal{V})^\otimes \cong \mathbb{D}(\mathcal{G}_{\tilde{x}})(\mathcal{O}_K)^\otimes$ corresponds to $s_{\alpha,p,\tilde{x}}$ via the p -adic comparison isomorphism. Hence $s_{\alpha,\text{dR},\tilde{x}}$ may be identified with the image of the elements $\tilde{s}_\alpha \in \mathbb{D}(\mathcal{G}_{\tilde{x}})(\widehat{W}(\mathcal{O}_K))^\otimes$ of Proposition 4.2.2 inside $\mathbb{D}(\mathcal{G}_{\tilde{x}})(\mathcal{O}_K)^\otimes$. The same Proposition implies that there is an \mathcal{O}_K -linear bijection

$$\mathbb{D}(\mathcal{G}_{\tilde{x}})(\mathcal{O}_K) \cong \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \mathcal{O}_K$$

taking $s_{\alpha, \text{dR}, \tilde{x}}$ to $s_{\alpha, 0, \bar{x}}$ and which lifts the identity over k . Thus there is a G -valued cocharacter μ_y which is G -conjugate to μ_h^{-1} and which induces a filtration on $\mathbb{D} \otimes_{\mathbb{Z}_p} \mathcal{O}_K$ lifting the filtration on $\mathbb{D} \otimes_{\mathbb{Z}_p} k$. We may therefore define the notion of $(\tilde{\mathcal{G}}, \mu_y)$ -adapted liftings as in §4 and it follows from Proposition 4.2.2 that $\mathcal{G}_{\tilde{x}}$ is a $(\tilde{\mathcal{G}}, \mu_y)$ -adapted lifting.

5.1.6. Note that $G \subset \text{GL}(V_{\mathbb{Q}_p})$ contains the scalars since it contains the image of w_h . It follows that under our assumptions, conditions (4.1.5.1)–(4.1.5.3) are satisfied. We let $P \subset \text{GL}(\mathbb{D})$ be a parabolic lifting P_0 as in §4.1. We obtain formal local models $\hat{M}^{\text{loc}} = \text{Spf} A$ and $\hat{M}_{\tilde{\mathcal{G}}}^{\text{loc}} = \text{Spf} A_{\tilde{\mathcal{G}}} \cong \hat{M}_{\tilde{\mathcal{G}}, \{\mu_h\}}^{\text{loc}}$, and the filtration corresponding to μ_y is given by a point $y : A_{\tilde{\mathcal{G}}} \rightarrow \mathcal{O}_K$.

Proposition 5.1.7. *Assume K^p is neat. Let $\hat{U}_{\bar{x}}$ be the completion of $\mathcal{S}_{\tilde{K}}(\mathbf{G}, X)^-$ at the image of \bar{x} .*

- (1) $\hat{U}_{\bar{x}}$ can be identified with a closed subspace of $\text{Spf} A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\tilde{E}}$ containing $\text{Spf} A_{\tilde{\mathcal{G}}}$.
- (2) A deformation \mathcal{G} of $\mathcal{G}_{\bar{x}}$ corresponds to a point on the irreducible component of $\hat{U}_{\bar{x}}$ containing \tilde{x} if and only if \mathcal{G} is $(\tilde{\mathcal{G}}, \mu_y)$ -adapted.
- (3) Let $\bar{x}' \in \mathcal{S}_{\tilde{K}}(\mathbf{G}, X)(k)$ whose image in $\mathcal{S}_{\tilde{K}}(\mathbf{G}, X)^-(k)$ coincides with that of \bar{x} . Then $s_{\alpha, 0, \bar{x}'} = s_{\alpha, 0, \bar{x}} \in \mathbb{D}^{\otimes}$ if and only if $\bar{x} = \bar{x}'$.

Proof. Since the conditions (4.1.5.1)–(4.1.5.3) are satisfied, we may apply the construction of Proposition 4.1.7; this allows us to view $\text{Spf} A$ as a versal deformation space for $\mathcal{G}_{\bar{x}}$ and hence we obtain a map $\Theta : \hat{U}_{\bar{x}} \rightarrow \text{Spf} A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\tilde{E}}$ such that the universal p -divisible group over $\text{Spf} A \otimes_{\mathbb{Z}_p} \mathcal{O}_{\tilde{E}}$ pulls back to the one over $\hat{U}_{\bar{x}}$ arising from the universal abelian scheme over $\hat{U}_{\bar{x}}$. The map Θ is a closed immersion by the Serre–Tate theorem.

Let $Z \subset \hat{U}_{\bar{x}}$ denote the irreducible component of $\hat{U}_{\bar{x}}$ containing \tilde{x} . Let K' be a finite extension of \tilde{E} and let $\tilde{x}' \in Z(K')$. Then the tensors $s_{\alpha, p, \tilde{x}'}$ correspond to $s_{\alpha, 0, \bar{x}}$ under the p -adic comparison isomorphism. Moreover the filtration on $\mathbb{D} \otimes_{\mathbb{Z}_p} K'$ corresponding to $\mathcal{G}_{\tilde{x}'}$ is induced by a G -valued cocharacter conjugate to μ_h^{-1} , and hence conjugate to μ_y . By Proposition 4.2.4, $\mathcal{G}_{\tilde{x}'}$ is a $(\tilde{\mathcal{G}}, \mu_y)$ -adapted deformation of $\mathcal{G}_{\bar{x}}$ and hence \tilde{x}' corresponds to a point of $\text{Spf} A_{\tilde{\mathcal{G}}}$. Since this is true for any \tilde{x}' , it follows that $\Theta|_Z$ factors through $\text{Spf} A_{\tilde{\mathcal{G}}}$. Since Z and $\text{Spf} A_{\tilde{\mathcal{G}}}$ have the same dimension, it follows that $Z \cong \text{Spf} A_{\tilde{\mathcal{G}}}$. We thus obtain (1) and (2).

One direction of (3) is clear. For the other direction, let $\tilde{x}' \in \mathcal{S}_{\tilde{K}}(\mathbf{G}, X)(\mathcal{O}_{K'})$ be a lift of \bar{x}' . Then by Proposition 4.2.2, $s_{\alpha, 0, \bar{x}'}$ arises from the specialization of tensors $\tilde{s}_{\alpha} \in \mathbb{D}(\mathcal{G}_{\tilde{x}'})(\widehat{W}(\mathcal{O}_K))$. By Assumption, we have $s_{\alpha, 0, \bar{x}'} = s_{\alpha, 0, \bar{x}}$. It follows that $\mathcal{G}_{\tilde{x}'}$ corresponds to a $(\tilde{\mathcal{G}}, \mu_y)$ -adapted lifting and hence to a point of $\text{Spf} A_{\tilde{\mathcal{G}}}$. By what we have seen, \tilde{x}' corresponds to a point in the same irreducible component $Z \subset \hat{U}_{\bar{x}}$ containing \tilde{x} and hence $\bar{x} = \bar{x}'$. \square

5.1.8. The above description of the local structure of $\mathcal{S}_{\tilde{K}}(\mathbf{G}, X)$ may be globalized as follows.

Theorem 5.1.9. (1) $\mathcal{S}_{\tilde{K}_p}(\mathbf{G}, X)$ is an $\mathcal{O}_{\mathbf{E}(v)}$ -flat, $\mathbf{G}(\mathbb{A}_f^p)$ -equivariant extension of $\text{Sh}_{\tilde{K}_p}(\mathbf{G}, X)$.

- (2) Assume K^p is neat. Let $\widehat{U}_{\bar{x}}$ be the completion of $\mathcal{S}_{\bar{K}}(\mathbf{G}, X)$ at some k -point \bar{x} . Then there exists a point $\bar{z} \in M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}}(k)$ such that $\widehat{U}_{\bar{x}}$ is isomorphic to the completion of $M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}}$ at \bar{z} .
- (3) $\mathcal{S}_{\bar{K}}(\mathbf{G}, X)$ fits in a local model diagram:

$$\begin{array}{ccc} & \widetilde{\mathcal{S}}_{\bar{K}}(\mathbf{G}, X)_{\mathcal{O}_E} & \\ \pi \swarrow & & \searrow q \\ \mathcal{S}_{\bar{K}}(\mathbf{G}, X)_{\mathcal{O}_E} & & M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}} \end{array}$$

where π is a $\widetilde{\mathcal{G}}$ -torsor and q is smooth of relative dimension $\dim G$.

Proof. (1) is clear and (2) follows from Proposition 5.1.7.

For (3), we first assume K^p is neat. Recall we have the vector bundle \mathcal{V} over $\mathcal{S}_{\bar{K}}(\mathbf{G}, X)$ corresponding to the de Rham cohomology of the universal abelian variety over $\mathcal{S}_{\bar{K}}(\mathbf{G}, X)$. Its generic fiber $\mathcal{V}_{\mathbf{E}}$ is equipped with tensors $s_{\alpha, \text{dR}} \in \mathcal{V}_{\mathbf{E}}^{\otimes}$ and these extend to \mathcal{V} by the same argument as [KP18, Proposition 4.2.6]. Moreover the argument of *loc. cit.* also shows that the scheme classifying isomorphisms $f : V_{\mathcal{O}_{\mathbf{E}(v)}}^{\vee} \cong \mathcal{V}$ which take s_{α} to $s_{\alpha, \text{dR}}$ is a $\widetilde{\mathcal{G}}$ -torsor $\widetilde{\mathcal{S}}_{\bar{K}}(\mathbf{G}, X)$.

Let (x, f) be an S -point of $\widetilde{\mathcal{S}}_{\bar{K}}(\mathbf{G}, X)_{\mathcal{O}_E}$. The map q is defined by sending (x, f) to the inverse image $f^{-1}(\mathcal{F}) \subset V_{\mathcal{O}_{\mathbf{E}(v)}}^{\vee} \otimes_{\mathcal{O}_{\mathbf{E}(v)}} \mathcal{O}_S$ of the Hodge filtration $\mathcal{F} \subset \mathcal{V}_x$. This gives us a map $\widetilde{\mathcal{S}}_{\bar{K}}(\mathbf{G}, X)_{\mathcal{O}_E} \rightarrow \text{Gr}(V_{\mathbb{Z}_p}^{\vee}) \otimes_{\mathbb{Z}_p} \mathcal{O}_E$ which factors through $M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}}$ by the argument of [KP18, Theorem 4.2.7], which also shows that q is smooth.

Now for a general (not necessarily neat) K^p , we let $K_1^p \subset K^p$ be a neat compact open subgroup which is normal in K^p . The action of K^p/K_1^p on $\mathcal{S}_{\bar{K}_p K_1^p}(\mathbf{G}, X)$ naturally extends to $\widetilde{\mathcal{S}}_{\bar{K}_p K_1^p}(\mathbf{G}, X)$, and the map

$$q_1 : \widetilde{\mathcal{S}}_{\bar{K}_p K_1^p}(\mathbf{G}, X)_{\mathcal{O}_E} \rightarrow M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}}$$

is compatible with this action. We thus obtain a diagram of stacks

(5.1.9.1)

$$\begin{array}{ccccc} & \widetilde{\mathcal{S}}_{\bar{K}_p K_1^p}(\mathbf{G}, X)_{\mathcal{O}_E} & \xrightarrow{\tilde{p}} & \widetilde{\mathcal{S}}_{\bar{K}}(\mathbf{G}, X)_{\mathcal{O}_E} & \\ \pi_1 \swarrow & & & \searrow \pi & \\ \mathcal{S}_{\bar{K}_p K_1^p}(\mathbf{G}, X)_{\mathcal{O}_E} & \xrightarrow{p} & \mathcal{S}_{\bar{K}}(\mathbf{G}, X)_{\mathcal{O}_E} & & \\ & & & & \searrow q \\ & & & & M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}} \end{array}$$

as desired. \square

5.1.10. We now use the above to study integral models for parahoric level structure. Let \mathbf{G}_{sc} denote the simply connected cover of \mathbf{G}_{der} and we set $\mathbf{C} := \ker(\mathbf{G}_{\text{sc}} \rightarrow \mathbf{G}_{\text{der}})$. For $c \in H^1(\mathbb{Q}, \mathbf{C})$ and ℓ a finite prime, we write c_{ℓ} for the image of c in $H^1(\mathbb{Q}_{\ell}, \mathbf{C})$. We introduce the following assumption.

(5.1.10.1) If $c \in H^1(\mathbb{Q}, \mathbf{C})$ satisfies $c_{\ell} = 0$ for all $\ell \neq p$, then $c_p = 0$.

There is a natural finite map of Shimura varieties $\text{Sh}_{\mathbf{K}}(\mathbf{G}, X) \rightarrow \text{Sh}_{\bar{\mathbf{K}}}(\mathbf{G}, X)$ and we define the integral model for parahoric level $\mathcal{S}_{\mathbf{K}}(\mathbf{G}, X)$ to be the normalization of $\mathcal{S}_{\bar{\mathbf{K}}}(\mathbf{G}, X)$ inside $\text{Sh}_{\mathbf{K}}(\mathbf{G}, X)$. We similarly write $\mathcal{S}_{K_p}(\mathbf{G}, X)$ for the inverse limit

over the prime to p levels. The discussion in [KP18, §4.3] extends verbatim to the current situation and we obtain the following proposition; cf. [KP18, Proposition 4.3.7, Corollary 4.3.9]

Proposition 5.1.11. *Assume (5.1.10.1) is satisfied.*

- (1) *The covering $\mathcal{S}_K(\mathbf{G}, X) \rightarrow \mathcal{S}_{\tilde{K}}(\mathbf{G}, X)$ is étale, and for K^p sufficiently small, this covering splits over an unramified extension.*
- (2) *The geometrically connected components of $\mathcal{S}_K(\mathbf{G}, X)$ are defined over the maximal extension \mathbf{E}^p of \mathbf{E} unramified at all primes above p .*

□

5.2. Integral models for Shimura varieties of abelian type. We now use the previous results to construct integral models for Shimura varieties of abelian type under the assumption of *acceptability* (cf. Definition 5.2.6). In particular, this will allow us to construct integral models for general Hodge type Shimura varieties without the assumptions in §5.1. This last case is all that is needed for our main application on ℓ -independence. However, since the general abelian type case is no more difficult, we also include this case for completeness. As many of the arguments are exactly the same as in [KP18, §4], in what follows we will refer to relevant statements in [KP18] if the argument in *loc. cit.* carries over directly and only give details for those points which do not.

5.2.1. We keep the notation of §5.1, so that (\mathbf{G}, X) is a Shimura datum of Hodge type and we set $G = \mathbf{G}_{\mathbb{Q}_p}$. Assume that (\mathbf{G}, X) satisfies the following conditions.

- The pair $(G, \{\mu_h\})$ is regular and $p \nmid |\pi_1(G_{\text{der}})|$.
- \mathbf{G} satisfies (5.1.10.1).
- The center Z of $G := \mathbf{G}_{\mathbb{Q}_p}$ is an R -smooth torus.

As before, we let $\mathcal{G} = \mathcal{G}_x$ be a parahoric group scheme corresponding to a point $x \in \mathcal{B}(G, \mathbb{Q}_p)$ which is generic in the facet containing it.

Let (\mathbf{G}_2, X_2) be a Shimura datum which is equipped with a central isogeny $\alpha : \mathbf{G}_{\text{der}} \rightarrow \mathbf{G}_{2, \text{der}}$ inducing an isomorphism $(\mathbf{G}_{\text{ad}}, X_{\text{ad}}) \cong (\mathbf{G}_{2, \text{ad}}, X_{2, \text{ad}})$. The parahoric \mathcal{G} determines a parahoric \mathcal{G}_2 of $G_2 := \mathbf{G}_2 \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and we set $K_{2, p} := \mathcal{G}_2(\mathbb{Z}_p)$. We write \mathbf{E}_2 for the reflex field of (\mathbf{G}_2, X_2) and we let $\mathbf{E}' := \mathbf{E} \cdot \mathbf{E}_2$. Our choice of embedding i_p induces a place v' (resp. v_2) of \mathbf{E}' (resp. \mathbf{E}_2) and we set $E' := \mathbf{E}'_{v'}$ and $E_2 := \mathbf{E}_{2, v_2}$ to be the completions.

Fix a connected component $X^+ \subset X$. By real approximation, upon modifying the isomorphism $\mathbf{G}_{\text{ad}} \cong \mathbf{G}_{2, \text{ad}}$ by an element of $\mathbf{G}_{\text{ad}}(\mathbb{Q})$, we may assume that the image of $X_2 \subset X_{2, \text{ad}}$ contains the image of X^+ . We write X_2^+ for X^+ viewed as a subset of X_2 . We denote by $\text{Sh}_{K_p}(\mathbf{G}, X)^+ \subset \text{Sh}_{K_p}(\mathbf{G}, X)$ and $\text{Sh}_{K_{2, p}}(\mathbf{G}_2, X_2)^+ \subset \text{Sh}_{K_{2, p}}(\mathbf{G}_2, X_2)$ the geometrically connected components corresponding to X^+ and X_2^+ . These are defined over extensions of \mathbf{E} and \mathbf{E}' respectively, which are unramified at primes above p . The identification $X_2^+ \simeq X^+$ induces a finite map

$$(5.2.1.1) \quad \text{Sh}_{K_p}(\mathbf{G}, X)^+ \rightarrow \text{Sh}_{K_{2, p}}(\mathbf{G}_2, X_2)^+$$

Let x_{ad} be the image of x in $\mathcal{B}(G_{\text{ad}}, \mathbb{Q}_p)$ and we denote by \mathcal{G}_{ad} the parahoric model of G_{ad} corresponding to x_{ad} . We then have the following generalization of [KP18, Corollary 4.6.18].

Proposition 5.2.2. *Under the assumptions above, there is a $\mathbf{G}_2(\mathbb{A}_f^p)$ -equivariant extension of $\mathrm{Sh}_{K_{2,p}}(\mathbf{G}_2, X_2)$ to an $\mathcal{O}_{E'}$ -scheme with $\mathbf{G}_2(\mathbb{A}_f^p)$ -action $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)$ such that*

- (1) *For any discrete valuation ring R of mixed characteristic the map*

$$\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)(R) \rightarrow \mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X)(R[\frac{1}{p}])$$

is a bijection

- (2) *The map (5.2.1.1) induces a finite map of $\mathcal{O}_{E'_{\mathrm{ur}}}$ -schemes*

$$\mathcal{S}_{K_p}(\mathbf{G}, X)^+ \rightarrow \mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)^+,$$

where $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)^+$ denotes the closure of $\mathrm{Sh}_{K_{2,p}}(\mathbf{G}_2, X_2)^+$ in the $\mathcal{O}_{E'_{\mathrm{ur}}}$ -scheme $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'_{\mathrm{ur}}}}$, and similarly for $\mathcal{S}_{K_p}(\mathbf{G}, X)^+$.

- (3) *If $\tilde{\mathcal{G}} = \mathcal{G}$, then there exists a diagram*

$$(5.2.2.1) \quad \begin{array}{ccc} & \widetilde{\mathcal{S}}_{K_{2,p}}^{\mathrm{ad}}(\mathbf{G}_2, X_2) & \\ \pi \swarrow & & \searrow q \\ \mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2) & & \mathrm{M}_{\mathcal{G}, \{\mu_h\}}^{\mathrm{loc}} \end{array}$$

where π is a $\mathbf{G}_2(\mathbb{A}_f^p)$ -equivariant $\mathcal{G}_{\mathrm{ad}}$ -torsor and q is smooth of relative dimension $\dim \mathbf{G}_{\mathrm{ad}}$, and $\mathbf{G}_2(\mathbb{A}_f^p)$ -equivariant, when $\mathrm{M}_{\mathcal{G}, \{\mu_h\}}^{\mathrm{loc}}$ is equipped with the trivial $\mathbf{G}_2(\mathbb{A}_f^p)$ -action.

Proof. This can be deduced from Theorem 5.1.9, as in [KP18, §4.4-4.6]. We explain only how the assumption of R -smoothness of Z is used.

Let $G_{\mathbb{Z}_{(p)}}$ (resp. $G_{\mathrm{ad}, \mathbb{Z}_{(p)}}$) denote the $\mathbb{Z}_{(p)}$ -model of \mathbf{G} (resp. \mathbf{G}_{ad}) associated to \mathcal{G} (resp. $\mathcal{G}_{\mathrm{ad}}$). Let \mathbf{Z} denote the center of \mathbf{G} and $Z_{\mathbb{Z}_{(p)}}$ the closure of \mathbf{Z} in $G_{\mathbb{Z}_{(p)}}$. By Proposition 2.4.12, the assumption of R -smoothness on $Z = \mathbf{Z}_{\mathbb{Q}_p}$ and descent implies that the natural map $G_{\mathbb{Z}_{(p)}}/Z_{\mathbb{Z}_{(p)}} \rightarrow G_{\mathrm{ad}, \mathbb{Z}_{(p)}}$ is an isomorphism. This gives us the analogue of [KP18, Lemma 4.6.2(2)], and allows us to carry out the constructions of §4.6 of *loc. cit.* \square

Let $K_2^p \subset \mathbf{G}_2(\mathbb{A}_f^p)$ be a compact open subgroup, and we write $K_2 := K_{2,p} K_2^p \subset \mathbf{G}_2(\mathbb{A}_f)$. Taking the quotient of the diagram (5.2.2.1) by K_2^p , we obtain

$$q : \widetilde{\mathcal{S}}_{K_2}^{\mathrm{ad}}(\mathbf{G}_2, X_2) \rightarrow \mathrm{M}_{\mathcal{G}, \{\mu_h\}}^{\mathrm{loc}},$$

a smooth morphism of $\mathcal{O}_{E'}$ -stacks of relative dimension $\dim \mathbf{G}_{\mathrm{ad}}$.

5.2.3. We recall some features of the construction in Proposition 5.2.2 which will be needed later. As in [KP18, §4.5.6], we set

$$\mathcal{A}(\mathbf{G}) := \mathbf{G}(\mathbb{A}_f)/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Q})_+/\mathbf{Z}(\mathbb{Q})} \mathbf{G}_{\mathrm{ad}}(\mathbb{Q})^+$$

$$\mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}}) := \mathbf{G}(\mathbb{A}_f^p)/\mathbf{Z}(\mathbb{Z}_{(p)})^- *_{\mathbf{G}(\mathbb{Z}_{(p)})_+/\mathbf{Z}(\mathbb{Z}_{(p)})} \mathbf{G}_{\mathrm{ad}}(\mathbb{Z}_{(p)})^+,$$

and as in [KP18, §4.6.3], we set

$$\mathcal{A}(\mathbf{G})^\circ := \mathbf{G}(\mathbb{Q})^-/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Q})/\mathbf{Z}(\mathbb{Q})} \mathbf{G}_{\mathrm{ad}}(\mathbb{Z}_{(p)})^+$$

$$\mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ := \mathbf{G}(\mathbb{Z}_{(p)})^-/\mathbf{Z}(\mathbb{Q})^- *_{\mathbf{G}(\mathbb{Z}_{(p)})_+/\mathbf{Z}(\mathbb{Z}_{(p)})} \mathbf{G}_{\mathrm{ad}}(\mathbb{Z}_{(p)})^+.$$

We refer to *loc. cit.* for an explanation of this notation. We obtain an $\mathcal{A}(\mathbf{G})$ -action (resp. $\mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})$ -action) on $\mathrm{Sh}(\mathbf{G}, X)$ (resp. $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$).

The assumption that the center of G is an R -smooth torus implies that the $\mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})$ -action on $\mathrm{Sh}_{K_p}(\mathbf{G}, X)$ extends to an $\mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})$ -action on $\mathcal{S}_{K_p}(\mathbf{G}, X)$. As in [KP18, Lemma 4.6.10], the natural map

$$(5.2.3.1) \quad \mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ \backslash \mathcal{A}(\mathbf{G}_{2, \mathbb{Z}_{(p)}}) \rightarrow \mathcal{A}(\mathbf{G})^\circ \backslash \mathcal{A}(\mathbf{G}_2) / K_{2,p}$$

is an injection, and we fix $J \subset \mathbf{G}_2(\mathbb{Q}_p)$ a set of coset representatives for the image of (5.2.3.1). Then $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)$ is constructed as

$$(5.2.3.2) \quad \mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2) = [[\mathcal{S}_{K_p}(\mathbf{G}, X)^+ \times \mathcal{A}(\mathbf{G}_{2, \mathbb{Z}_{(p)}})] / \mathcal{A}(\mathbf{G}_{\mathbb{Z}_{(p)}})^\circ]^{[J]}$$

5.2.4. Let H be a simple, adjoint, reductive group over \mathbb{R} , which is of classical type, and is associated to a Hermitian symmetric domain; in particular $H(\mathbb{R})$ is not compact. Thus H is of type $A, B, C, D^\mathbb{R}, D^\mathbb{H}$ in the classification of [Del79, 1.3.9], with the type A case including unitary groups of any signature $U(p, q)$ with $p, q \neq 0$. We set $H^\sharp = H_{\mathrm{sc}}$, the simply connected cover of H , unless H is of type $D^\mathbb{H}$, in which case we set H^\sharp equal to the image of H_{sc} in the representation corresponding to the standard representation of the orthogonal group.

Now let F be a totally real field, and \mathbf{H} a simple, adjoint reductive group of classical type over F . Assume that

- for every embedding $\sigma : F \hookrightarrow \mathbb{R}$, $\mathbf{H} \otimes_{\sigma, F} \mathbb{R}$ is either compact or associated to a Hermitian symmetric domain.
- $\mathbf{H} \otimes_{\sigma, F} \mathbb{R}$ is non-compact for some σ .
- If \mathbf{H} is of type D , then for those σ such that $\mathbf{H} \otimes_{\sigma, F} \mathbb{R}$ is non-compact, the associated Hermitian symmetric domain does not depend on σ . That is, it is always of type $D^\mathbb{R}$ or always of type $D^\mathbb{H}$.

We define \mathbf{H}^\sharp to be \mathbf{H}_{sc} unless \mathbf{H} is of type D , in which case we define \mathbf{H}^\sharp to be the unique quotient of \mathbf{H}_{sc} such that $\mathbf{H}^\sharp \otimes_{\sigma, F} \mathbb{R} = (\mathbf{H} \otimes_{\sigma, F} \mathbb{R})^\sharp$ whenever $\mathbf{H} \otimes_{\sigma, F} \mathbb{R}$ is non-compact.

Now suppose \mathbf{H} is a reductive group over F , with $\mathbf{H}^{\mathrm{ad}} = \prod_{i=1}^s \mathbf{H}_i$ where each \mathbf{H}_i is a simple, adjoint reductive group of classical type over F satisfying the three conditions above. Then we set $\mathbf{H}^\sharp = \prod_{i=1}^s \mathbf{H}_i^\sharp$.

Now let (\mathbf{H}, Y) be a Shimura datum such that $(\mathbf{H}_{\mathrm{ad}}, Y_{\mathrm{ad}})$ is of abelian type. Recall [Del79] that in this case the three conditions above are satisfied, so \mathbf{H}^\sharp is well defined², and (\mathbf{H}, Y) is of abelian type if and only if $\mathbf{H}_{\mathrm{der}}$ is a quotient of \mathbf{H}^\sharp .

5.2.5. Proposition 5.2.2 shows that we can construct good integral models for Shimura data (\mathbf{G}_2, X_2) of abelian type provided we can relate it to a Shimura datum (\mathbf{G}, X) of Hodge type satisfying good properties. Those (\mathbf{G}_2, X_2) for which we can do this are essentially the following

Definition 5.2.6. Let (\mathbf{G}_2, X_2) be a Shimura datum. We say that (\mathbf{G}_2, X_2) is *acceptable* if it is of abelian type and there is an isomorphism $G_{2, \mathrm{ad}} := \mathbf{G}_{2, \mathrm{ad}, \mathbb{Q}_p} \cong \prod_{i=1}^r \mathrm{Res}_{F_i/\mathbb{Q}_p} H_i$ where F_i/\mathbb{Q}_p is a finite extension and H_i is a reductive group over F_i which splits over a tamely ramified extension of F_i .

²In [KP18, 4.6.21] it is incorrectly asserted that \mathbf{H}^\sharp is defined for any (H, Y) with H of classical type, however H may not satisfy the third condition above. This is however satisfied if $(\mathbf{H}_{\mathrm{ad}}, Y_{\mathrm{ad}})$ is of abelian type.

The following lemma is the analogue of [KP18, Lemma 4.6.22], and is the key input that will allow us to deduce the existence of good integral models for acceptable Shimura data.

Proposition 5.2.7. *Let (\mathbf{G}_2, X_2) be an acceptable Shimura datum.*

Then there exists a Shimura datum (\mathbf{G}, X) of Hodge type together with a central isogeny $\mathbf{G}_{\text{der}} \rightarrow \mathbf{G}_{2,\text{der}}$ which induces an isomorphism $(\mathbf{G}_{\text{ad}}, X_{\text{ad}}) \cong (\mathbf{G}_{2,\text{ad}}, X_{2,\text{ad}})$. Moreover, (\mathbf{G}, X) may be chosen to satisfy the following conditions.

- (1) $\pi_1(\mathbf{G}_{\text{der}})$ is a 2-group and is trivial if $(\mathbf{G}_{2,\text{ad}}, X_{2,\text{ad}})$ has no factors of type $D^{\mathbb{H}}$. Moreover \mathbf{G} satisfies assumption (5.1.10.1).
- (2) Any prime $v_2|p$ of \mathbf{E}_2 splits in the composite $\mathbf{E}' := \mathbf{E}.\mathbf{E}_2$.
- (3) The center Z of G is an R -smooth torus over \mathbb{Q}_p .
- (4) $X_*(G_{\text{ab}})_I$ is torsion free.
- (5) The pair $(G, \{\mu_h\})$ is regular and $p \nmid |\pi_1(\mathbf{G}_{\text{der}})|$.

Proof. We follow the proof of [KP18, Lemma 4.6.22].

Let $\mathbf{G}_{2,\text{ad}} \cong \prod_{j=1}^s \text{Res}_{F_j/\mathbb{Q}} \mathbf{H}_j$, where F_j is a totally real field and \mathbf{H}_j is an absolutely simple F_j -group. By [Del79, 2.3.10], we may choose (\mathbf{G}, X) a Shimura datum of Hodge type with $\mathbf{G}_{\text{der}} \cong \mathbf{G}_{2,\text{ad}}^{\sharp}$, and such that the central isogeny $\mathbf{G}_{\text{der}} \rightarrow \mathbf{G}_{2,\text{der}}$ induces an isomorphism of Shimura data $(\mathbf{G}_{\text{ad}}, X_{\text{ad}}) \cong (\mathbf{G}_{2,\text{ad}}, X_{2,\text{ad}})$. Then \mathbf{G}_{der} has the form $\prod_{j=1}^s \text{Res}_{F_j/\mathbb{Q}} \mathbf{H}_j^{\sharp}$. As in [KP18, Lemma 4.6.22], it follows that (\mathbf{G}, X) satisfies (1).

In the course of constructing (\mathbf{G}, X) satisfying the other conditions, we will keep track of a certain group \mathbf{G}' containing \mathbf{G} such that the Hodge embedding $(\mathbf{G}, X) \rightarrow (\mathbf{GSp}(V), S^{\pm})$ extends to a representation $\mathbf{G}' \rightarrow \mathbf{GL}(V)$; this will be needed in the verification of (5).

We now explain how to choose (\mathbf{G}, X) satisfying (2). We first assume $s = 1$ so that $\mathbf{G}_{2,\text{ad}} \cong \text{Res}_{F/\mathbb{Q}} \mathbf{H}$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_d$ denote the primes of F above p and write F_i for the completion of F at \mathfrak{p}_i . Then $\mathbf{G}_{2,\text{ad},\mathbb{Q}_p} \cong \prod_{i=1}^d \text{Res}_{F_i/\mathbb{Q}_p} \mathbf{H}_{F_i}$, and our assumptions imply that $H_i := \mathbf{H}_{F_i}$ splits over a tamely ramified extension of F_i . We choose K/F a quadratic imaginary extension of F such that all primes of F above p split in K . We fix a set T of embeddings $K \rightarrow \mathbb{C}$ satisfying the same conditions as in [KP18, §4.6.22]. The construction of [Del79, Proposition 2.3.10] then gives a Shimura datum (\mathbf{G}, X) of Hodge type such that any prime $v_2|p$ of \mathbf{E}_2 splits in \mathbf{E}' . Moreover (\mathbf{G}, X) is constructed as a subgroup of a group \mathbf{G}' with $\mathbf{G}_{\text{der}} \simeq \mathbf{G}'_{\text{der}}$, $\mathbf{G}' \cong \text{Res}_{F/\mathbb{Q}} \mathbf{H}'$ and such that the Hodge embedding $(\mathbf{G}, X) \rightarrow (\mathbf{GSp}(V), S^{\pm})$ extends to a representation $\mathbf{G}' \rightarrow \mathbf{GL}(V)$. The group \mathbf{G}' splits over the composite of K and the splitting field of \mathbf{G} . It follows that $\mathbf{G}'_{\mathbb{Q}_p} \cong \prod_{i=1}^d \text{Res}_{F_i/\mathbb{Q}_p} H'_i$ where H'_i splits over a tamely ramified extension of F_i . In general for $s > 1$, we apply the above to each of the individual factors.

We now show that we can arrange so that (3) is satisfied. Let $\mathbf{G} \subset \mathbf{G}'$ as above and set $G' := \mathbf{G}'_{\mathbb{Q}_p}$. Let T' denote the centralizer of a maximal $\check{\mathbb{Q}}_p$ -split torus in G' defined over \mathbb{Q}_p and let $T := G \cap T'$ which is a maximal torus of G . Then $T' \cong \prod_{i=1}^r \text{Res}_{F_i/\mathbb{Q}_p} S'_i$ where F_i/\mathbb{Q}_p is finite and S'_i is a torus over F_i which splits over a tamely ramified extension. By construction of \mathbf{G} in [Del79, Proposition 2.3.10], for $i = 1, \dots, r$ there are induced tori S''_i over F_i which split over a tamely ramified extension and maps $S'_i \rightarrow S''_i$ which induce a map $T' \rightarrow T'' := \prod_{i=1}^r \text{Res}_{F_i/\mathbb{Q}_p} S''_i$ such that T is the identity component of the pullback $T' \times_{T''} \mathbb{G}_m$. Here $\mathbb{G}_m \rightarrow T''$

is the diagonal map. Thus T arises from the construction in Corollary 2.4.7 and hence is R -smooth. Arguing as in [Kis10, Proof of Prop 2.2.4], we may choose a maximal torus \mathbf{T} of \mathbf{G} such that $\mathbf{T}_{\mathbb{Q}_p}$ is $\mathbf{G}(\mathbb{Q}_p)$ -conjugate to T , and there exists $h \in X$ such that h factors through $\mathbf{T}_{\mathbb{R}}$. In fact, we may choose \mathbf{T} to be given by $\mathbf{T}' \cap \mathbf{G}$, where $\mathbf{T}' \subset \mathbf{G}'$ is a torus such that $\mathbf{T}'_{\mathbb{Q}_p}$ is $\mathbf{G}'(\mathbb{Q}_p)$ -conjugate to T' . We set $\mathbf{G}_1 := (\mathbf{G} \times \mathbf{T})/\mathbf{Z}$ and $\mathbf{G}'_1 := (\mathbf{G}' \times \mathbf{T}')/\mathbf{Z}'$, where \mathbf{Z} and \mathbf{Z}' are the centers of \mathbf{G} and \mathbf{G}' respectively. Then the center Z_1 of $G_1 := \mathbf{G}_{1,\mathbb{Q}_p}$ is isomorphic to T and hence an R -smooth torus.

We let X_1 denote the conjugacy class of Deligne homomorphisms for \mathbf{G}_1 determined by $h \times 1$ for $h \in X$. As in [KP18, Lemma 4.6.22], we let W denote the \mathbf{G}_1 -representation $\mathrm{Hom}_{\mathbf{Z}}(V, V)$, and we may equip W with an alternating form such that there is a Hodge embedding $(\mathbf{G}_1, X_1) \rightarrow (\mathbf{GSp}(W), S_1^{\pm})$. By construction, this extends to a homomorphism $\mathbf{G}'_1 \rightarrow \mathbf{GL}(W)$. Moreover, if we let $Z = \mathbf{Z}_{\mathbb{Q}_p}$ we take $T_1 := (T \times T)/Z \subset G_1$ which is the centralizer of a maximal \check{F} -split torus in G_1 , then T_1 also arises from the construction in Corollary 2.4.7; it is the identity component of the pullback $(T' \times T')/Z' \times_{T''} \mathbb{G}_m$ where $Z' := \mathbf{Z}'_{\mathbb{Q}_p}$. Thus T_1 is R -smooth. This observation will be needed below to insure that (5) is satisfied. Upon replacing (\mathbf{G}, X) by (\mathbf{G}_1, X_1) we may assume (\mathbf{G}, X) satisfies (3).

To show we can arrange so that (4) and (5) are satisfied, we may apply the same construction as in [KP18, Lemma 4.6.22] to (\mathbf{G}, X) . This gives a Shimura datum (\mathbf{G}_1, X_1) of Hodge type with $X_*(G_{1,\mathrm{ab}})_I$ torsion free, i.e. condition (4) is satisfied. A similar argument as the one in the previous paragraph shows that the Hodge embedding $(\mathbf{G}_1, X_1) \rightarrow (\mathbf{GSp}(V), S^{\pm})$ extends to an embedding $\mathbf{G}'_1 \rightarrow \mathbf{GL}(V)$ for a suitable \mathbf{G}'_1 of the form $\prod_{j=1}^s \mathrm{Res}_{F_j/\mathbb{Q}} \mathbf{H}'_j$. Moreover, the explicit description of \mathbf{G}_1 shows that both the center Z_1 of $G_1 = \mathbf{G}_{1,\mathbb{Q}_p}$ and the centralizer T of a maximal $\check{\mathbb{Q}}_p$ -split torus in G_1 arise from the construction in Corollary 2.4.7. It follows that $(G_1, \{\mu_{h_1}\})$ is regular. Since we have assumed $p > 2$, condition (1) implies $p \nmid |\pi_1(G_{\mathrm{der}})|$ and hence condition (5) is satisfied. \square

5.2.8. For later applications to constructing canonical liftings, we introduce the following additional condition on the parahoric.

Definition 5.2.9. Let (\mathbf{G}_2, X_2) be an acceptable Shimura datum and \mathcal{G}_2 a parahoric group scheme for $G_2 = \mathbf{G}_{2,\mathbb{Q}_p}$. We say the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is *acceptable* if we can choose a Shimura datum as in Proposition 5.2.7 such that the corresponding parahoric \mathcal{G} of $G = \mathbf{G}_{\mathbb{Q}_p}$ is connected.

Corollary 5.2.10. Let (\mathbf{G}_2, X_2) be an acceptable Shimura datum and \mathcal{G}_2 any parahoric group scheme of G_2 . Assume $(\mathbf{G}_{2,\mathrm{ad}}, X_{2,\mathrm{ad}})$ has no factors of type $D^{\mathbb{H}}$. Then the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is acceptable.

Proof. Let (\mathbf{G}, X) be as in Proposition 5.2.7 and \mathcal{G} the corresponding parahoric group scheme of G . Since $\pi_1(G_{\mathrm{der}})$ is trivial, we have $\pi_1(G) \cong X_*(G_{\mathrm{ab}})$. Thus $\pi_1(G)_I \cong X_*(G_{\mathrm{ab}})_I$ is torsion free and hence the Kottwitz map $\tilde{\kappa}_G$ is trivial on $\tilde{\mathcal{G}}$. It follows that \mathcal{G} is a connected parahoric. \square

Remark 5.2.11. The assumption of acceptability on the triple above is what is needed to construct canonical liftings in §5.3. We remark that it is possible for a triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ to be acceptable even if $(\mathbf{G}_{2,\mathrm{ad}}, X_{2,\mathrm{ad}})$ has factors of type $D^{\mathbb{H}}$, cf. Proposition 7.2.3; thus it is a more general notion than just excluding $D^{\mathbb{H}}$ factors.

5.2.12. Proposition 5.2.7 shows that if (\mathbf{G}_2, X_2) is acceptable, it can be related to a Hodge type Shimura datum (\mathbf{G}, X) satisfying the assumptions in §5.2.1. We thus obtain the following theorem; the argument is the same as [KP18, Theorem 4.6.23].

Theorem 5.2.13. *Let (\mathbf{G}_2, X_2) be an acceptable Shimura datum. Let \mathcal{G}_2 be a parahoric group scheme of G_2 and set $K_{2,p} := \mathcal{G}_2(\mathbb{Z}_p)$.*

Then there exists a Shimura datum of Hodge type (\mathbf{G}, X) such that the conditions of Proposition 5.2.2 are satisfied and such that all primes $v_2|p$ of \mathbf{E}_2 split completely in $\mathbf{E}' = \mathbf{E}.\mathbf{E}_2$. In particular for any prime $v_2|p$ of \mathbf{E}_2 , we obtain a $\mathbf{G}_2(\mathbb{A}_f^p)$ -equivariant \mathcal{O}_{E_2} -scheme $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)$ with the following properties.

- (1) $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)$ is étale locally isomorphic to $M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}}$, where \mathcal{G} is the parahoric group scheme of G corresponding to \mathcal{G}_2 .
- (2) For any discrete valuation ring R of mixed characteristic the map

$$\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)(R) \rightarrow \mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)(R[\frac{1}{p}])$$

is a bijection.

- (3) If the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is acceptable, then (\mathbf{G}, X) can be chosen so that for any compact open subgroup $K_2 = K_{2,p}K_2^p \subset \mathbf{G}_2(\mathbb{A}_f)$, there exists a diagram of \mathcal{O}_{E_2} -stacks

$$\begin{array}{ccc} & \widetilde{\mathcal{S}}_{K_2}(\mathbf{G}_2, X_2) & \\ \pi \swarrow & & \searrow q \\ \mathcal{S}_{K_2}(\mathbf{G}_2, X_2) & & M_{\mathcal{G}, \{\mu_h\}}^{\text{loc}} \end{array}$$

where $\mathcal{S}_{K_2}(\mathbf{G}_2, X_2) := \mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)/K_2^p$, π is a \mathcal{G}_{ad} -torsor and the map q is smooth of relative dimension $\dim \mathbf{G}_{\text{ad}}$. In particular, such a diagram exists if $(\mathbf{G}_{2,\text{ad}}, X_{2,\text{ad}})$ has no factors of type D^{H} .

□

Remark 5.2.14. (1) If $p > 2$, then every abelian type Shimura datum (\mathbf{G}_2, X_2) is acceptable. Thus this Theorem essentially completes the construction of integral models for abelian type Shimura varieties with parahoric level over primes $p > 3$. Moreover for $p = 3$, only the case when $G_{2,\text{ad}}$ has a factor of type D_4 needs to be excluded.

- (2) The local model diagram in Theorem 5.2.13 (3), is a weaker form of the diagram postulated in [HR17]. However, for our applications, the important property is that $\widetilde{\mathcal{S}}_{K_2}(\mathbf{G}_2, X_2) \rightarrow \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)$ is a torsor for a *connected* smooth \mathcal{O}_{E_2} -group scheme.

5.3. μ -ordinary locus and canonical liftings.

5.3.1. We keep the notation of §5.2. We let (\mathbf{G}_2, X_2) be an acceptable Shimura datum (cf. Definition 5.2.6) and $K_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ where \mathcal{G}_2 is a parahoric group scheme of $G_2 := \mathbf{G}_{2,\mathbb{Q}_p}$. In this subsection, we study the μ -ordinary locus in the special fiber of integral models and show that when (\mathbf{G}_2, X_2) is Hodge type and the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is acceptable (cf. Definition 5.2.9), each μ -ordinary point contains a special lifting. Similar to the construction of integral models, the result is deduced from the corresponding statement in some special Hodge type cases.

By Theorem 5.2.13, we may construct an integral model $\mathcal{S}_{K_2}(\mathbf{G}_2, X_2)/\mathcal{O}_{E_2}$ for $\mathrm{Sh}_{K_2}(\mathbf{G}_2, X_2)$ from an auxiliary Shimura datum (\mathbf{G}, X) of Hodge type as in the conclusion of Proposition 5.2.7 equipped with a good Hodge embedding $\iota : (\mathbf{G}, X) \rightarrow (\mathrm{GSp}(V), S^\pm)$. In particular (\mathbf{G}, X) satisfies the conditions in §5.2.1. We fix such a (\mathbf{G}, X) and ι for the rest of this section.

We assume that (\mathbf{G}_2, X_2) is Hodge type and we fix a Hodge embedding $\iota_2 : (\mathbf{G}_2, X_2) \rightarrow (\mathbf{GSp}(V_2), S_2^\pm)$. By the main theorem of [Lan00], there is a $G_2(\mathbb{Q}_p^{\mathrm{ur}})$ -equivariant embedding of buildings $\mathcal{B}(G_2, \mathbb{Q}_p^{\mathrm{ur}}) \rightarrow \mathcal{B}(\mathrm{GSp}(V_2, \mathbb{Q}_p), \mathbb{Q}_p^{\mathrm{ur}})$. Upon replacing ι_2 with a new Hodge embedding, we may assume there is a \mathbb{Z}_p -lattice $V_{2, \mathbb{Z}_p} \subset V_{2, \mathbb{Q}_p}$ with $V_{2, \mathbb{Z}_p} \subset V_{2, \mathbb{Z}_p}^\vee$ such that $G_2 \rightarrow \mathrm{GSp}(V_{2, \mathbb{Q}_p})$ extends to a morphism of Bruhat–Tits stabilizer schemes $\tilde{G}_2 \rightarrow \mathcal{GSP}$, where \mathcal{GSP} is the group scheme stabilizer of V_{2, \mathbb{Z}_p} (cf. [BT84, Proposition 1.7.6]). We set $K'_{2, p} := \mathcal{GSP}(\mathbb{Z}_p)$ and we let $K_2'^p \subset \mathbf{GSp}(V_{2, \mathbb{A}_f^p})$ a compact open subgroup containing K_2^p .

Proposition 5.3.2. *There is a map of \mathcal{O}_{E_2} -stacks*

$$(5.3.2.1) \quad \mathcal{S}_{K_2}(\mathbf{G}_2, X_2) \rightarrow \mathcal{S}_{K_2'}(\mathbf{GSp}(V_2), S_2^\pm)_{\mathcal{O}_{E_2}}$$

extending the natural map on the generic fiber.

Proof. Let \mathbf{Z} denote the center of \mathbf{G} and we write \mathbf{Z}^{sp} for the connected component of the identity of the kernel of the multiplier homomorphism $c : \mathbf{GSp}(V) \rightarrow \mathbb{G}_m$ restricted to \mathbf{Z} . We define a subgroup $\mathbf{G}_3 \subset \mathbf{GL}(V) \times \mathbf{GL}(V_2)$ generated by $\mathbf{Z}^{\mathrm{sp}} \times 1$, the image of $\mathbf{G}_{\mathrm{der}}$ under the the product of ι and $\mathbf{G}_{\mathrm{der}} \rightarrow \mathbf{G}_{2, \mathrm{der}} \xrightarrow{\iota} \mathbf{GSp}(V_2)$, and the diagonal torus $\mathbb{G}_m \subset \mathbf{GL}(V) \times \mathbf{GL}(V_2)$. Set $V_3 = V \oplus V_2$ which we may equip with a perfect alternating bilinear form induced from V and V_2 . As in [Zha, §4.3], there is a conjugacy class of Deligne homomorphisms X_3 for \mathbf{G}_3 such that (\mathbf{G}_3, X_3) is a Shimura datum, and there are natural morphisms of Shimura data

$$(\mathbf{G}, X) \longleftarrow (\mathbf{G}_3, X_3) \longrightarrow (\mathbf{G}_2, X_2) \longrightarrow (\mathbf{GSp}(V_2), S_2^\pm).$$

Moreover using the explicit description of \mathbf{G}_3 and our assumption on ι_2 above, one checks that the Hodge embedding $\iota_3 : (\mathbf{G}_3, X_3) \rightarrow (\mathbf{GSp}(V_3), S_3^\pm)$ is a good Hodge embedding.

We can now conclude the proof by applying the arguments of [Zha]. More precisely, when \mathbf{G}_2 is tamely ramified the result follows from [Zha, Proposition 5.4], but the same arguments work since we have constructed integral models in a more general situation: Let \mathcal{G} and \mathcal{G}_3 denote the parahoric group schemes of $G = \mathbf{G}_{\mathbb{Q}_p}$ and $G_3 = \mathbf{G}_{3, \mathbb{Q}_p}$ corresponding to \mathcal{G}_2 , and set $K_p = \mathcal{G}(\mathbb{Z}_p)$, $K_{3, p} = \mathcal{G}_3(\mathbb{Z}_p)$. Arguing as in [Zha, Theorem 4.6], we obtain maps on connected components

$$(5.3.2.2) \quad \begin{aligned} \mathcal{S}_{K_p}(\mathbf{G}, X)_{\mathcal{O}_{E_2}^{\mathrm{ur}}}^+ &\simeq \mathcal{S}_{K_{3, p}}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E_2}^{\mathrm{ur}}}^+ \\ &\rightarrow \mathcal{S}_{K_{2, p}}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E_2}^{\mathrm{ur}}}^+ \rightarrow \mathcal{S}_{K_p'}(\mathbf{GSp}(V_2), S_2^\pm)_{\mathcal{O}_{E_2}^{\mathrm{ur}}}^+. \end{aligned}$$

We may then apply the argument of [Zha, Proposition 5.4], noting that the diagram (5.3.1) of *loc. cit.* exists in our setting. \square

5.3.3. Let $h : \mathcal{A}^2 \rightarrow \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)$ denote the pullback of the universal abelian variety along (5.3.2.1). Let $s_\alpha \in V_2^\otimes$ be a collection of tensors whose stabilizer is \mathbf{G}_2 . Then as in §5.1.4, these give rise to tensors $s_{\alpha, B} \in V_B := R^1 h_{\mathrm{an}*} \mathbb{Q}$, $s_{\alpha, \ell} \in \mathcal{V}_\ell(\mathcal{A}^2) := R^1 h_{\mathrm{ét}*} \mathbb{Q}_\ell$ for all $\ell \neq p$ and $s_{\alpha, p} \in \mathcal{V}_p(\mathcal{A}^2) := R^1 h_{\eta, \mathrm{ét}*} \mathbb{Q}_p$. For any \mathcal{O}_{E_2} -scheme T and $x \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)$, we write \mathcal{A}_x^2 for the pullback of \mathcal{A}^2 to x .

For K/\mathbb{Q}_p finite and $\tilde{x} \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ with special fiber \bar{x} , we let $s_{\alpha,0,\tilde{x}} \in \mathbb{D}(\mathcal{A}_{\bar{x}}^2[p^\infty])[1/p]^\otimes$ denote the images of $s_{\alpha,p,\tilde{x}}$ under the p -adic comparison isomorphism. As in §5.1.5, these tensors depend only on \bar{x} and not on \tilde{x} ; we thus write $s_{\alpha,0,\bar{x}}$ for these tensors. Note that [KPS, Proposition 1.3.7] applies here since the morphism $\mathcal{S}_{K_2}(\mathbf{G}_2, X_2) \rightarrow \mathcal{S}_{K'_2}(\mathbf{GSp}(V_2), S_2^\pm)_{\mathcal{O}_{E_2}}$ factors through the normalization of its scheme theoretic image, and all objects are pulled back from this normalization.

5.3.4. Let $\bar{x} \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(k)$, and set $\mathbb{D} := \mathbb{D}(\mathcal{A}_{\bar{x}}^2[p^\infty])$. We fix an isomorphism

$$V_{2,\mathbb{Z}_p}^\vee \otimes_{\mathbb{Z}_p} \check{\mathbb{Q}}_p \cong \mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p,$$

taking s_α to $s_{\alpha,0,\bar{x}}$; such an isomorphism exists by Steinberg's theorem (cf. [KPS, 1.3.8]). Then the Frobenius on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$ is given by $b\sigma$ for some $b \in G_2(\check{\mathbb{Q}}_p)$. By [KPS, Lemma 1.3.9], we have $[b] \in B(G_2, \{\mu_2\})$ where $\{\mu_2\} = \{\mu_{h_2}^{-1}\}$. We write \mathcal{S}_{K_2} (resp. $\mathcal{S}_{K_{2,p}}$) for the special fiber of $\mathcal{S}_{K_2}(\mathbf{G}_2, X_2)$ (resp. $\mathcal{S}_{K_{2,p}}(\mathbf{G}_2, X_2)$) over the residue field k_{E_2} of \mathcal{O}_{E_2} . The map $\mathcal{S}_{K_2}(k) \rightarrow B(G_2, \{\mu_2\})$ sending \bar{x} to the σ -conjugacy class $[b]$ of the associated element b induces the Newton stratification of $\mathcal{S}_{K_{2,k}} := \mathcal{S}_{K_2} \otimes_{k_{E_2}} k$. Let $[b] \in B(G_2, \{\mu_2\})$, we write $\mathcal{S}_{K_{2,[b]}} \subset \mathcal{S}_{K_{2,k}}$ for the strata corresponding to $[b]$; if K_2^p is neat, it is a locally closed subscheme of $\mathcal{S}_{K_{2,k}}$. Similarly, we write $\mathcal{S}_{K_{2,p,[b]}} = \lim_{\leftarrow K_2^p} \mathcal{S}_{K_{2,p}K_2^p,[b]}$; such a definition makes sense since $\mathcal{S}_{K_{2,[b]}}$ is compatible with the prime to p level. For the rest of §5.3 we assume the existence of the class $[b]_{\mu_2} \in B(G_2, \{\mu_2\})$ as in Definition 2.2.4.

Definition 5.3.5. We define the μ_2 -ordinary locus of $\mathcal{S}_{K_{2,k}}$ to be $\mathcal{S}_{K_{2,[b]_{\mu_2}}}$.

5.3.6. We say that a parahoric subgroup $K_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ is very special if $\mathcal{G}_2(\check{\mathbb{Z}}_p)$ is a special parahoric subgroup of $G_2(\check{\mathbb{Q}}_p)$. Note that such a parahoric exists if and only if G_2 is quasi-split (cf. [Zhu14, Lemma 6.1]). The following is deduced easily from [KPS, Corollary 1.3.16].

Theorem 5.3.7. *Assume G_2 is quasi-split, $K_{2,p} = \mathcal{G}_2(\mathbb{Z}_p)$ is a very special parahoric subgroup and K_2^p is neat. Then*

- (1) \mathcal{S}_{K_2} is normal.
- (2) The μ_2 -ordinary locus $\mathcal{S}_{K_{2,[b]_{\mu_2}}}$ is Zariski open and dense in $\mathcal{S}_{K_{2,k}}$.

Proof. To show (1), it suffices by Theorem 5.2.13 to show that the special fiber of $M_{\mathcal{G},\{\mu_h\}}^{\text{loc}}$ is normal. For this, it suffices by Theorem 3.1.6 to show that the special fiber is integral. This follows from the argument in [PZ13, Corollary 9.4], noting that as in *loc. cit.* the μ -admissible set $\text{Adm}(\{\mu\})_J$ has a single extremal element when $J \subset \mathbb{S}$ corresponds to a very special standard parahoric of $G(\check{\mathbb{Q}}_p)$.

(2) follows from (1) by [KPS, Corollary 1.3.16]. \square

5.3.8. Let $\bar{x} \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(k)$. Define $\text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}}^2)$ to be the \mathbb{Q} -group whose points in a \mathbb{Q} -algebra R is given by

$$\text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}}^2)(R) = (\text{End}(\mathcal{A}_{\bar{x}}^2) \otimes_{\mathbb{Z}} R)^\times$$

By functoriality, $\text{Aut}_{\mathbb{Q}}\mathcal{A}_{\bar{x}}^2$ acts on $T_\ell\mathcal{A}_{\bar{x}}^2 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ for $\ell \neq p$ and on $\mathbb{D} \otimes_{\check{\mathbb{Z}}_p} \check{\mathbb{Q}}_p$, and we write $I_{\bar{x}}$ for the closed subgroup of $\text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}}^2)$ consisting of automorphisms which preserve $s_{\alpha,\ell,\bar{x}}$ and $s_{\alpha,0,\bar{x}}$. There is a canonical inclusion $I_{\bar{x}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \subset J_b$, where J_b is the σ -centralizer group for $b \in G_2(\check{\mathbb{Q}}_p)$.

The goal of the rest of this section is to prove the following theorem.

Theorem 5.3.9. *Assume the triple $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is acceptable. Let $\bar{x} \in \mathcal{S}_{K_2, [b]_{\mu_2}}(k)$. Then \bar{x} admits a lifting to a special point $\tilde{x} \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(K)$ for some K/\mathbb{Q}_p finite such that the action of $I_{\bar{x}}(\mathbb{Q})$ on $\mathcal{A}_{\bar{x}}^2$ lifts to an action (in the isogeny category) on $\mathcal{A}_{\tilde{x}}^2$.*

Remark 5.3.10. The statement of the Theorem and all the constructions above implicitly depend on the choice of auxiliary Shimura datum (\mathbf{G}, X) and the choice of Hodge embeddings ι and ι_2 . It is possible to show that they are independent of the choices, but we will not consider this and always work with a fixed choice of (\mathbf{G}, X) and ι, ι_2 .

5.3.11. Note that $(\mathbf{G}, X, \mathcal{G})$ is also an acceptable triple with (\mathbf{G}, X) Hodge type. Theorem 5.3.9 will be reduced to the following special case.

Proposition 5.3.12. *Assume $(\mathbf{G}_2, X_2, \mathcal{G}_2) = (\mathbf{G}, X, \mathcal{G})$ and the Hodge embeddings ι and ι_2 coincide. Then Theorem 5.3.9 holds.*

Proof. Under these assumptions, we have $\mathcal{S}_K(\mathbf{G}, X) = \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)$ and the integral model is constructed as in §5.1.3. Moreover \mathcal{G}_2 is a connected parahoric. Since the definition of $I_{\bar{x}}$ is independent of the prime to p level, it suffices to consider the case of neat K_2^p . Applying the construction in §4.3, we obtain a parahoric model \mathcal{M} of a Levi subgroup $M \subset G_2$, and an M -valued cocharacter $\tilde{\lambda}$ lying in the G_2 -conjugacy class of μ_2 and such that $\tilde{\lambda}$ is central in M . Let \mathcal{G} be the $(\mathcal{M}, \tilde{\lambda})$ -adapted deformation to \mathcal{O}_K constructed in Theorem 4.3.6. By Proposition 5.1.7, \mathcal{G} corresponds to a point $\tilde{x} \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ lifting \bar{x} and hence to an abelian variety $\mathcal{A}_{\tilde{x}}^2$ over K . By Theorem 4.3.6, the action of $J_b(\mathbb{Q}_p)$ on $\mathcal{G}_{\tilde{x}}$ lifts to \mathcal{G} . Since $I_{\bar{x}}(\mathbb{Q}) \subset J_b(\mathbb{Q}_p)$, by the Serre–Tate theorem, the action of $I_{\bar{x}}$ lifts to $\mathcal{A}_{\tilde{x}}^2$ in the isogeny category.

We now show \tilde{x} is a special point. Since $I_{\bar{x}}$ fixes the tensors $s_{\alpha, 0, \bar{x}}$, it also fixes $s_{\alpha, p, \tilde{x}}$, and hence it fixes $s_{\alpha, B}$. Thus we may consider $I_{\bar{x}}$ as a subgroup of \mathbf{G}_2 . By [KPS, Theorem 6], the absolute rank of $I_{\bar{x}}$ is equal to the absolute rank of \mathbf{G}_2 . Let \mathbf{T} be a maximal torus of $I_{\bar{x}}$, which is therefore a maximal torus of \mathbf{G}_2 . The Mumford–Tate group of $\mathcal{A}_{\tilde{x}}^2$ is a subgroup of \mathbf{G}_2 which commutes with \mathbf{T} hence must be contained in \mathbf{T} . Therefore \tilde{x} is a special point. \square

5.3.13. To prove Theorem 5.3.9 in general, we make use of the following auxiliary construction. For notational convenience, we write (\mathbf{G}_1, X_1) for (\mathbf{G}, X) and $\iota_1 : (\mathbf{G}_1, X_1) \rightarrow (\mathbf{GSp}(V_1), S_1^{\pm})$ for the good Hodge embedding ι .

We define \mathbf{G}_3 to be the identity component of $(\mathbf{G}_1 \times_{\mathbf{G}_{2, \text{ad}}} \mathbf{G}_2) \times_{\mathbb{G}_m \times \mathbb{G}_m} \mathbb{G}_m$, where $\mathbf{G}_1 \times_{\mathbf{G}_{2, \text{ad}}} \mathbf{G}_2 \rightarrow \mathbb{G}_m \times \mathbb{G}_m$ is induced by composing with the multiplier homomorphisms $c_1 : \mathbf{GSp}(V_1) \rightarrow \mathbb{G}_m$, $c_2 : \mathbf{GSp}(V_2) \rightarrow \mathbb{G}_m$, and $\mathbb{G}_m \rightarrow \mathbb{G}_m \times \mathbb{G}_m$ is the diagonal embedding. Let $h_1 \in X_1$ and $h_2 \in X_2$ which have the same image in $X_{2, \text{ad}}$; such a pair exists by our choice of $\mathbf{G}_{1, \text{ad}} \cong \mathbf{G}_{2, \text{ad}}$ (cf. §5.2.1). Then $h_1 \times h_2$ factors through \mathbf{G}_3 and determines a $\mathbf{G}_{3, \mathbb{R}}$ conjugacy class of Deligne homomorphisms X_3 such that (\mathbf{G}_3, X_3) is a Shimura datum. There are natural morphisms of Shimura data

$$(\mathbf{G}_1, X_1) \longleftarrow (\mathbf{G}_3, X_3) \longrightarrow (\mathbf{G}_2, X_2).$$

For $i = 1, 2, 3$, let \mathbf{E}_i denote the reflex field of (\mathbf{G}_i, X_i) ; then we have $\mathbf{E}_3 \subset \mathbf{E}' := \mathbf{E}_1 \mathbf{E}_2$. We let v_i (resp. v') denote the place of \mathbf{E}_i (resp. \mathbf{E}') induced by the

embedding i_p and we let E_i (resp. E') denote the completion. By construction, we have $E' = E_2$. Set $G_i := \mathbf{G}_{i, \mathbb{Q}_p}$, and let \mathcal{G}_1 (resp. \mathcal{G}_3) denote the parahoric subgroup of G_1 (resp. G_3) determined by \mathcal{G}_2 . For $i = 1, 2, 3$, we set $K_{i,p} := \mathcal{G}_i(\mathbb{Z}_p)$ and we fix compact open subgroups $K_i^p \subset \mathbf{G}_i(\mathbb{A}_f^p)$ such that K_3^p maps to K_1^p and K_2^p . We set $K_i := K_{i,p} K_i^p$.

5.3.14. Let \mathbf{H} denote the subgroup of $\mathbf{GSp}(V_1) \times \mathbf{GSp}(V_2)$ consisting of elements (g_1, g_2) such that $c_1(g_1) = c_1(g_2)$. Then the natural map $\mathbf{G}_3 \rightarrow \mathbf{GSp}(V_1) \times \mathbf{GSp}(V_2)$ factors through \mathbf{H} and we let S' denote the $\mathbf{H}_{\mathbb{R}}$ -conjugacy class of homomorphisms $\mathbb{S} \rightarrow \mathbf{H}_{\mathbb{R}}$ induced by X_3 .

Set $V_3 := V_1 \oplus V_2$. We equip V_3 with a perfect alternating bilinear form given by the sum of the forms on V_1 and V_2 . Then there are natural morphisms of Shimura data $(\mathbf{H}, S') \rightarrow (\mathbf{GSp}(V_i), S_i^{\pm})$ for $i = 1, 2, 3$. Recall we have fixed a \mathbb{Z}_p -lattice $V_{2, \mathbb{Z}_p} \subset V_{2, \mathbb{Q}_p}$; we let $V_{1, \mathbb{Z}_p} \subset V_{1, \mathbb{Q}_p}$ be a \mathbb{Z}_p -lattice such that ι_1 is good with respect to V_{1, \mathbb{Z}_p} . We set $V_{3, \mathbb{Z}_p} := V_{1, \mathbb{Z}_p} \oplus V_{2, \mathbb{Z}_p} \subset V_{3, \mathbb{Q}_p}$. For $i = 1, 2, 3$, we let $K'_{i,p}$ denote the stabilizer of V_{i, \mathbb{Z}_p} inside $\mathbf{GSp}(V_{i, \mathbb{Q}_p})$ and let H_p denote the stabilizer of V_{3, \mathbb{Z}_p} inside $\mathbf{H}(\mathbb{Q}_p)$. We also fix compact open subgroups $K_i'^p \subset \mathbf{GSp}(V_{i, \mathbb{A}_f^p})$ containing the image of K_i^p for $i = 1, 2, 3$, $H^p \subset \mathbf{H}(\mathbb{A}_f^p)$ containing the image of K_3^p , and we set $K'_i = K'_{i,p} K_i'^p$, $H = H_p H^p$.

The Shimura variety $\mathrm{Sh}_{\mathbf{H}}(\mathbf{H}, S')$ has a moduli interpretation as pairs of tuples $(A^i, \lambda_i, \epsilon_i^p)$, $i = 1, 2$, where A^i is an abelian variety up to prime to p isogeny, λ_i is a weak polarization and ϵ_i^p is a prime to p level structure and hence extends to an integral model $\mathcal{S}_{\mathbf{H}}(\mathbf{H}, S')$ over $\mathbb{Z}_{(p)}$.

Proposition 5.3.15. *There is a commutative diagram of $\mathcal{O}_{E'}$ -stacks*
(5.3.15.1)

$$\begin{array}{ccccc} \mathcal{S}_{K_1}(\mathbf{G}_1, X_1)_{\mathcal{O}_{E'}} & \xleftarrow{j_1} & \mathcal{S}_{K_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}} & \xrightarrow{j_2} & \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'}} \\ \downarrow i_1 & & \downarrow i_3 & & \downarrow i_2 \\ \mathcal{S}_{K'_1}(\mathbf{GSp}(V_1), S_1^{\pm})_{\mathcal{O}_{E'}} & \xleftarrow{\quad} & \mathcal{S}_{K'_3}(\mathbf{H}, S')_{\mathcal{O}_{E'}} & \xrightarrow{\quad} & \mathcal{S}_{\mathbf{H}}(\mathbf{GSp}(V_2), S_2^{\pm})_{\mathcal{O}_{E'}} \end{array}$$

Proof. It suffices to consider the case of neat prime to p level structure so that we may assume all objects are schemes. The existence of the bottom row follows from the moduli interpretations of the integral models. The morphisms in the top row can be constructed using the same argument as [Zha, Proposition 5.4] noting that all the models are constructed via (\mathbf{G}_1, X_1) .

The morphism i_1 exists by construction of $\mathcal{S}_{K_1}(\mathbf{G}_1, X_1)_{\mathcal{O}_{E'}}$. The morphism i_2 is constructed in Proposition 5.3.2 and i_3 can be constructed in the same way. The commutativity then follows from the commutativity on the generic fiber. \square

5.3.16. Composing i_3 and the natural map $\mathcal{S}_{\mathbf{H}}(\mathbf{H}, S')_{\mathcal{O}_{E'}} \rightarrow \mathcal{S}_{K'_2}(\mathbf{GSp}(V_3), S_3^{\pm})_{\mathcal{O}_{E'}}$ we obtain a map $\mathcal{S}_{K_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}} \rightarrow \mathcal{S}_{K'_3}(\mathbf{GSp}(V_3), S_3^{\pm})_{\mathcal{O}_{E'}}$. Therefore we may apply the constructions of §5.3.3 to $\mathcal{S}_{K_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}}$.

Let $\mathcal{A}^i \rightarrow \mathcal{S}_{K_i}(\mathbf{G}_i, X_i)_{\mathcal{O}_{E'}}$, denote the pullback of the universal abelian variety along $\mathcal{S}_{K_i}(\mathbf{G}_i, X_i)_{\mathcal{O}_{E'}} \rightarrow \mathcal{S}_{K'_i}(\mathbf{GSp}(V_i), S_i^{\pm})_{\mathcal{O}_{E'}}$. For $i = 3$, this map factors through $\mathcal{S}_{\mathbf{H}}(\mathbf{H}, S')_{\mathcal{O}_{E'}}$ and there is an identification

$$(5.3.16.1) \quad \mathcal{A}^3 \cong j_1^* \mathcal{A}^1 \times j_2^* \mathcal{A}^2.$$

Let $\bar{x}_3 \in \mathcal{S}_{K_3}(\mathbf{G}_3, X_3)(k)$ and write $\bar{x}_1 \in \mathcal{S}_{K_1}(\mathbf{G}_1, X_1)(k)$, $\bar{x}_2 \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(k)$ for the image of \bar{x}_3 under j_1 and j_2 . The isomorphism (5.3.16.1) implies we have an isomorphism $\mathcal{A}_{\bar{x}_3}^3 \cong \mathcal{A}_{\bar{x}_1}^1 \times \mathcal{A}_{\bar{x}_2}^2$. We let $I_{\bar{x}_3} \subset \text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}_3}^3)$, $I_{\bar{x}_2} \subset \text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}_1}^1)$ denote the groups constructed in the same way as §5.3.8.

Proposition 5.3.17. *There are natural exact sequences:*

$$0 \longrightarrow \mathbf{C}_1 \longrightarrow I_{\bar{x}_3} \longrightarrow I_{\bar{x}_1} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{C}_2 \longrightarrow I_{\bar{x}_3} \longrightarrow I_{\bar{x}_2} \longrightarrow 0$$

where \mathbf{C}_1 (resp. \mathbf{C}_2) is the kernel of the map $f : \mathbf{G}_3 \rightarrow \mathbf{G}_1$ (resp. $g : \mathbf{G}_3 \rightarrow \mathbf{G}_2$).

Proof. Since $\mathbf{G}_3 \subset \mathbf{H}$, we may assume that the set of tensors defining $\mathbf{G}_3 \subset \mathbf{GL}(V_3)$ includes tensors corresponding to the projections of $V_{3, \mathbb{Z}_{(p)}}$ onto the direct summands $V_{i, \mathbb{Z}_{(p)}} \subset V_{3, \mathbb{Z}_{(p)}}$ for $i = 1, 2$. It follows that $I_{\bar{x}_3}$ respects the product decomposition $\mathcal{A}_{\bar{x}_3}^3 \cong \mathcal{A}_{\bar{x}_1}^1 \times \mathcal{A}_{\bar{x}_2}^2$ and hence we obtain a natural map $I_{\bar{x}_3} \rightarrow \text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}_1}^1)$. Similarly, by considering the pullback to V_3 of tensors defining \mathbf{G}_1 , one can show that $I_{\bar{x}_3} \rightarrow \text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\bar{x}_1}^1)$ factors through $I_{\bar{x}_1}$. We obtain a natural map $I_{\bar{x}_3} \rightarrow I_{\bar{x}_1}$.

Let $\tilde{x}_3 \in \mathcal{S}_{K_3}(\mathbf{G}_3, X_3)(\mathcal{O}_K)$ denote a lift of \bar{x}_3 . Since \mathbf{C}_1 lies in the center of \mathbf{G}_3 , we have natural maps

$$\mathbf{C}_1 \rightarrow \text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\tilde{x}_3}^3 \otimes_K \overline{K}) \rightarrow \text{Aut}_{\mathbb{Q}}(\mathcal{A}_{\tilde{x}_3, k}^3)$$

whose image lies in $I_{\bar{x}_3}$.

We thus obtain a sequence $\mathbf{C}_1 \rightarrow I_{\bar{x}_3} \rightarrow I_{\bar{x}_1}$ and it suffices to check the exactness upon base changing to \mathbb{Q}_{ℓ} for some prime $\ell \neq p$. By [KPS, Theorem 6] there is a semisimple element $\gamma_{\ell} \in \mathbf{G}_3(\mathbb{Q}_{\ell})$ such that the natural inclusion $I_{\bar{x}_3} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \subset \mathbf{G}_{3, \mathbb{Q}_{\ell}}$ (resp. $I_{\bar{x}_1} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell} \subset \mathbf{G}_{1, \mathbb{Q}_{\ell}}$) identifies $I_{\bar{x}_3} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$ (resp. $I_{\bar{x}_1} \otimes_{\mathbb{Q}} \mathbb{Q}_{\ell}$) with the centralizer of γ_{ℓ} in $\mathbf{G}_{3, \mathbb{Q}_{\ell}}$ (resp. $f(\gamma_{\ell})$ in $\mathbf{G}_{1, \mathbb{Q}_{\ell}}$). We thus obtain the first exact sequence and the argument for $I_{\bar{x}_2}$ is analogous. \square

5.3.18. We can now prove the general case of Theorem 5.3.9.

Proof of Theorem 5.3.9. It suffices to consider the case of neat prime to p level structure. For $i = 1, 2, 3$, we write \mathcal{S}_{K_i} for the special fiber of the integral model $\mathcal{S}_{K_i}(\mathbf{G}_i, X_i)$. Let $\bar{x}_2 \in \mathcal{S}_{K_2, [b]_{\mu_2}}(k)$. We first assume $\bar{x}_2 = j_2(\bar{x}_3)$ for some $\bar{x}_3 \in \mathcal{S}_{K_3}(k)$; by Lemma 2.2.8 we have $\bar{x}_3 \in \mathcal{S}_{K_3, [b]_{\mu_3}}(k)$. Let $\bar{x}_1 \in \mathcal{S}_{K_1, [b]_{\mu_1}}(k)$ denote the image of \bar{x}_3 . By Proposition 5.3.12, there exists $K/\check{\mathbb{Q}}_p$ finite and $\tilde{x}_1 \in \text{Sh}_{K_1}(\mathbf{G}_1, X_1)(K)$ lifting \bar{x}_1 such that the action of $I_{\bar{x}_1}(\mathbb{Q})$ lifts to $\mathcal{A}_{\tilde{x}_1}^1$. Then we may consider $I_{\bar{x}_1}$ as a subgroup of \mathbf{G}_1 and we let \mathbf{T}_1 denote the connected component of the center of $I_{\bar{x}_1}$. The Mumford–Tate group of $\mathcal{A}_{\tilde{x}_1}^1$ is a connected subgroup of \mathbf{G}_1 which commutes with $I_{\bar{x}_1}$, hence is contained in \mathbf{T}_1 , as $I_{\bar{x}_1}$ and \mathbf{G}_1 have the same rank.

Let $\mathbf{T}_3 \subset \mathbf{G}_3$ denote the identity component of the preimage of \mathbf{T}_1 in \mathbf{G}_3 and \mathbf{T}_2 the image of \mathbf{T}_3 in \mathbf{G}_2 . By construction, the morphisms of integral models

$$\mathcal{S}_{K_1}(\mathbf{G}_1, X_1)_{\mathcal{O}_{E'}} \leftarrow \mathcal{S}_{K_3}(\mathbf{G}_3, X_3)_{\mathcal{O}_{E'}} \rightarrow \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)_{\mathcal{O}_{E'}}$$

induce isomorphisms of the completions at geometric points in the special fiber. Thus let \tilde{x}_3 (resp. \tilde{x}_2) denote the point lifting \bar{x}_3 (resp. \bar{x}_2) corresponding to \tilde{x}_1 . Then the Mumford–Tate group for $\mathcal{A}_{\tilde{x}_3}^3$ (resp. $\mathcal{A}_{\tilde{x}_2}^2$) is contained in \mathbf{T}_3 (resp. \mathbf{T}_2). It follows from Proposition 5.3.17 that $I_{\bar{x}_3}$ (resp. $I_{\bar{x}_2}$) is contained in the centralizer

of \mathbf{T}_3 in \mathbf{G}_3 (resp. \mathbf{T}_2 in \mathbf{G}_2), and hence the action of $I_{\bar{x}_2}(\mathbb{Q})$ lifts to an action on $\mathcal{A}_{\bar{x}_2}$.

Now let $\bar{x}_2 \in \mathcal{S}_{K_2, [b]_{\mu_2}}(k)$ be any point. It suffices to prove the result with $\mathcal{S}_{K_2, p}(\mathbf{G}_2, X_2)$ in place of $\mathcal{S}_{K_2}(\mathbf{G}_2, X_2)$, and with \bar{x}_2 replaced by a lift to a point of $\mathcal{S}_{K_2, p, [b]_{\mu_2}}(k)$, which we will again denote \bar{x}_2 . Recall $J \subset \mathbf{G}_2(\mathbb{Q}_p)$ is a set of coset representatives for the image of (5.2.3.1). Then by the construction of $\mathcal{S}_{K_2, p}(\mathbf{G}_2, X_2)$ via $\mathcal{S}_{K_1, p}(\mathbf{G}_1, X_1)$ in §5.2.3, there exists $j \in J$ such that $\bar{x}_2 \in [\mathcal{S}_{K_1, p}(\mathbf{G}_1, X_1)^+ \times \mathcal{A}(\mathbf{G}_{2, \mathbb{Z}(p)})j] / \mathcal{A}(\mathbf{G}_{1, \mathbb{Z}(p)})^\circ$. We let $\bar{x}'_2 \in [\mathcal{S}_{K_1, p}(\mathbf{G}_1, X_1)^+ \times \mathcal{A}(\mathbf{G}_{2, \mathbb{Z}(p)})] / \mathcal{A}(\mathbf{G}_{1, \mathbb{Z}(p)})^\circ$ be the point corresponding to \bar{x}_2 under the isomorphism induced by j . Then upon modifying \bar{x}_2 by an element of $\mathbf{G}_2(\mathbb{A}_f^p)$ which only changes the abelian variety $\mathcal{A}_{\bar{x}_2}^2$ up to prime to p isogeny, we may assume $\bar{x}'_2 = j_2(\bar{x}_3)$ for some $\bar{x}_3 \in \mathcal{S}_{K_3, p}(\mathbf{G}_3, X_3)(k)$.

Let $\tilde{x}'_2 \in \mathcal{S}_{K_2, p}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ be a lift of \bar{x}'_2 , for some finite extension $K/\check{\mathbb{Q}}_p$. By construction, corresponding to the element j , there is (after possibly increasing K) a point $\tilde{x}_2 \in \mathcal{S}_{K_2, p}(\mathbf{G}_2, X_2)(\mathcal{O}_K)$ lifting \bar{x}_2 , and a p -power quasi-isogeny $\mathcal{A}_{\tilde{x}_2}^2 \rightarrow \mathcal{A}_{\tilde{x}'_2}^2$ taking $s_{\alpha, 0, \tilde{x}_2}$ to $s_{\alpha, 0, \tilde{x}'_2}$ (resp. $s_{\alpha, \ell, \tilde{x}_2}$ to $s_{\alpha, \ell, \tilde{x}'_2}$ for $\ell \neq p$). By considering the reduction of this quasi-isogeny one sees that $\bar{x}'_2 \in \mathcal{S}_{K_2, p, [b]_{\mu}}(k)$, and one also obtains an induced isomorphism $I_{\bar{x}_2} \cong I_{\bar{x}'_2}$. From what we saw above, it follows that we may choose \tilde{x}'_2 such that the action of $I_{\bar{x}'_2}$ lifts to $\mathcal{A}_{\tilde{x}'_2}^2$. Then the action of $I_{\bar{x}_2} \cong I_{\bar{x}'_2}$ lifts to $\mathcal{A}_{\tilde{x}_2}^2$. \square

5.3.19. We will use the above to deduce properties about the conjugacy class of Frobenius as in [Kis17, §2.3]. Assume $\bar{x} \in \mathcal{S}_{K_2, [b]_{\mu_2}}(k)$ arises from an \mathbb{F}_q -point $x \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(\mathbb{F}_q)$ where \mathbb{F}_q is a finite extension of k_{E_2} . For $\ell \neq p$ a prime, let γ_ℓ denote the geometric q -Frobenius in $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the dual of the ℓ -adic Tate module $T_\ell \mathcal{A}_{\bar{x}}^{2\vee}$. Since the tensors $s_{\alpha, \ell, \bar{x}} \in T_\ell \mathcal{A}_{\bar{x}}^{2, \otimes}$ are Galois-invariant, we may consider γ_ℓ as an element of $\mathbf{G}_2(\mathbb{Q}_\ell)$ via the level structure $V_{\mathbb{Q}_\ell} \cong T_\ell \mathcal{A}_{\bar{x}}^2 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$.

Corollary 5.3.20. *Assume $(\mathbf{G}_2, X_2, \mathcal{G}_2)$ is an acceptable triple of Hodge type. Suppose $\bar{x} \in \mathcal{S}_{K_2, [b]_{\mu_2}}(k)$ arises from $x \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(\mathbb{F}_q)$. There exists an element $\gamma_0 \in \mathbf{G}_2(\mathbb{Q})$, such that*

- (1) *For $\ell \neq p$, γ_0 is conjugate to γ_ℓ in $\mathbf{G}_2(\mathbb{Q}_\ell)$.*
- (2) *γ_0 is elliptic in $\mathbf{G}_2(\mathbb{R})$.*

Proof. The proof is the same as in [Kis17, Corollary 2.3.1]. Since $\mathcal{A}_{\bar{x}}^2$ is defined over \mathbb{F}_q , the q -Frobenius γ lies in $I_{\bar{x}}(\mathbb{Q})$. Let $\tilde{x} \in \mathcal{S}_{K_2}(\mathbf{G}_2, X_2)(K)$ denote the lifting constructed in Theorem 5.3.9. Then by considering the action of $I_{\bar{x}}(\mathbb{Q})$ on the Betti cohomology of $\mathcal{A}_{\bar{x}}$, we may consider $I_{\bar{x}}(\mathbb{Q})$ as a subgroup of $\mathbf{G}_2(\mathbb{Q})$. Defining γ_0 to be the image of γ inside $\mathbf{G}_2(\mathbb{Q})$, we have that γ_0 is conjugate to γ_ℓ in $\mathbf{G}_2(\mathbb{Q}_\ell)$ by the Betti-étale comparison isomorphism. If \mathbf{T} is any torus in $I_{\bar{x}}$ containing γ_0 , the positivity of the Rosati involution implies $\mathbf{T}(\mathbb{R})/w_{h_2}(\mathbb{R}^\times)$ is compact. Hence $\gamma_0 \in \mathbf{T}(\mathbb{Q})$ is elliptic in $\mathbf{G}(\mathbb{R})$. \square

6. INDEPENDENCE OF ℓ FOR SHIMURA VARIETIES

We apply the results of the previous section to deduce an ℓ -independence result for the conjugacy class of Frobenius at all points on the special fiber of Shimura varieties.

6.1. Frobenius conjugacy classes.

6.1.1. We keep the notation of the previous section but now (\mathbf{G}, X) will be an acceptable Shimura datum of Hodge type. As before we let \mathcal{G} be a parahoric group scheme of $G = \mathbf{G}_{\mathbb{Q}_p}$ and set $K_p = \mathcal{G}(\mathbb{Z}_p)$. Then we have the integral model $\mathcal{S}_K(\mathbf{G}, X)$ over \mathcal{O}_E constructed from a fixed auxiliary Hodge type Shimura datum (\mathbf{G}_1, X_1) as in Proposition 5.2.7 and a good Hodge embedding ι_1 . The auxiliary Shimura datum (\mathbf{G}_1, X_1) plays a minor role in what follows.

Let $p > 2$ and $\ell \neq p$ be primes and suppose that in addition the compact open subgroup $K \subset \mathbf{G}(\mathbb{A}_f)$ is of the form $K_\ell K^\ell$. We let $\tilde{\mathbb{L}}_\ell$ denote the $\mathbf{G}(\mathbb{Q}_\ell)$ -local system on $\mathcal{S}_K(\mathbf{G}, X)$ arising from the pro-étale covering

$$\mathcal{S}_{K^\ell}(\mathbf{G}, X) := \varprojlim_{K'_\ell \subset K_\ell} \mathcal{S}_{K'_\ell K^\ell}(\mathbf{G}, X) \rightarrow \mathcal{S}_K(\mathbf{G}, X)$$

and we write \mathbb{L}_ℓ denote the induced local system on the special fiber \mathcal{S}_K over k_E . If $\iota : (\mathbf{G}, X) \rightarrow (\mathbf{GSp}(V), S^\pm)$ is a Hodge embedding as in §5.3.1 then we have an identification

$$(6.1.1.1) \quad \mathbb{L}_\ell = \underline{\text{Isom}}_{(s_\alpha, s_{\alpha, \ell})}(V_{\mathbb{Q}_\ell}, \mathcal{V}_\ell^\vee)$$

where the scheme classifies \mathbb{Q}_ℓ -linear isomorphisms taking s_α to $s_{\alpha, \ell}$; here the notation is as in §5.3.3.

6.1.2. Let $y \in \mathcal{S}_K(\mathbb{F}_q)$ and we write \bar{y} for the induced geometric point of \mathcal{S}_K . We let \mathcal{S}_K^0 denote the connected component of \mathcal{S}_K containing y and $\bar{x} \in \mathcal{S}_K^0(k)$ a fixed geometric point. Over \mathcal{S}_K^0 , the $\mathbf{G}(\mathbb{Q}_\ell)$ -local system \mathbb{L}_ℓ corresponds to a homomorphism

$$\rho_\ell^0 : \pi_1(\mathcal{S}_K^0, \bar{x}) \rightarrow \mathbf{G}(\mathbb{Q}_\ell).$$

We have a map

$$\text{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \rightarrow \pi_1(\mathcal{S}_K^0, \bar{y}) \xrightarrow{\sim} \pi_1(\mathcal{S}_K^0, \bar{x}),$$

where the isomorphism $\pi_1(\mathcal{S}_K^0, \bar{y}) \xrightarrow{\sim} \pi_1(\mathcal{S}_K^0, \bar{x})$ is well-defined up to conjugation. We thus obtain a well defined conjugacy class in $\pi_1(\mathcal{S}_K^0, \bar{x})$ corresponding to the image of the geometric q -Frobenius and we write Frob_y for a representative of this conjugacy class.

6.1.3. For any reductive group H over a field F of characteristic 0, we write Conj_H for the variety of semisimple conjugacy classes in H . Explicitly, if $H = \text{Spec } R$, then we have $\text{Conj}_H \cong \text{Spec } R^H$, where H acts on R via conjugation. The set $\text{Conj}_H(\overline{F})$ can be identified with the set of semisimple $H(\overline{F})$ conjugacy classes in $H(\overline{F})$. We write $\chi_H : H \rightarrow \text{Conj}_H$ for the projection map. For example if $H = \text{GL}_n$, $\text{Conj}_{\text{GL}_n}$ is the variety $\mathbb{A}_F^{n-1} \times \mathbb{G}_{m, F}$ and the map χ takes an element of GL_n to its associated characteristic polynomial.

In our setting, we thus obtain for each prime $\ell \neq p$, a well-defined element $\gamma_{y, \ell} \in \text{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ corresponding to $\chi_{\mathbf{G}}(\rho_\ell^0(\text{Frob}_y))$. Our main Theorem concerning the ℓ -independence property of Shimura varieties is the following.

Theorem 6.1.4. *Let $p > 2$. Assume $G = \mathbf{G}_{\mathbb{Q}_p}$ is quasi-split, \mathcal{G} is a very special parahoric group scheme and that $(\mathbf{G}, X, \mathcal{G})$ is an acceptable triple of Hodge type. Let $y \in \mathcal{S}_K(\mathbb{F}_q)$ where \mathbb{F}_q/k_E is a finite extension. Then there exists an element $\gamma_0 \in \text{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that $\gamma_0 = \gamma_{y, \ell} \in \text{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.*

Remark 6.1.5. Unlike in Corollary 5.3.20, it is not always possible to lift γ to an element of $\mathbf{G}(\mathbb{Q})$.

The rest of §6 will be devoted to the proof of Theorem 6.1.4.

6.2. Explicit curves in the special fiber of local models.

6.2.1. We begin by recalling the local model diagram and certain properties of the Kottwitz–Rapoport stratification. By Theorem 5.2.13 (3), there exists a diagram of stacks

$$(6.2.1.1) \quad \begin{array}{ccc} & \widetilde{\mathcal{S}}_{\mathbf{K}}^{\text{ad}}(\mathbf{G}, X) & \\ \pi \swarrow & & \searrow q \\ \mathcal{S}_{\mathbf{K}}(\mathbf{G}, X) & & \mathcal{M}_{\mathcal{G}_1, \{\mu_{h_1}\}}^{\text{loc}} \end{array}$$

where $\pi : \widetilde{\mathcal{S}}_{\mathbf{K}}^{\text{ad}}(\mathbf{G}, X) \rightarrow \mathcal{S}_{\mathbf{K}}(\mathbf{G}, X)$ is a \mathcal{G}_{ad} -torsor. Here \mathcal{G}_{ad} is the parahoric group scheme of $G_{1,\text{ad}} \cong G_{\text{ad}}$ corresponding to \mathcal{G} .

Let \mathcal{M} denote the special fiber of $\mathcal{M}_{\mathcal{G}_1, \{\mu_{h_1}\}}^{\text{loc}}$; it is a scheme over k_E . Recall the local model is defined using a group $G' \cong \prod_{i=1}^r \text{Res}_{F_i/\mathbb{Q}_p} H$ such that there exists a central extension $G'_{\text{der}} \rightarrow G_{\text{der}}$, and the parahoric group scheme \mathcal{G}' of G' is determined by \mathcal{G} ; then the geometric special fiber \mathcal{M}_k has a stratification indexed by $\text{Adm}_{G'}(\{\mu\})_{J'}$. Here we consider $\text{Adm}_{G'}(\{\mu\})_{J'} \subset W'_{J'} \backslash W'/W'_{J'}$ where W' is the Iwahori Weyl group for G' and $J' \subset \mathbb{S}'$ is the subset of simple reflections for G' determined by \mathcal{G}' . We write \mathcal{M}_k^w for the strata corresponding to $w \in \text{Adm}_{G'}(\{\mu\})_{J'}$. It follows formally from the existence of the diagram (6.2.1.1) that $\mathcal{S}_{\mathbf{K},k}$ admits a stratification by $\text{Adm}_{G'}(\{\mu\})_{J'}$. This is known as the Kottwitz–Rapoport stratification and we write $\mathcal{S}_{\mathbf{K},k}^w$ for the strata corresponding to $w \in \text{Adm}_{G'}(\{\mu\})_{J'}$. From the definition of this stratification, for $\bar{x} \in \mathcal{S}_{\mathbf{K}}(k)$ the complete local ring of $\mathcal{S}_{\mathbf{K},k}^w$ at \bar{x} is identified with the complete local ring at a point $\bar{x}' \in \mathcal{M}_k^w(k)$. The closure relations for this stratification is given by the Bruhat order on $W'_{J'} \backslash W'/W'_{J'}$.

6.2.2. For the rest of §6, we assume $(\mathbf{G}, X, \mathcal{G})$ satisfies the assumptions in Theorem 6.1.4. In this case, \mathcal{M}_k and $\mathcal{S}_{\mathbf{K},k}$ are normal schemes; cf. Theorem 5.3.7.

We let $\mathfrak{s} \in \mathcal{B}(G, \check{\mathbb{Q}}_p)$ denote the special vertex associated to \mathcal{G} . This determines a special vertex $\mathfrak{s}' \in \mathcal{B}(G', \check{\mathbb{Q}}_p)$. In this case the set $\text{Adm}_{G'}(\{\mu\})_{J'}$ has the following alternative description. Let S' denote a maximal $\check{\mathbb{Q}}_p$ -split torus of G' defined over \mathbb{Q}_p such that $\mathfrak{s}' \in \mathcal{A}(G', S', \check{\mathbb{Q}}_p)$ and T' the centralizer of S' . Fix a Borel subgroup of G' defined over \mathbb{Q}_p and assume we have identified $X_*(T')_I \otimes_{\mathbb{Z}} \mathbb{R}$ with $\mathcal{A}(G', S', \check{\mathbb{Q}}_p)$ via the choice of special vertex \mathfrak{s}' . We may consider μ as an element of $X_*(T')_I$. For $\lambda, \lambda' \in X_*(T')_I^+$, we write $\lambda \preceq \lambda'$ if $\lambda' - \lambda$ is an *integral* linear combination of positive coroots in the reduced root system Σ' associated to G' ; we write $\lambda \prec \lambda'$ if in addition $\lambda \neq \lambda'$. Then there is an identification

$$W'_{J'} \backslash W'/W'_{J'} \cong X_*(T')_I^+,$$

and the ordering \preceq agrees with the Bruhat order on $W'_{J'} \backslash W'/W'_{J'}$, under this identification (cf. [Lus83]). It follows that we have an identification

$$\text{Adm}_{G'}(\{\mu\})_{J'} = \{t_{\lambda} | \lambda \in X_*(T')_I^+, \lambda \preceq \mu\}.$$

We will write \mathcal{M}_k^{λ} (resp. $\mathcal{S}_{\mathbf{K},k}^{\lambda}$) for the strata $\mathcal{M}_k^{t_{\lambda}}$ (resp. $\mathcal{S}_{\mathbf{K},k}^{t_{\lambda}}$).

6.2.3. For notational simplicity, we will use $\underline{\mathcal{G}}$ to denote the group $\underline{\mathcal{G}}'_{\mathbb{F}_p[[t]]}$ defined in §3.1.9. Its generic fiber will be denoted \underline{G} and the Iwahori Weyl group $W_{\underline{G}}$ may be identified with the Iwahori Weyl group for G' . As in Theorem 3.1.12, we may identify \mathcal{M}_k with a union of Schubert varieties corresponding to $\text{Adm}_{G'}(\{\mu\})_{J'}$ in $\mathcal{FL}_{\underline{G}}$. The strata \mathcal{M}_k^λ may be identified with the $\underline{G}(k[[t]])$ -orbit of the element \dot{t}_λ considered as an element in $\mathcal{FL}_{\underline{G}}$ and by the above discussion, the closure relations between the strata are given by the partial ordering \preceq . Since $t_\mu \in \text{Adm}_{G'}(\{\mu\})_{J'}$ is the unique maximal element, it follows that \mathcal{M}_k^μ is contained in the smooth locus of \mathcal{M} and hence $\mathcal{S}_{K,k}^\mu$ is contained in the smooth locus of $\mathcal{S}_{K,k}$.

The strata \mathcal{M}_k^λ and $\mathcal{S}_{K,k}^\lambda$ are both defined over the field of definition of $\lambda \in W'_{J'} \backslash W' / W'_{J'}$. In other words, if n is the smallest integer such that $\sigma^n(\lambda) = \lambda$, then \mathcal{M}_k^λ and $\mathcal{S}_{K,k}^\lambda$ are both defined over \mathbb{F}_{p^n} ; we write \mathcal{M}^λ and \mathcal{S}_K^λ for the models over \mathbb{F}_{p^n} .

6.2.4. The key geometric property of the Kottwitz–Rapoport stratification on \mathcal{M}_k that we will need is the following.

Proposition 6.2.5. *Let $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$ with $\lambda \in \text{Adm}_{G'}(\{\mu\})_{J'}$ and $\lambda \neq \mu$. There exists a smooth, geometrically connected curve C over \mathbb{F}_q and a map $\phi : C \rightarrow \mathcal{M}_{\mathbb{F}_q}$ such that*

- (i) *There exists $y' \in C(\mathbb{F}_q)$ such that $\phi(y') = y$.*
- (ii) *$\phi^{-1}(\mathcal{M}_k^{\lambda'})$ is open and dense in C for some $\lambda' \in \text{Adm}_{G'}(\{\mu\})_{J'}$ with $\lambda \prec \lambda'$.*

Remark 6.2.6. Using an ampleness argument, it is easy to show that such a map always exists if we replace \mathbb{F}_q by its algebraic closure k . The key property is that for \mathcal{M} , this map exists without extending the residue field. By [Dri12, §6], there are normal and Cohen–Macaulay schemes where this property fails.

Proof of Proposition 6.2.5. The statement depends only on \mathcal{G}' and not on \mathcal{G} , so we may assume (for notational simplicity) that $G = G'$. We first show using the $\underline{\mathcal{G}}$ -action on \mathcal{M} that it suffices to consider the case

$$y = \dot{t}_\lambda \in \underline{G}(k((t))) / \underline{G}(k[[t]]).$$

Let σ_q denote the q -Frobenius; then since $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$, we have $\sigma_q(\lambda) = \lambda$. Therefore we may choose the lift $\dot{t}_\lambda \in \underline{G}(\mathbb{F}_q((t)))$ so that $\dot{t}_\lambda \in \mathcal{M}^\lambda(\mathbb{F}_q)$. By Lemma 6.2.7 below, there exists $g \in \underline{G}(\mathbb{F}_q[[t]])$ such that $g\dot{t}_\lambda = y$ in $\mathcal{FL}_{\underline{G}}$. Therefore if C satisfies the conditions (i) and (ii) for the point \dot{t}_λ , gC satisfies (i) and (ii) for the point y . It therefore suffices to prove the case $y = \dot{t}_\lambda$; we make this assumption from now on.

Now since $\lambda \prec \mu$, by Stembridge’s Lemma [Rap00, Lemma 2.3], there exists a positive root $\alpha \in \Sigma$ such that $\lambda + \alpha^\vee \preceq \mu$. Since $\lambda, \mu \in X_*(T)_I^{\sigma_q}$, it follows that

$$\lambda + \sigma_q^i(\alpha^\vee) \preceq \mu$$

for all i . If $\{\alpha, \sigma_q(\alpha), \dots, \sigma_q^{m-1}(\alpha)\}$ denotes the orbit of α under σ_q , it follows that

$$\lambda' := \lambda + \sum_{i=0}^{m-1} \sigma_q^i(\alpha^\vee) \preceq \mu,$$

and hence $\lambda' \in \text{Adm}_G(\{\mu\})_J$. Now α determines a relative root $\tilde{\alpha}$ of \underline{G} over $\mathbb{F}_q((t))$ which we always take to be the long root; then $\tilde{\alpha}$ is either divisible or non-divisible. We let $U_{\tilde{\alpha}}$ denote the relative root subgroup corresponding to $\tilde{\alpha}$ and $\underline{G}_{\tilde{\alpha}}$ the simply

connected covering of the (semi-simple) group generated by $U_{\tilde{\alpha}}$ and $U_{-\tilde{\alpha}}$; it is a reductive group over $\mathbb{F}_q((t))$. We will identify $U_{\tilde{\alpha}}$ with the corresponding unipotent subgroup of $\underline{G}_{\tilde{\alpha}}$. The parahoric $\underline{\mathcal{G}}$ determines a parahoric model $\underline{\mathcal{G}}_{\tilde{\alpha}}$ of $\underline{G}_{\tilde{\alpha}}$ and there is a closed immersion

$$\iota_{\tilde{\alpha}} : \mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}} \rightarrow \mathcal{FL}_{\underline{\mathcal{G}}, \mathbb{F}_q}$$

defined over \mathbb{F}_q , where $\mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}}$ is the affine flag variety associated to $\underline{\mathcal{G}}_{\tilde{\alpha}}$. We write $\mathcal{U}_{\tilde{\alpha}}$ (resp. $\mathcal{U}_{-\tilde{\alpha}}$) for the group schemes over $\mathbb{F}_q[[t]]$ corresponding to $U_{\tilde{\alpha}}(\mathbb{F}_q((t))) \cap \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$ (resp. $U_{-\tilde{\alpha}}(\mathbb{F}_q((t))) \cap \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$). Then we claim that for each positive α , there exists a morphism

$$f : \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}}$$

defined over \mathbb{F}_q satisfying the following two conditions

- (i') $f(0) = \dot{e}$, where \dot{e} is the base point in $\mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}}$.
- (ii') $f(\mathbb{A}_{\mathbb{F}_q}^1 \setminus \{0\}) \in L^+ \mathcal{U}_{\tilde{\alpha}} \dot{e}_{\alpha^\vee} L^+ \underline{\mathcal{G}}_{\tilde{\alpha}} / L^+ \underline{\mathcal{G}}_{\tilde{\alpha}} \cup L^+ \mathcal{U}_{\tilde{\alpha}/2} \dot{e}_{\alpha^\vee} L^+ \underline{\mathcal{G}}_{\tilde{\alpha}} / L^+ \underline{\mathcal{G}}_{\tilde{\alpha}}$.

Here the second term in the union in (ii') is to be read as empty if $\tilde{\alpha}$ is not divisible. Assuming the claim we may prove the proposition as follows. We consider the morphism

$$\phi : \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathcal{FL}_{\underline{\mathcal{G}}}, \quad x \mapsto \dot{e}_\lambda(\iota_{\tilde{\alpha}} \circ f)(x),$$

in other words we translate the composition $\iota_{\tilde{\alpha}} \circ f$ by \dot{e}_λ . Then condition (i) follows from (i') and condition (ii) follows from (ii') using the fact that λ is dominant.

It remains to prove the existence of f satisfying (i') and (ii'). We will construct f explicitly using a presentation of the group $\underline{G}_{\tilde{\alpha}}$; it turns out that by [BT84, §4.1.4] there are essentially three distinct cases to consider which we now describe.

If $\tilde{\alpha}$ is a non-divisible root then there is an identification

$$\underline{G}_{\tilde{\alpha}} \cong \text{Res}_{K/\mathbb{F}_q((t))} \text{SL}_2$$

where K is some finite separable extension of $\mathbb{F}_q((t))$ and the parahoric $\underline{\mathcal{G}}_{\tilde{\alpha}}$ is characterized by the property

$$\underline{\mathcal{G}}_{\tilde{\alpha}}(k[[t]]) = \text{SL}_2(\mathcal{O}_K \otimes_{\mathbb{F}_q[[t]]} k[[t]]).$$

If $\tilde{\alpha}/2$ is also a relative root, then there is an identification

$$\underline{G}_{\tilde{\alpha}} \cong \text{Res}_{K/\mathbb{F}_q((t))} \text{SU}_3$$

where $K/\mathbb{F}_q((t))$ is finite separable and SU_3 is the special unitary group associated to a hermitian space over a (separable)³ quadratic extension K'/K . We recall the presentation of the K -group SU_3 in [Tit79, Example 1.15]. We let $\tau \in \text{Gal}(K'/K)$ denote the non-trivial element and we consider the hermitian form on K'^3 given by

$$\langle (x_{-1}, x_0, x_1), (y_{-1}, y_0, y_1) \rangle = \tau(x_{-1})y_1 + \tau(x_0)y_0 + \tau(x_1)y_{-1}.$$

The group SU_3 is the special unitary group attached to this form. For $i = -1, 1$ and $c, d \in K'$ such that $\tau(c)c + d + \tau(d) = 0$, we define

$$u_i(c, d) = I_3 + (g_{rs})$$

where I_3 is the identity matrix and (g_{rs}) is the matrix with entries $g_{-i,0} = -\tau(c)$, $g_{0,i} = c$, $g_{-i,i} = d$ and $g_{rs} = 0$ otherwise. The root subgroups are then given by

$$U_{\pm\tilde{\alpha}/2}(K) = \{u_{\pm 1}(c, d) | c, d \in K', \tau(c)c + \tau(d) + d = 0\}$$

$$U_{\pm\tilde{\alpha}}(K) = \{u_{\pm 1}(0, d) | c, d \in K', \tau(d) + d = 0\}.$$

³Since we have assumed $p > 2$, this is automatic.

Then we may consider the parahoric

$$\underline{\mathcal{G}}_{\tilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap \mathrm{GL}_3(\mathcal{O}_{K'});$$

we call this the standard parahoric.

When K'/K is unramified this is the only very special parahoric (up to conjugacy). When K'/K is ramified, there is another conjugacy class of very special parahorics in addition to the standard parahoric which we shall call the non-standard parahoric. We let u be a uniformizer of K' and we define $s \in \mathrm{GL}_3(K')$ to be the element $\mathrm{diag}(1, 1, u)$. Then the non-standard parahoric $\underline{\mathcal{G}}_{\tilde{\alpha}}$ is given by

$$\underline{\mathcal{G}}_{\tilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap s\mathrm{GL}_3(\mathcal{O}_{K'})s^{-1}.$$

We label the cases as follows.

Case (1): $\tilde{\alpha}$ is non-divisible, $\underline{\mathcal{G}}_{\tilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SL}_2$ and $\underline{\mathcal{G}}_{\tilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SL}_2(\mathcal{O}_K)$.

Case (2): $\tilde{\alpha}$ is divisible, $\underline{\mathcal{G}}_{\tilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SU}_3$ and $\underline{\mathcal{G}}_{\tilde{\alpha}}$ is the standard parahoric.

Case (3): $\tilde{\alpha}$ is divisible, $\underline{\mathcal{G}}_{\tilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SU}_3$ with K'/K ramified and $\underline{\mathcal{G}}_{\tilde{\alpha}}$ is the non-standard parahoric.

We now proceed with the construction of f in each of the three cases.

Case (1). In this case the isomorphism $\underline{\mathcal{G}}_{\tilde{\alpha}} \cong \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathrm{SL}_2$ induces identifications

$$u_{\pm\tilde{\alpha}} : \mathrm{Res}_{K/\mathbb{F}_q((t))}\mathbb{G}_a \xrightarrow{\sim} U_{\pm\tilde{\alpha}}.$$

Let u be a uniformizer of K ; then we may define a map

$$f : \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}}, \quad x \mapsto u_{-\tilde{\alpha}}(u^{-1}x).$$

Clearly (i') is satisfied, and a simple calculation in SL_2 shows that for $0 \neq x$, we have

$$u_{-\tilde{\alpha}}(u^{-1}x) \in u_{\tilde{\alpha}}(ux^{-1})\dot{t}_{\alpha^\vee}L^+\underline{\mathcal{G}}_{\tilde{\alpha}}$$

so that (ii') also holds.

Case (2). Recall in this case, the parahoric $\underline{\mathcal{G}}_{\tilde{\alpha}}$ is characterized by $\underline{\mathcal{G}}_{\tilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap \mathrm{GL}_3(\mathcal{O}_{K'})$. We fix a uniformizer u of K' such that $\tau(u) = -u$ and define

$$f : \mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}}, \quad x \mapsto u_{-1}(0, u^{-1}x).$$

A calculation using the presentation recalled above shows that for $x \neq 0$, we have

$$u_{-1}(0, u^{-1}x) \in u_1(0, ux^{-1})\dot{t}_{\alpha^\vee}L^+\underline{\mathcal{G}}_{\tilde{\alpha}};$$

as in Case (1), it follows that (i') and (ii') are satisfied.

Case (3). Recall K'/K is ramified and $\underline{\mathcal{G}}_{\tilde{\alpha}}(\mathbb{F}_q[[t]]) = \mathrm{SU}_3(K) \cap s\mathrm{GL}_3(\mathcal{O}_{K'})s^{-1}$. We consider the map

$$\mathbb{A}_{\mathbb{F}_q}^1 \rightarrow \mathcal{FL}_{\underline{\mathcal{G}}_{\tilde{\alpha}}}, \quad x \mapsto u_{-1}(x, -\frac{x^2}{2}).$$

A calculation using the presentation above shows that for $x \neq 0$, we have

$$u_{-1}(x, -\frac{x^2}{2}) \in u_1(2x^{-1}, 2x^{-2})\dot{t}_{\alpha^\vee}L^+\underline{\mathcal{G}}_{\tilde{\alpha}};$$

as in the previous two cases it follows that (i') and (ii') are satisfied. \square

Lemma 6.2.7. *Let $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$ and assume $\dot{t}_\lambda \in \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$. Then there exists $g \in \underline{\mathcal{G}}(\mathbb{F}_q[[t]])$ such that $g\dot{t}_\lambda L^+\underline{\mathcal{G}} = y$ in $\mathcal{FL}_{\underline{\mathcal{G}}}$.*

Proof. By definition, there exists $h \in \underline{\mathcal{G}}(k[[t]])$ such that $h\dot{t}_\lambda = y$. We consider the subgroup

$$\underline{\mathcal{G}}(k[[t]]) \cap \dot{t}_\lambda \underline{\mathcal{G}}(k[[t]]) \dot{t}_\lambda^{-1} \subset \underline{\mathcal{G}}(k((t)));$$

it is the intersection of the kernel of the Kottwitz homomorphism $\tilde{\kappa}_{\underline{\mathcal{G}}}$ and the stabilizer of a bounded subset of the building $\mathcal{B}(\underline{\mathcal{G}}, k((t)))$. Thus by [HR08, Prop. 3 and Remark 4], it arises as the k -points of a smooth connected group scheme $\underline{\mathcal{K}}_\lambda$ defined over $\mathbb{F}_q[[t]]$.

The element h is defined up to right multiplication by $\underline{\mathcal{K}}_\lambda(k[[t]])$; hence since $\sigma_q(y) = y$, we have $\sigma_q(h) = hk$ for some $k \in \underline{\mathcal{K}}_\lambda(k[[t]])$. By Lang's theorem applied to $\underline{\mathcal{K}}_\lambda$, there exists $k_1 \in \underline{\mathcal{K}}_\lambda(k[[t]])$ such that $g := hk_1$ is fixed by σ_q , and we have $g\dot{t}_\lambda = y$ in $\mathcal{FL}_{\underline{\mathcal{G}}}$. \square

6.2.8. Using Theorem 6.2.1.1, we may deduce the following result about the local structure of the Shimura stack \mathcal{S}_K .

Corollary 6.2.9. *Let $x \in \mathcal{S}_K^\lambda(\mathbb{F}_q)$ with $\lambda \in \text{Adm}_{G'}(\{\mu\})_{J'}$ and $\lambda \neq \mu$. There exists a smooth, geometrically connected curve C' over \mathbb{F}_q and a map $\phi' : C' \rightarrow \mathcal{S}_{\mathbb{F}_q}$ such that*

- (i) *There exists $x' \in C'(\mathbb{F}_q)$ such that $\phi'(x') = x$.*
- (ii) *$\phi'^{-1}(\mathcal{S}_{K,k}^{\lambda'}) \subset C'$ is an open dense subscheme for some $\lambda' \in \text{Adm}_{G'}(\{\mu\})_{J'}$ with $\lambda \prec \lambda'$.*

Proof. We write

$$\begin{array}{ccc} & \widetilde{\mathcal{S}}_K & \\ \pi_{k_E} \swarrow & & \searrow q_{k_E} \\ \mathcal{S}_K & & \mathcal{M} \end{array}$$

for the special fiber of (6.2.1.1). Since π_{k_E} is a torsor for the smooth connected group scheme $\mathcal{G}_{\text{ad}, k_E}$, the point x lifts to a point $\tilde{x} \in \widetilde{\mathcal{S}}_K(\mathbb{F}_q)$ and we write y for its image in $\mathcal{M}(\mathbb{F}_q)$. By definition of the stratification on \mathcal{S}_K , we have $y \in \mathcal{M}^\lambda(\mathbb{F}_q)$. We apply Proposition 6.2.5 to y to obtain a map $\phi' : C \rightarrow \mathcal{M}_{\mathbb{F}_q}$ satisfying (i) and (ii) in Proposition 6.2.5 for some $\lambda' \in \text{Adm}_{G'}(\{\mu\})_{J'}$ with $\lambda \prec \lambda'$; we let $y' \in C(\mathbb{F}_q)$ mapping to y .

Consider the pullback $\widetilde{\mathcal{S}}_{K, \mathbb{F}_q} \times_{\mathcal{M}_{\mathbb{F}_q}} C$ which is a smooth stack over \mathbb{F}_q . By [LMB00, Théorème 6.3], there exists a smooth scheme Y/\mathbb{F}_q and a smooth map $Y \rightarrow \widetilde{\mathcal{S}}_{K, \mathbb{F}_q} \times_{\mathcal{M}_{\mathbb{F}_q}} C$ defined over \mathbb{F}_q such that \tilde{x} lies in the image of a point $\tilde{y} \in Y(\mathbb{F}_q)$. Now let $Y^{\lambda'}$ denote the preimage of $\mathcal{M}^{\lambda'}$ in Y ; by the assumption on C , it is a dense open subscheme of Y . By [Poo04, Theorem 1.1], there exists a smooth geometrically connected curve $C' \subset Y$ such that $\tilde{y} \in C'(\mathbb{F}_q)$ and $C' \cap Y^{\lambda'} \neq \emptyset$ so that the preimage of $Y^{\lambda'}$ in C' is open and dense. We write $\phi' : C' \rightarrow \mathcal{S}_{K, \mathbb{F}_q}$ for the composition

$$C' \rightarrow Y \rightarrow \widetilde{\mathcal{S}}_{K, \mathbb{F}_q} \times_{\mathcal{M}_{\mathbb{F}_q}} C \rightarrow \widetilde{\mathcal{S}}_{K, \mathbb{F}_q} \rightarrow \mathcal{S}_{K, \mathbb{F}_q}.$$

Then setting $x' = \tilde{y} \in C'(\mathbb{F}_q)$, we have $\phi'(x') = x$, so (i) is satisfied, and property (ii) follows by the construction. \square

6.3. Compatible local systems and ℓ -independence.

6.3.1. We recall the theory of compatible local systems. Let X be a normal scheme over \mathbb{F}_q where q is a power of p and let \mathcal{L}_ℓ be a $\overline{\mathbb{Q}}_\ell$ -local system (lisse sheaf) on X . For $x \in X(\mathbb{F}_{q^n})$, we write Frob_x for the local Frobenius automorphism acting on the stalk $\mathcal{L}_{\ell, \bar{x}}$ of \mathcal{L}_ℓ at a geometric point \bar{x} lying over x . Suppose that for every closed point $x \in X(\mathbb{F}_{q^n})$ the characteristic polynomial $\det(1 - \text{Frob}_x t | \mathcal{L}_{\ell, \bar{x}})$, has coefficients in a number field $E \subset \overline{\mathbb{Q}}_\ell$ (this is conjectured to be the case if \mathcal{L}_ℓ has determinant of finite order). Let ℓ' be a prime not equal to p or ℓ . A $\overline{\mathbb{Q}}_{\ell'}$ -local system $\mathcal{K}_{\ell'}$ is said to be a *compatible local system* for \mathcal{L}_ℓ if there is some possibly larger number field E' and embeddings $E' \subset \overline{\mathbb{Q}}_\ell, E' \subset \overline{\mathbb{Q}}_{\ell'}$ such that for every closed point $x \in X(\mathbb{F}_{q^n})$, the characteristic polynomials $\det(1 - \text{Frob}_x t | \mathcal{L}_{\ell, \bar{x}})$, $\det(1 - \text{Frob}_x t | \mathcal{K}_{\ell', \bar{x}})$ have coefficients in E' and there is an equality

$$\det(1 - \text{Frob}_x t | \mathcal{L}_{\ell, \bar{x}}) = \det(1 - \text{Frob}_x t | \mathcal{K}_{\ell', \bar{x}}) \in E'[t].$$

The existence of compatible local systems over smooth curves is due to Lafforgue [Laf02, Théorème VII.6], and the case of smooth schemes is due to Drinfeld [Dri12, Theorem 1.1].

6.3.2. We now continue with the notations of §6.1. For the rest of this section, it will be convenient to fix a Hodge embedding $\iota : (\mathbf{G}, X) \rightarrow (\mathbf{GSp}(V), S^\pm)$ as in §5.3.1.

The element $\gamma_{y, \ell} \in \text{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ arises as an element of $\text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$. Indeed the image of $\gamma_{y, \ell}$ in $\text{Conj}_{\mathbf{GL}(V)}(\mathbb{Q}_\ell)$ under the map induced by ι lies in $\text{Conj}_{\mathbf{GL}(V)}(\mathbb{Q})$ since it corresponds to the action of Frobenius on the ℓ -adic Tate module of an abelian variety. Since $\text{Conj}_{\mathbf{G}} \rightarrow \text{Conj}_{\mathbf{GL}(V)}$ is a finite map, $\gamma_{y, \ell} \in \text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$. Similarly if $\ell' \nmid p\ell$ is another prime, $\gamma_{y, \ell'}$ arises as an element of $\text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$.

We let F be a finite extension of \mathbb{Q} such that $\gamma_{y, \ell}, \gamma_{y, \ell'} \in \text{Conj}_{\mathbf{G}}(F)$; such an extension exists since $\text{Conj}_{\mathbf{G}}$ is a \mathbb{Q} -variety. Let λ, λ' be the two places over F induced by the fixed embeddings $i_\ell : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_\ell$ and $i_{\ell'} : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_{\ell'}$. We take $\vartheta : \mathbf{G}_F \rightarrow \mathbf{GL}_{\mathbf{n}_F}$ to be a representation over F ; then the $\mathbf{G}(\mathbb{Q}_\ell)$ -local system \mathbb{L}_ℓ induces an F_λ -adic local system \mathcal{L}_ℓ over \mathcal{S}_K . Similarly we obtain an $F_{\lambda'}$ -adic local system $\mathcal{L}_{\ell'}$.

Lemma 6.3.3. *For any closed point $x \in \mathcal{S}_K(\mathbb{F}_q)$, the eigenvalues of Frob_x acting on $\mathcal{L}_{\ell, \bar{x}}$ are ℓ -adic units.*

Proof. It suffices to prove this for a single faithful representation of \mathbf{G} . For the representation $\iota : \mathbf{G} \rightarrow \mathbf{GL}(V)$, the action of Frob_x on $\mathcal{L}_{\ell, \bar{x}}$ corresponds to the action of Frobenius on the ℓ -adic Tate module of an abelian variety and hence its eigenvalues are all ℓ -adic units. \square

6.3.4. We let $\vartheta(\gamma_{y, \ell}) \in \text{Conj}_{\mathbf{GL}_{\mathbf{n}}}(F) \subset \text{Conj}_{\mathbf{GL}_{\mathbf{n}}}(F_\lambda)$ denote the image of the conjugacy class of Frob_y under ϑ and we similarly define $\vartheta(\gamma_{y, \ell'}) \in \text{Conj}_{\mathbf{GL}_{\mathbf{n}}}(F) \subset \text{Conj}_{\mathbf{GL}_{\mathbf{n}}}(F_{\lambda'})$.

Proposition 6.3.5. *$\vartheta(\gamma_{y, \ell}) = \vartheta(\gamma_{y, \ell'})$ in $\text{Conj}_{\mathbf{GL}_{\mathbf{n}}}(F)$.*

Proof. Note that if $y \in \mathcal{S}_{K, [b]_\mu}(\mathbb{F}_q)$, where $\mathcal{S}_{K, [b]_\mu}$ denotes the μ -ordinary locus of \mathcal{S}_K , then the result follows from Corollary 5.3.20. The proof then proceeds in two steps. We first prove the result for $y \in \mathcal{S}_K^\mu(\mathbb{F}_q)$ using the result for the μ -ordinary locus. We then deduce the result for general y by descending induction on the strata λ for which $y \in \mathcal{S}_K^\lambda(\mathbb{F}_q)$.

Step (1): Let $y \in \mathcal{S}_K^\mu(\mathbb{F}_q)$. Recall that \mathcal{S}_K^μ is a smooth algebraic stack over k_E and that $\mathcal{S}_{K,[b]_\mu} \cap \mathcal{S}_K^\mu$ is a dense and open substack of \mathcal{S}_K^μ (in fact one can show $\mathcal{S}_{K,[b]_\mu} \subset \mathcal{S}_K^\mu$). Using the same argument as in the proof of 6.2.9 (i.e. applying [LMB00, Théorème 6.3] and [Poo04, Theorem 1.1]), we may find a smooth geometrically connected curve C over \mathbb{F}_q and a map $\psi : C \rightarrow \mathcal{S}_{K,\mathbb{F}_q}^\mu$ defined over \mathbb{F}_q such that there exists a point $x' \in C(\mathbb{F}_q)$ with $\psi(x') = x$ and such that the preimage $C_{[b]_\mu}$ of $\mathcal{S}_{K,[b]_\mu}$ in C is open and dense. We write \mathcal{L}_ℓ^C (resp. $\mathcal{L}_{\ell'}^C$) for the pullback $\psi^* \mathcal{L}_\ell$ of \mathcal{L}_ℓ (resp. $\psi^* \mathcal{L}_{\ell'}$ of $\mathcal{L}_{\ell'}$) to C . By Lemma 6.3.3, \mathcal{L}_ℓ^C satisfies the conditions in Chin's refinement of Lafforgue's Theorem [Chi04, Theorem 4.6]. Thus there exists a $\overline{\mathbb{Q}}_{\ell'}$ -local system $\mathcal{K}_{\ell'}^C$ over C which is compatible for \mathcal{L}_ℓ^C . Upon possibly enlarging F , we have that for any closed point $x \in C(\mathbb{F}_{q^s})$,

$$\det(1 - \text{Frob}_x t | \mathcal{L}_{\ell,\bar{x}}^C) = \det(1 - \text{Frob}_x t | \mathcal{K}_{\ell',\bar{x}}^C) \in F[t].$$

Hence, by Step (1), for any closed point $x \in C_{[b]_\mu}(\mathbb{F}_{q^s})$, we have

$$\det(1 - \text{Frob}_x t | \mathcal{L}_{\ell',\bar{x}}^C) = \det(1 - \text{Frob}_x t | \mathcal{L}_{\ell,\bar{x}}^C) = \det(1 - \text{Frob}_x t | \mathcal{K}_{\ell',\bar{x}}^C).$$

Therefore, by the Chebotarev density Theorem, the semisimplifications of $\mathcal{K}_{\ell'}^C$ and $\mathcal{L}_{\ell'}^C$ are isomorphic, and hence

$$\vartheta(\gamma_{y,\ell}) = \det(1 - \text{Frob}_y t | \mathcal{L}_{\ell,\bar{y}}^C) = \det(1 - \text{Frob}_y t | \mathcal{L}_{\ell',\bar{y}}^C) = \vartheta(\gamma_{y,\ell'})$$

which is what we wanted to show.

Step (2): Let $y \in \mathcal{S}_K^\lambda(\mathbb{F}_q)$. We proceed by descending induction on λ ; by part (2) we know the result for the maximal element $\lambda = \mu$. Thus suppose the result is true for all $\lambda' \succ \lambda$.

Let $\phi : C \rightarrow \mathcal{S}_{K,\mathbb{F}_q}$ be a map as in Corollary 6.2.9 where C is a smooth geometrically connected curve over \mathbb{F}_q . We write \mathcal{L}_ℓ^C (resp. $\mathcal{L}_{\ell'}^C$) for the local system $\phi^* \mathcal{L}_\ell$ (resp. $\phi^* \mathcal{L}_{\ell'}$) on C . We let $\mathcal{K}_{\ell'}^C$ be a compatible $\overline{\mathbb{Q}}_{\ell'}$ -local system for \mathcal{L}_ℓ^C which exists as above. We let $U \subset C$ denote the open subscheme

$$U := \phi^{-1} \left(\bigcup_{\lambda \prec \lambda'} \mathcal{S}_{K,\mathbb{F}_q}^{\lambda'} \right).$$

By property (ii) in Corollary 6.2.9, U is a non-empty dense open subscheme of C . Applying the induction hypothesis we see that for all $x \in U(\mathbb{F}_{q^s})$, we have

$$\det(1 - \text{Frob}_x t | \mathcal{L}_{\ell',\bar{x}}^C) = \det(1 - \text{Frob}_x t | \mathcal{K}_{\ell',\bar{x}}^C).$$

Arguing as in Step (2) we find that

$$\vartheta(\gamma_{y,\ell}) = \det(1 - \text{Frob}_y t | \mathcal{L}_{\ell,\bar{y}}^C) = \det(1 - \text{Frob}_y t | \mathcal{L}_{\ell',\bar{y}}^C) = \vartheta(\gamma_{y,\ell'}).$$

This completes the proof of the Proposition. \square

6.3.6. We may now prove Theorem 6.1.4.

Proof of Theorem 6.1.4. For all $\ell, \ell' \neq p$, and ϑ as above, we have $\vartheta(\gamma_{y,\ell}) = \vartheta(\gamma_{y,\ell'})$ by Proposition 6.3.5. This implies that $\gamma_{y,\ell} = \gamma_{y,\ell'} \in \text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$, by a result of Steinberg [Ste65, 6.6]. Hence, there exists $\gamma_y \in \text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$ such that $\gamma_y = \gamma_{y,\ell}$ for all $\ell \neq p$. It suffices to show γ_y is defined over \mathbb{Q} .

Since $\text{Conj}_{\mathbf{G}}$ is a \mathbb{Q} -variety, the residue field of the point γ_y is a finite extension F/\mathbb{Q} . Since $\gamma_y \in \text{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$ for all ℓ , each finite prime of \mathbb{Q} has a split prime in F above it; hence the Chebotarev density theorem implies $\gamma_y \in \text{Conj}_{\mathbf{G}}(\mathbb{Q})$. Indeed let F'/\mathbb{Q} be the Galois closure of F . Then for every prime $\ell \neq p$, there exists l a

prime of F' above ℓ such that the Frobenius Frob_l lies in $\text{Gal}(F'/F) \subset \text{Gal}(F'/\mathbb{Q})$. It follows that $\text{Gal}(F'/F)$ intersects every conjugacy class of $\text{Gal}(F'/\mathbb{Q})$ and hence these groups are equal. \square

Remark 6.3.7. The proof of Theorem 6.1.4 uses Theorem 5.3.9 and hence depends on a choice Hodge embedding ι for (\mathbf{G}, X) . The statement of Theorem 6.1.4 itself does not depend on such a choice since the local system $\tilde{\mathbb{L}}_\ell$ is intrinsic to $\mathcal{S}_K(\mathbf{G}, X)$. The Hodge embedding is used to deduce properties of $\tilde{\mathbb{L}}_\ell$ via the isomorphism (6.1.1.1).

7. CONJUGACY CLASS OF FROBENIUS FOR ABELIAN VARIETIES

We now apply the results of §6 to prove our main result concerning abelian varieties.

7.1. Mumford–Tate groups.

7.1.1. Let A be an abelian variety over a number field E . Recall we have fixed an embedding $i_\infty : \mathbb{Q} \rightarrow \mathbb{C}$; using this we may consider E as a subfield of \mathbb{C} . We write V_B for the Betti cohomology $H_B^1(A(\mathbb{C}), \mathbb{Q})$ which is equipped with a Hodge structure of type $((0, -1), (-1, 0))$. This Hodge structure is induced by a morphism

$$h : \mathbb{S} := \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \rightarrow \text{GL}(V_B)$$

We write

$$\mu : \mathbb{C}^\times \xrightarrow{z \mapsto (z, 1)} \mathbb{C}^\times \times c^*(\mathbb{C}^\times) \xrightarrow{h} \text{GL}(V_B \otimes \mathbb{C})$$

for the Hodge cocharacter.

Definition 7.1.2. The Mumford–Tate group \mathbf{G} of A is the smallest algebraic subgroup of $\text{GL}(V_B)$ defined over \mathbb{Q} such that $\mathbf{G}(\mathbb{C})$ contains the image of μ .

The group \mathbf{G} can also be characterized as the algebraic subgroup of $\text{GL}(V_B)$ that stabilizes all Hodge cycles; it is known that \mathbf{G} is a reductive group. We remark that \mathbf{G} depends on the embedding $E \hookrightarrow \mathbb{C}$; indeed different embeddings will give rise to an inner form of \mathbf{G} .

7.1.3. For a prime number ℓ , we write $T_\ell A$ for the Tate module of A . The action of the absolute Galois group $\Gamma_E := \text{Gal}(\bar{E}/E)$ on $T_\ell A^\vee$ gives rise to a representation $\rho_\ell : \Gamma_E \rightarrow \mathbf{GL}(T_\ell A^\vee)$ and the Betti-étale comparison gives us a canonical isomorphism

$$H_B^1(A(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong T_\ell A^\vee \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Deligne’s theorem that Hodge cycles are absolutely Hodge [Del82], implies that upon replacing E by a finite extension, the map ρ_ℓ factors through $\mathbf{G}(\mathbb{Q}_\ell)$; see [Noo09, Remarque 1.9]. In fact this condition does not depend on ℓ .

Lemma 7.1.4. ρ_ℓ factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for some prime ℓ , if and only if it factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for all primes ℓ .

Proof. The subgroup $\mathbf{G} \subset \text{GL}(V_B)$ is the stabilizer of a collection of Hodge cycles $(s_\alpha)_\alpha$. We consider the ℓ -adic components $(s_{\alpha, \ell})_\ell$, as in §5.1.4. For $\sigma \in \Gamma_E$, $(\sigma(s_{\alpha, \ell}))_\ell$, is again a Hodge cycle, by Deligne’s theorem [Del82, Theorem 2.11]. In particular, if $(\sigma(s_{\alpha, \ell}))_\ell$, and $(s_{\alpha, \ell})_\ell$ have equal components at some prime ℓ , then they are equal. \square

The Lemma shows that the condition that Γ_E fixes $(s_{\alpha,\ell})_\alpha$ pointwise does not depend on ℓ . This condition is equivalent to asking that Γ_E maps to $\mathbf{G}(\mathbb{Q}_\ell)$.

7.1.5. We replace E by the smallest extension such that Γ_E maps to $\mathbf{G}(\mathbb{Q}_\ell)$, and we write $\rho_\ell^\mathbf{G}$ for the induced map $\Gamma_E \rightarrow \mathbf{G}(\mathbb{Q}_\ell)$ and ι_ℓ for the inclusion $\mathbf{G}(\mathbb{Q}_\ell) \rightarrow \mathbf{GL}(T_\ell A^\vee)$.

Let v be a prime of E lying above a prime p such that A has good reduction at v . Upon modifying the embedding $i_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ fixed in §5.1.1, we may assume that v is induced by i_p . We write $E = E_v$, and we let \mathbb{F}_q denote the residue field of E at v . For $\ell \neq p$ a prime, the criterion of Néron–Ogg–Shafarevich implies the representation ρ_ℓ is unramified at v . Let Fr_v be a geometric Frobenius element at v , we write $\gamma_\ell(v) = \chi_\mathbf{G}(\rho_\ell^\mathbf{G}(\text{Fr}_v)) \in \text{Conj}_\mathbf{G}(\mathbb{Q}_\ell)$ for the conjugacy class of $\rho_\ell^\mathbf{G}(\text{Fr}_v)$ which only depends on v and not the choice of Frobenius element. We write $P_{v,\ell}(t)$ for the characteristic polynomial of Fr_v acting on $T_\ell A^\vee$, which has coefficients in \mathbb{Z} and is independent of ℓ .

7.1.6. We will make use of the following auxiliary construction. Let F/\mathbb{Q} be a totally real field, and let $\mathbf{H}' := \text{Res}_{F/\mathbb{Q}} \mathbf{G}$. There is a canonical inclusion $\mathbf{G} \hookrightarrow \mathbf{H}'$. We let (V, ψ) be the symplectic space corresponding to $H_1(A(\mathbb{C}), \mathbb{Q})$ where ψ is a Riemann form for A and $\mathbf{G} \rightarrow \mathbf{GSp}(V)$ is the natural map. We let W denote the symplectic space over \mathbb{Q} whose underlying vector space is $V \otimes_{\mathbb{Q}} F$ and whose alternating form ψ' is given by the composition

$$W \times W \xrightarrow{\psi \otimes_{\mathbb{Q}} F} F \xrightarrow{\text{Tr}_{F/\mathbb{Q}}} \mathbb{Q}.$$

Let $c_\mathbf{G} : \mathbf{G} \rightarrow \mathbb{G}_m$ denote the restriction of the multiplier homomorphism $c : \mathbf{GSp}(V) \rightarrow \mathbb{G}_m$ to \mathbf{G} . We form the fiber product

$$\begin{array}{ccc} \mathbf{H}'' & \longrightarrow & \mathbb{G}_m \\ \downarrow & & \downarrow \Delta \\ \mathbf{H}' & \xrightarrow{\text{Res}_{F/\mathbb{Q}} c_\mathbf{G}} & \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \end{array}$$

where the map Δ is the diagonal map and we let \mathbf{H} denote the neutral connected component of \mathbf{H}'' . Thus \mathbf{H} is a connected reductive group over \mathbb{Q} . The inclusion $\mathbf{G} \hookrightarrow \mathbf{H}'$ factors through \mathbf{H} and we let h' denote the composition

$$\mathbb{S} \xrightarrow{h} \mathbf{G}_\mathbb{R} \rightarrow \mathbf{H}_\mathbb{R}.$$

Write X for the $\mathbf{G}(\mathbb{R})$ conjugacy class of h and $X_\mathbf{H}$ for the $\mathbf{H}(\mathbb{R})$ -conjugacy class of h' .

Consider the composition

$$\iota' : \mathbf{H}' \xrightarrow{\text{Res}_{F/\mathbb{Q}} \iota} \text{Res}_{F/\mathbb{Q}} \mathbf{GSp}(V) \xrightarrow{f} \mathbf{GL}(W)$$

where f is induced by the forgetful functor from F -vector spaces to \mathbb{Q} -vector spaces. It is easy to see that the restriction of ι' to \mathbf{H} factors through $\mathbf{GSp}(W)$, and we also denote by ι' the induced map. We write S'^\pm for the Siegel half space corresponding to W . One checks easily that (\mathbf{G}, X) , and $(\mathbf{H}, X_\mathbf{H})$ are Shimura data, and that we have embeddings of Shimura data

$$(\mathbf{G}, X) \hookrightarrow (\mathbf{H}, X_\mathbf{H}) \hookrightarrow (\mathbf{GSp}(W), S'^\pm).$$

7.1.7. The following lemma will be used to show that for the ℓ -independence of $\gamma_\ell(v)$ in $\text{Conj}_{\mathbf{G}}$, it suffices to show the ℓ -independence in $\text{Conj}_{\mathbf{H}}$.

Lemma 7.1.8. *The natural inclusion $\mathbf{G} \rightarrow \mathbf{H}$ induces a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -equivariant injection*

$$\text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}}) \rightarrow \text{Conj}_{\mathbf{H}}(\overline{\mathbb{Q}}).$$

Proof. Let $h, h' \in \mathbf{G}(\overline{\mathbb{Q}})$ such that there exists $g \in \mathbf{H}(\overline{\mathbb{Q}})$ such that $g^{-1}hg = h'$. We consider \mathbf{H} as a subgroup of \mathbf{H}' . Then under the identification

$$\mathbf{H}'_{\overline{\mathbb{Q}}} \cong \prod_{\iota: \mathbb{F} \rightarrow \overline{\mathbb{Q}}} \mathbf{G}_{\overline{\mathbb{Q}}},$$

h, h' correspond to the elements $(h, \dots, h), (h', \dots, h')$ respectively and we write $g = (g_1, \dots, g_n)$. Then $g^{-1}hg = h'$ implies $g_1 h g_1^{-1} = h'$. Thus h and h' have the same image in $\text{Conj}_{\mathbf{G}}(\overline{\mathbb{Q}})$. The $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -equivariance follows from the fact that $\mathbf{G} \rightarrow \mathbf{H}$ is defined over \mathbb{Q} . \square

7.2. The main theorem. We now prove our main theorem (cf. Theorem 1.1). We need the following preliminary result.

Lemma 7.2.1. *Let G be a connected reductive group over \mathbb{Q}_p . If $g \in G(\mathbb{Q}_p)$ lies in some compact open subgroup of $G(\mathbb{Q}_p)$, then there exists a finite extension F/\mathbb{Q}_p over which G splits and such that g lies in the parahoric subgroup of $G(F)$ associated to a very special vertex in the building $\mathcal{B}(G, F)$.*

Proof. Write $g = g_s g_u$ for the Jordan decomposition of g so that g_s is semisimple and g_u is unipotent. Since g lies in a compact open subgroup of $G(\mathbb{Q}_p)$, g is power bounded and hence g_s and g_u are power bounded. Let $T \subset G$ be a maximal torus defined over \mathbb{Q}_p such that $g_s \in T(\mathbb{Q}_p)$. We will take F to be the splitting field of T .

Since $g_s \in T(F)$ is power bounded, it is contained in $\mathcal{T}_{F,0}(\mathcal{O}_F)$ where $\mathcal{T}_{F,0}$ is the connected Néron model for the base change T_F . If we let $\mathcal{A}(G, T, F) \subset \mathcal{B}(G, F)$ be the apartment corresponding to T_F , then g_s acts trivially on $\mathcal{A}(G, T, F)$.

Now $g_u \in U(F)$ where U is the unipotent radical of some Borel subgroup B of G_F containing T . Let $\mathfrak{s} \in \mathcal{A}(G, T, F)$ be any special vertex and we use this vertex to identify $\mathcal{A}(G, T, F)$ with $X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Since each affine root subgroup of G_F fixes a half apartment in $\mathcal{A}(G, T, F)$, there exists a sufficiently dominant (with respect to the choice of Borel B) very special vertex \mathfrak{s}' which is fixed by g_u . It follows that \mathfrak{s}' is fixed by g . We write $\tilde{\mathcal{G}}$ for the Bruhat–Tits stabilizer scheme over \mathcal{O}_F corresponding to \mathfrak{s}' ; by the above discussion we have $g \in \tilde{\mathcal{G}}(\mathcal{O}_F)$. Since G is split over F , $\tilde{\mathcal{G}}$ is equal to the parahoric group-scheme \mathcal{G} associated to \mathfrak{s}' . \square

7.2.2. We now return to the assumptions and notation of §7.1. Thus we have an abelian variety A/E , such that $\rho_\ell : \Gamma_E \rightarrow \text{GL}(T_\ell A^\vee)$ factors through $\mathbf{G}(\mathbb{Q}_\ell)$ for all ℓ . Recall $E = E_v$ and \mathbb{F}_q is its residue field. The map $i_p : \overline{\mathbb{Q}} \rightarrow \overline{\mathbb{Q}}_p$ determines an inclusion

$$(7.2.2.1) \quad \text{Gal}(\overline{E}/E) \rightarrow \text{Gal}(\overline{\mathbb{E}}/\mathbb{E}).$$

We let $\tilde{\sigma}_q \in \Gamma_E$ be the image under (7.2.2.1) of a lift of the geometric Frobenius in $\text{Gal}(\overline{E}/E)$.

Proposition 7.2.3. *Let $p > 2$. There exists a totally real field F such that if $(\mathbf{H}, X_{\mathbf{H}})$ denotes the Shimura datum of Hodge type coming from the construction in §7.1.6, we have $H := \mathbf{H}_{\mathbb{Q}_p}$ is quasi-split and there exists a very special parahoric group scheme \mathcal{H} for H such that*

- (1) *The image of $\rho_p^{\mathbf{G}}(\tilde{\sigma}_q)$ in $H(\mathbb{Q}_p)$ lies in $\mathcal{H}(\mathbb{Z}_p)$.*
- (2) *The triple $(\mathbf{H}, X_{\mathbf{H}}, \mathcal{H})$ is acceptable.*

Proof. Let $G = \mathbf{G}_{\mathbb{Q}_p}$. By Lemma 7.2.1 applied to the element $\rho_p^{\mathbf{G}}(\tilde{\sigma}_q) \in G(\mathbb{Q}_p)$, there exists a finite extension F/\mathbb{Q}_p such that G_F is split and there exists a very special parahoric \mathcal{G} of G_F such that the image of $\rho_p^{\mathbf{G}}(\tilde{\sigma}_q)$ in $G(F)$ lies in $\mathcal{G}(\mathcal{O}_F)$. We let F be a totally real field such that $F_w \cong F$ for all places $w|p$ of F . By construction $\mathbf{H} \subset \mathbf{H}' = \text{Res}_{F/\mathbb{Q}} \mathbf{G}$ and we have an isomorphism

$$H' := \mathbf{H}'_{\mathbb{Q}_p} \cong \prod_{w|p} \text{Res}_{F_w/\mathbb{Q}_p} \mathbf{G}_{F_w} \cong \prod_{w|p} \text{Res}_{F/\mathbb{Q}_p} G_F$$

We let \mathcal{H}' denote the parahoric group scheme of H' corresponding to $\prod_{w|p} \mathcal{G}$. Then $\mathcal{H}'(\mathbb{Z}_p) \cap H(\mathbb{Q}_p)$ arises as the \mathbb{Z}_p -points of a parahoric group scheme \mathcal{H} for $H := \mathbf{H}_{\mathbb{Q}_p}$. By construction H' is quasi-split since it is the restriction of scalars of a split group, and hence H is quasi-split. Since $G(\mathbb{Q}_p) \subset H(\mathbb{Q}_p)$, the image of $\rho_p^{\mathbf{G}}(\tilde{\sigma}_q)$ in $H(\mathbb{Q}_p)$ lies in $\mathcal{H}(\mathbb{Z}_p)$ so that (1) is satisfied.

To show (2) is satisfied, we let (\mathbf{H}_1, X_1) be an auxiliary Shimura datum of Hodge type as constructed in Proposition 5.2.7 so that there is a central extension $\mathbf{H}_{1\text{der}} \rightarrow \mathbf{H}_{1\text{der}}$ and we write $H_1 := \mathbf{H}_{1,\mathbb{Q}_p}$. The parahoric \mathcal{H} of H determines a very special parahoric group scheme of \mathcal{H}_1 of H_1 . It suffices to show \mathcal{H}_1 is a connected parahoric.

Note that there is an isomorphism $H_{\text{ad}} \cong H_{1,\text{ad}} \cong \prod_{i=1}^r \text{Res}_{F_i/\mathbb{Q}_p} G_i$ where G_i is a *split* reductive group over F_i . It follows that any parahoric of H_{ad} is connected. There is a natural map $\tilde{\mathcal{H}}_1 \rightarrow \tilde{\mathcal{H}}_{\text{ad}}$ and a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{H}}_1(\check{\mathbb{Z}}_p) & \longrightarrow & \tilde{\mathcal{H}}_{\text{ad}}(\check{\mathbb{Z}}_p) \\ \tilde{\kappa}_{H_1} \downarrow & & \downarrow \tilde{\kappa}_{H_{\text{ad}}} \\ \pi_1(H_1)_I & \longrightarrow & \pi_1(H_{\text{ad}})_I. \end{array}$$

Therefore $\tilde{\mathcal{H}}_1(\check{\mathbb{Z}}_p)$ maps to $\ker(\pi_1(H_1)_I \rightarrow \pi_1(H_{\text{ad}})_I)$ and it suffices to show this group is torsion free.

We have a commutative diagram with exact rows.

$$\begin{array}{ccccccc} \pi_1(H_{1\text{der}})_I & \longrightarrow & \pi_1(H_1)_I & \longrightarrow & X_*(H_{1\text{ab}})_I & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(H_{\text{ad}})_I & \xrightarrow{\sim} & \pi_1(H_{\text{ad}})_I & \longrightarrow & \{1\} \longrightarrow 0 \end{array}$$

Since $\pi_1(H_{1,\text{der}}) \rightarrow \pi_1(H_{\text{ad}})$ is injective and these are induced modules, it follows that $\pi_1(H_{1,\text{der}})_I \rightarrow \pi_1(H_{\text{ad}})_I$ is injective. By construction, $X_*(H_{1\text{ab}})_I$ is torsion free, and hence so is $\ker(\pi_1(H_1)_I \rightarrow \pi_1(H_{\text{ad}})_I)$ by the snake Lemma. \square

Theorem 7.2.4. *Let $p > 2$ be a prime and $v|p$ a place of E where A has good reduction. Then there exists an element $\gamma \in \text{Conj}_{\mathbf{G}}(\mathbb{Q})$ such that for all $\ell \neq p$, we have $\gamma = \gamma_\ell(v)$ in $\text{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$.*

Remark 7.2.5. As remarked above, the group \mathbf{G} depends on the embedding $E \hookrightarrow \mathbb{C}$ up to inner automorphism. However, this does not change the \mathbb{Q} -variety $\text{Conj}_{\mathbf{G}}$, and it can be checked that the statement of the theorem can be made independent of the choice of embedding.

Proof of 7.2.4. We may assume that \mathbf{G} is not a torus as in this case A has complex multiplication and the result is a theorem of Shimura–Taniyama. We choose a totally real field F as in Proposition 7.2.3 and let $(\mathbf{H}, X_{\mathbf{H}})$ be the associated Shimura datum of Hodge type arising from the construction in §7.1.6. By construction, there is a very special parahoric \mathcal{H} of $\mathbf{H}_{\mathbb{Q}_p}$ such that the image of $\rho_p^{\mathbf{G}}(\bar{\sigma}_q)$ inside $\mathbf{H}(\mathbb{Q}_p)$ lies in $K_p := \mathcal{H}(\mathbb{Z}_p)$. Hence, there exists a finite extension E' of E such that $\rho_p^{\mathbf{G}}|_{\Gamma_{E'}}$ factors through K_p , and such that there is a prime $v'|v$ of E' such that $E'_{v'}$ has residue field \mathbb{F}_q . We may thus replace E by E' , without changing the statement of the theorem, and assume that the image of $\rho_p^{\mathbf{G}}$ in $\mathbf{H}(\mathbb{Q}_p)$ factors through K_p .

Now let $(s_{\alpha, \ell})_{\ell \neq p} \in \widehat{V}^p(A)^{\otimes}$ denote the ℓ -adic realizations of the absolute Hodge cycles for A . By our assumption on E , the representation $\rho^p : \Gamma_E \rightarrow \text{GL}(\widehat{V}^p(A))$ factors through $\mathbf{G}(\mathbb{A}_f^p) \subset \mathbf{H}(\mathbb{A}_f^p)$, and hence through a compact open subgroup $K^p \subset \mathbf{H}(\mathbb{A}_f^p)$. Write $K := K_p K^p$.

We now define a point of $\text{Sh}_K(\mathbf{H}, X_{\mathbf{H}})$ using the Hodge embedding $\iota' : (\mathbf{H}, X_{\mathbf{H}}) \rightarrow \mathbf{GSp}(W, S^{\pm})$. Consider the abelian variety up to isogeny $A^F = A \otimes_{\mathbb{Q}} F$, equipped with the isomorphism $\varepsilon : \widehat{V}(A^F) \simeq V \otimes_{\mathbb{Q}} \mathbb{A}_f \otimes_{\mathbb{Q}} F$ induced by the identity on V . Since $\rho_p^{\mathbf{G}}$ and ρ^p act via K , the K -orbit of ε is Γ_E -invariant. Thus, the triple $(A^F, \lambda \otimes F, \varepsilon)$, defines a point $\tilde{x}_A \in \text{Sh}_K(\mathbf{H}, X_{\mathbf{H}})(E)$. (Note that, since ψ is \mathbf{H} -invariant, up to scalars, λ is defined over E as a weak polarization).

By our choice of F , the triple $(\mathbf{H}, X_{\mathbf{H}}, \mathcal{H})$ satisfies the assumptions of Theorem 6.1.4. Thus we may apply it to the reduction $x_A \in \mathcal{S}_K(\mathbf{H}, X_{\mathbf{H}})(\mathbb{F}_q)$, where $\mathcal{S}_K(\mathbf{H}, X_{\mathbf{H}})$ is the integral model constructed from a choice of auxiliary Hodge type Shimura datum. This implies that there exists $\gamma \in \text{Conj}_{\mathbf{H}}(\mathbb{Q})$ such that for all $\ell \neq p$, we have $\gamma = \gamma_\ell(v)$ in $\text{Conj}_{\mathbf{H}}(\mathbb{Q}_\ell)$. By Lemma 7.1.8, it follows that $\gamma \in \text{Conj}_{\mathbf{G}}(\mathbb{Q})$ and $\gamma = \gamma_\ell(v)$ in $\text{Conj}_{\mathbf{G}}(\mathbb{Q}_\ell)$. \square

Remark 7.2.6. In the proof of Theorem 7.2.4, we used an integral $\mathcal{S}_K(\mathbf{H}, X_{\mathbf{H}})$ which depends on the choice of an auxiliary Shimura datum of Hodge type. As mentioned in Remark 5.3.10, such a model should be independent of choices. In any case, all we use is that such a model exists which satisfies the extension property in Theorem 5.2.13 (2) and the conclusion of Theorem 6.1.4.

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