# RATIONAL POINTS ON TWISTED K3 SURFACES AND DERIVED EQUIVALENCES

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Abstract. Using a construction of Hassett and Várilly-Alvarado, we produce derived equivalent twisted K3 surfaces over $\mathbf{Q}$, $\mathbf{Q}_2$, and $\mathbf{R}$, where one has a rational point and the other does not. This answers negatively a question recently raised by Hassett and Tschinkel.

## 1. Introduction

A twisted K3 surface is a pair $(X, \alpha)$, where $X$ is a K3 surface and $\alpha \in \mathrm{Br}(X)$ is a Brauer class. In a recent survey paper [5], Hassett and Tschinkel asked whether the existence of a rational point on a twisted K3 surface is invariant under derived equivalence. More precisely, they asked:

**Question.** *Let $(X_1, \alpha_1)$ and $(X_2, \alpha_2)$ be twisted K3 surfaces over a field $k$. Suppose there is a $k$-linear equivalence*

$$\mathrm{D^b}(X_1, \alpha_1) \simeq \mathrm{D^b}(X_2, \alpha_2)$$

*of twisted derived categories. Then is the existence of a $k$-point of $(X_1, \alpha_1)$ equivalent to the existence of a $k$-point of $(X_2, \alpha_2)$?*

By definition, a $k$-point of a twisted K3 surface $(X, \alpha)$ is a point $x \in X(k)$ such that the evaluation $\alpha(x) = 0 \in \mathrm{Br}(k)$. Equivalently, it is a $k$-point of the $\mathbf{G}_m$-gerbe over $X$ associated to $\alpha$.

In [5], it is shown that for the untwisted case of the question where $\alpha_1, \alpha_2$ vanish, the answer is positive over certain fields $k$, e.g. $\mathbf{R}$, finite fields, and $p$-adic fields (provided the $X_i$ have good reduction, or $p \geq 7$ and the $X_i$ have ADE reduction). The purpose of this paper is to show that if $\alpha_1, \alpha_2$ are allowed to be nontrivial, the answer to the question is negative for $k = \mathbf{Q}, \mathbf{Q}_2$, and $\mathbf{R}$.

We work over a field $k$ of characteristic not equal to 2, and consider a double cover $Y \to \mathbf{P}^2 \times \mathbf{P}^2$ ramified over a divisor of bidegree $(2, 2)$. The projection $\pi_i : Y \to \mathbf{P}^2$ onto the $i$-th $\mathbf{P}^2$ factor, $i = 1, 2$, realizes $Y$ as a quadric fibration. Provided that the discriminant divisor of $\pi_i$ is smooth, the Stein factorization of the relative Fano variety of lines of $\pi_i$ is a K3 surface $X_i$, which comes with a natural Brauer class $\alpha_i$. In this setup, we prove the following result.

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**Theorem 1.1.** *There is a $k$-linear equivalence* $\mathrm{D^b}(X_1, \alpha_1) \simeq \mathrm{D^b}(X_2, \alpha_2)$.

We note that this result seems to be known to the experts (at least for $k = \mathbf{C}$), but we could not find a proof in the literature.

Hassett and Várilly-Alvarado studied the above construction of twisted K3s in relation to rational points [6]. They show that over $k = \mathbf{Q}$, if certain conditions are imposed on the branch divisor $Z \subset \mathbf{P}^2 \times \mathbf{P}^2$ of $Y$, the class $\alpha_1$ gives a (transcendental) Brauer–Manin obstruction to the Hasse principle on $X_1$. A priori, $\alpha_2$ need not obstruct the existence of rational points on $X_2$. In fact, it is possible that $X_2$ has rational points, but the conditions imposed on $Z$ result in very large coefficients of the defining equation of $X_2$, making a computer search for points infeasible.

In this paper, we observe that the 2-adic condition imposed by Hassett and Várilly-Alvarado can be relaxed, while still guaranteeing $\alpha_1$ gives a Brauer–Manin obstruction (see Lemma 4.5). The upshot is that the defining coefficients of $X_2$ are much smaller, making it easy to find rational points with a computer. Up to modifying the $\alpha_i$ by a Brauer class pulled back from $k = \mathbf{Q}$, we obtain the desired example over $\mathbf{Q}$. We also check the example "localizes" over $\mathbf{Q}_2$ and $\mathbf{R}$. More precisely, we prove:

**Theorem 1.2.** *For $k = \mathbf{Q}, \mathbf{Q}_2$, or $\mathbf{R}$, the divisor $Z \subset \mathbf{P}^2 \times \mathbf{P}^2$ can be chosen so that there are Brauer classes $\alpha_i' \in \mathrm{Br}(X_i)$, congruent to $\alpha_i$ modulo $\mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X_i))$, such that:*

(1) *There is a $k$-linear equivalence* $\mathrm{D^b}(X_1, \alpha_1') \simeq \mathrm{D^b}(X_2, \alpha_2')$,
(2) $(X_1, \alpha_1')$ *has no $k$-point,*
(3) $(X_2, \alpha_2')$ *has a $k$-point.*

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## 2. Construction of the twisted K3 surfaces

In this section, $k$ denotes a base field of characteristic not equal to 2.

2.1. **Quadric fibrations.** We start by reviewing some terminology on quadric fibrations. Let $S$ be a variety over $k$, i.e. an integral, separated scheme of finite type over $k$. Let $\mathcal{E}$ be a rank $n \geq 2$ vector bundle on $S$, i.e. a locally free $\mathcal{O}_S$-module of rank $n$. Our convention is that the projective bundle of $\mathcal{E}$ is the morphism

$$p : \mathbf{P}(\mathcal{E}) = \mathrm{Proj}_S(\mathrm{Sym}^\bullet(\mathcal{E}^*)) \to S.$$

A quadric fibration is determined by a line bundle $\mathcal{L}$ on $S$ and a nonzero section

$$s \in \Gamma(\mathbf{P}(\mathcal{E}), \mathcal{O}_{\mathbf{P}(\mathcal{E})}(2) \otimes p^*\mathcal{L}) = \Gamma(S, \mathrm{Sym}^2(\mathcal{E}^*) \otimes \mathcal{L}).$$

Namely, the zero locus of $s$ on $\mathbf{P}(\mathcal{E})$ defines a subvariety $Q$, and the restriction $\pi : Q \to S$ of $p : \mathbf{P}(\mathcal{E}) \to S$ is the associated *quadric fibration*, which if flat is of relative dimension $n - 2$. Below we will be specifically interested in flat quadric fibrations of relative dimension 2, which we refer to as *quadric surface fibrations*.

Note that the section of $\mathrm{Sym}^2(\mathcal{E}^*) \otimes \mathcal{L}$ defining a quadric fibration corresponds to a morphism $q : \mathcal{E} \to \mathcal{E}^* \otimes \mathcal{L}$. Taking the determinant gives rise to a section of $\det(\mathcal{E}^*)^2 \otimes \mathcal{L}^n$ whose vanishing defines the *discriminant locus* $D \subset S$, which is a divisor provided $\pi : Q \to S$ is generically smooth. The fibration $\pi : Q \to S$ is said to have *simple degeneration* if the fiber over every closed point of $S$ is a quadric of corank $\leq 1$. We note that if $\pi : Q \to S$ is flat and generically smooth and $S$ is smooth over $k$, then the discriminant divisor $D$ is smooth over $k$ if and only if $Q$ is smooth over $k$ and $\pi$ has simple degeneration [1, Proposition 1.6].

2.2. **Twisted K3 surfaces.** Let $V_1$ and $V_2$ be 3-dimensional vector spaces over $k$. We denote by $H_i$ the hyperplane class on $\mathbf{P}(V_i)$; by abuse of notation, we denote by the same letter the pullback of $H_i$ to any variety mapping to $\mathbf{P}(V_i)$. Let

$$\pi : Y \to \mathbf{P}(V_1) \times \mathbf{P}(V_2)$$

be the double cover of $\mathbf{P}(V_1) \times \mathbf{P}(V_2)$ ramified over a smooth divisor $Z$ in the linear system $|2H_1 + 2H_2|$. Let $\mathrm{pr}_i : \mathbf{P}(V_1) \times \mathbf{P}(V_2) \to \mathbf{P}(V_i)$ be the $i$-th projection, and let $\pi_i = \mathrm{pr}_i \circ \pi : Y \to \mathbf{P}(V_i)$.

**Lemma 2.1.** *Let $\mathcal{E}_1 = (V_2 \otimes \mathcal{O}) \oplus \mathcal{O}(H_1)$ on $\mathbf{P}(V_1)$ and $\mathcal{E}_2 = (V_1 \otimes \mathcal{O}) \oplus \mathcal{O}(H_2)$ on $\mathbf{P}(V_2)$. Then for $i = 1, 2$ there is a commutative diagram*

$$\begin{array}{ccc} Y & \xrightarrow{\ j_i\ } & \mathbf{P}(\mathcal{E}_i) \\ {\scriptstyle \pi_i}\downarrow & \swarrow {\scriptstyle p_i} & \\ \mathbf{P}(V_i) & & \end{array}$$

*where $j_i$ is a closed immersion with $j_1^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_1)}(1) = \mathcal{O}_Y(H_2)$ and $j_2^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_2)}(1) = \mathcal{O}_Y(H_1)$. Moreover, $Y$ is cut out on $\mathbf{P}(\mathcal{E}_i)$ by a section of $\mathcal{O}_{\mathbf{P}(\mathcal{E}_i)}(2) \otimes \mathcal{O}(2H_i)$, so that $\pi_i$ is a quadric surface fibration.*

*Proof.* Consider the case $i = 1$. The morphism $j_1 : Y \to \mathbf{P}(\mathcal{E}_1)$ is given by the $\pi_1$-very ample line bundle $\mathcal{O}_Y(H_2)$. More precisely, using $\pi_*(\mathcal{O}_Y) = \mathcal{O} \oplus \mathcal{O}(-H_1 - H_2)$, we find

$$\begin{aligned} \pi_{1*}(\mathcal{O}_Y(H_2)) &= \mathrm{pr}_{1*}(\mathcal{O}(H_2) \oplus \mathcal{O}(-H_1)) \\ &= (V_2^* \otimes \mathcal{O}) \oplus \mathcal{O}(-H_1) \\ &= \mathcal{E}_1^*. \end{aligned}$$

Working locally on $\mathbf{P}(V_1)$, we see the canonical map $\pi_1^*\mathcal{E}_1^* = \pi_1^*\pi_{1*}(\mathcal{O}_Y(H_2)) \to \mathcal{O}_Y(H_2)$ is surjective and the corresponding morphism $j_1 : Y \to \mathbf{P}(\mathcal{E}_1)$ is an immersion. By construction $j_1^*\mathcal{O}_{\mathbf{P}(\mathcal{E}_1)}(1) = \mathcal{O}_Y(H_2)$. Moreover, if $\zeta$ denotes the class of $\mathcal{O}_{\mathbf{P}(\mathcal{E}_1)}(1)$ in $\mathrm{Pic}(\mathbf{P}(\mathcal{E}_1))$, then it is easy to compute

$$[Y] = 2\zeta + 2H_1 \in \mathrm{Pic}(\mathbf{P}(\mathcal{E}_1))$$

by using the intersection numbers $H_1^2H_2^2 = 2$ and $H_1H_2^3 = 0$ on $Y$. So $Y$ is indeed a quadric surface fibration, cut out by a section of $\mathcal{O}_{\mathbf{P}(\mathcal{E}_1)}(2) \otimes \mathcal{O}(2H_1)$ on $\mathbf{P}(\mathcal{E}_1)$. $\square$

Let $D_i$ denote the discriminant divisor of $\pi_i : Y \to \mathbf{P}(V_i)$. It follows from the lemma that $D_i$ is defined by a section of $\det(\mathcal{E}_i^*)^2 \otimes \mathcal{O}(8H_i) = \mathcal{O}(6H_i)$, i.e. $D_i \subset \mathbf{P}(V_i)$ is a sextic curve. Let $f_i : X_i \to \mathbf{P}(V_i)$ be the double cover of $\mathbf{P}(V_i)$ ramified over $D_i$. If $D_i$ is smooth (equivalently, if $\pi_i$ has simple degeneration), then $X_i$ is a smooth K3 surface. Moreover, $X_i$ comes equipped with an Azumaya algebra $\mathcal{A}_i$, as follows.

In general, consider a generically smooth quadric surface fibration $\pi : Q \to S$ over a smooth $k$-variety $S$, with smooth discriminant divisor and simple degeneration. Let $\mathcal{F} \to S$ be the relative Fano variety of lines of $\pi$. It follows from [7, Proposition 3.3] that Stein factorization gives morphisms

$$\mathcal{F} \xrightarrow{g} X \xrightarrow{f} S,$$

where $g$ is an étale locally trivial $\mathbf{P}^1$-bundle over $X$ and $f$ is the double cover of $S$ branched along the discriminant divisor $D$. The morphism $g$ corresponds to an Azumaya algebra $\mathcal{A}$ on $X$.

Applying this discussion to $\pi_i : Y \to \mathbf{P}(V_i)$, we see that if $D_i$ is smooth, then $X_i$ is equipped with an Azumaya algebra $\mathcal{A}_i$. Of course $\mathcal{A}_i$ represents a Brauer class $\alpha_i \in \mathrm{Br}(X_i)$, so we can regard the pair $(X_i, \mathcal{A}_i)$ as a twisted K3 surface.

## 3. Derived equivalence of the twisted K3 surfaces

In this section, we prove the twisted K3 surfaces $(X_i, \mathcal{A}_i)$ of the previous section are derived equivalent. Our proof works over any field $k$ of characteristic not equal to 2, and gives an explicit functor inducing the equivalence. The key tool is Kuznetsov's semiorthogonal decomposition of the derived category of a quadric fibration [9].

### 3.1. **Conventions.**

All triangulated categories appearing below will be $k$-linear, and functors between them will be $k$-linear and exact.

For a variety $X$, we denote by $\mathrm{D^b}(X)$ the bounded derived category of coherent sheaves on $X$, regarded as a triangulated category. More generally, for any sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$ which is coherent as an $\mathcal{O}_X$-module, we denote by $\mathrm{D^b}(X, \mathcal{A})$ the bounded derived category of coherent sheaves of right $\mathcal{A}$-modules on $X$. We note that if $\mathcal{A}$ is an Azumaya algebra corresponding to a Brauer class $\alpha \in \mathrm{Br}(X)$, then the bounded derived category of $\alpha$-twisted sheaves $\mathrm{D^b}(X, \alpha)$ is equivalent to $\mathrm{D^b}(X, \mathcal{A})$.

As a rule, all functors we consider are derived. More precisely, for a morphism of varieties $f : X \to Y$, we simply write $f_* : \mathrm{D^b}(X) \to \mathrm{D^b}(Y)$ for the derived pushforward (provided $f$ is proper) and $f^* : \mathrm{D^b}(Y) \to \mathrm{D^b}(X)$ for the derived pullback (provided $f$ has finite Tor-dimension). Similarly, for $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(X)$, we write $\mathcal{F} \otimes \mathcal{G} \in \mathrm{D^b}(X)$ for the derived tensor product.

### 3.2. **Semiorthogonal decompositions.**

One way to understand the derived category of a variety (or more generally a triangulated category) is by "decomposing" it into simpler pieces. This is formalized by the notion of a semiorthogonal decomposition, which plays a central role in the rest of this section. We summarize the rudiments of this theory; see e.g. [3] and [4] for a more detailed exposition.

**Definition 3.1.** Let $\mathcal{T}$ be a triangulated category. A *semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$$

is a sequence of full triangulated subcategories $\mathcal{A}_1, \dots, \mathcal{A}_n$ of $\mathcal{T}$ — called the *components* of the decomposition — such that:

(1) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{F} \in \mathcal{A}_i, \mathcal{G} \in \mathcal{A}_j$ if $i > j$.
(2) For any $\mathcal{F} \in \mathcal{T}$, there is a sequence of morphisms

$$0 = \mathcal{F}_n \to \mathcal{F}_{n-1} \to \cdots \to \mathcal{F}_1 \to \mathcal{F}_0 = \mathcal{F},$$

such that $\mathrm{Cone}(\mathcal{F}_i \to \mathcal{F}_{i-1}) \in \mathcal{A}_i$.

Semiorthogonal decompositions are closely related to the notion of an *admissible subcategory* of a triangulated category. Such a subcategory $\mathcal{A} \subset \mathcal{T}$ is by definition a full triangulated subcategory such that the inclusion $i : \mathcal{A} \hookrightarrow \mathcal{T}$ admits right and left adjoints $i^! : \mathcal{T} \to \mathcal{A}$ and $i^* : \mathcal{T} \to \mathcal{A}$. For $X$ a smooth proper variety over $k$, the components of any semiorthogonal decomposition of $\mathrm{D^b}(X)$ are in fact admissible subcategories.

The simplest examples of admissible subcategories come from exceptional objects. An object $\mathcal{F} \in \mathcal{T}$ of a triangulated category is called *exceptional* if

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}[p]) = \begin{cases} k & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

If $X$ is a proper variety and $\mathcal{F} \in \mathrm{D^b}(X)$ is exceptional, then the full triangulated subcategory $\langle \mathcal{F} \rangle \subset \mathrm{D^b}(X)$ generated by $\mathcal{F}$ is admissible and equivalent to the derived category of a point via $\mathrm{D^b}(\mathrm{Spec}(k)) \to \mathrm{D^b}(X) : V \mapsto V \otimes \mathcal{F}$. To simplify notation, we write $\mathcal{F}$ in place of $\langle \mathcal{F} \rangle$ when $\langle \mathcal{F} \rangle$ appears as a component in a semiorthogonal decomposition, i.e. instead of $\mathrm{D^b}(X) = \langle \dots, \langle \mathcal{F} \rangle, \dots \rangle$ we write $\mathrm{D^b}(X) = \langle \dots, \mathcal{F}, \dots \rangle$.

**Example 3.2.** It is easy to see any line bundle on projective space $\mathbf{P}^n$ is exceptional as an object of $\mathrm{D^b}(\mathbf{P}^n)$. In fact, Beilinson [2] showed $\mathrm{D^b}(\mathbf{P}^n)$ has a semiorthogonal decomposition into $n+1$ line bundles, namely

$$\mathrm{D^b}(\mathbf{P}^n) = \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n) \rangle.$$

Given one semiorthogonal decomposition of a triangulated category $\mathcal{T}$, others can be obtained via mutation functors. If $i : \mathcal{A} \hookrightarrow \mathcal{T}$ is the inclusion of an admissible subcategory, the *left* and *right mutation functors* $\mathrm{L}_{\mathcal{A}} : \mathcal{T} \to \mathcal{T}$ and $\mathrm{R}_{\mathcal{A}} : \mathcal{T} \to \mathcal{T}$ are defined by the formulas

$$\mathrm{L}_{\mathcal{A}}(\mathcal{F}) = \mathrm{Cone}(ii^!\mathcal{F} \to \mathcal{F}) \quad \text{and} \quad \mathrm{R}_{\mathcal{A}}(\mathcal{F}) = \mathrm{Cone}(\mathcal{F} \to ii^*\mathcal{F})[-1],$$

where $ii^!\mathcal{F} \to \mathcal{F}$ and $\mathcal{F} \to ii^*\mathcal{F}$ are the counit and unit morphisms of the adjunctions. These functors satisfy the following basic properties.

**Lemma 3.3.** *The mutation functors* $\mathrm{L}_{\mathcal{A}}$ *and* $\mathrm{R}_{\mathcal{A}}$ *annihilate* $\mathcal{A}$. *Moreover, they restrict to mutually inverse equivalences*

$$\mathrm{L}_{\mathcal{A}}|_{{}^\perp\mathcal{A}} : {}^\perp\mathcal{A} \xrightarrow{\sim} \mathcal{A}^\perp \quad \text{and} \quad \mathrm{R}_{\mathcal{A}}|_{\mathcal{A}^\perp} : \mathcal{A}^\perp \xrightarrow{\sim} {}^\perp\mathcal{A},$$

*where* $\mathcal{A}^\perp$ *and* ${}^\perp\mathcal{A}$ *are the* right *and* left orthogonal *categories to* $\mathcal{A}$*, i.e. the full subcategories of* $\mathcal{T}$ *defined by*

$$\mathcal{A}^\perp = \{\mathcal{F} \in \mathcal{T} \mid \operatorname{Hom}(\mathcal{G}, \mathcal{F}) = 0 \textit{ for all } \mathcal{G} \in \mathcal{A}\},$$
$${}^\perp\mathcal{A} = \{\mathcal{F} \in \mathcal{T} \mid \operatorname{Hom}(\mathcal{F}, \mathcal{G}) = 0 \textit{ for all } \mathcal{G} \in \mathcal{A}\}.$$

The following lemma describes the action of mutation functors on a semiorthogonal decomposition.

**Lemma 3.4.** *Let* $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ *be a semiorthogonal decomposition with admissible components. Then for* $1 \le i \le n-1$ *there is a semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-1}, \mathrm{L}_{\mathcal{A}_i}(\mathcal{A}_{i+1}), \mathcal{A}_i, \mathcal{A}_{i+2}, \dots, \mathcal{A}_n \rangle,$$

*and for* $2 \le i \le n$ *there is a semiorthogonal decomposition*

$$\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_{i-2}, \mathcal{A}_i, \mathrm{R}_{\mathcal{A}_i}(\mathcal{A}_{i-1}), \mathcal{A}_{i+1}, \dots, \mathcal{A}_n \rangle.$$

We will also need the following lemma, which allows us to compute the effect of a mutation functor in a special case. It follows easily from Serre duality.

**Lemma 3.5.** *Let* $X$ *be a smooth projective variety over* $k$*, and let* $\mathrm{D^b}(X) = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$ *be a semiorthogonal decomposition. Then* $\mathrm{L}_{\langle \mathcal{A}_1, \dots, \mathcal{A}_{n-1} \rangle}(\mathcal{A}_n) = \mathcal{A}_n \otimes \omega_X$*, where* $\mathcal{A}_n \otimes \omega_X$ *denotes the image of* $\mathcal{A}_n$ *under the autoequivalence* $\mathcal{F} \mapsto \mathcal{F} \otimes \omega_X$ *of* $\mathrm{D^b}(X)$*.*

3.3. **Derived categories of quadric fibrations.** Let $\pi : Q \to S$ be a quadric fibration associated to a rank $n$ vector bundle $\mathcal{E}$ and a section of $\mathrm{Sym}^2(\mathcal{E}^*) \otimes \mathcal{L}$, as in Section 2.1. Then there is an associated *even Clifford algebra* $\mathscr{C}\!\ell_0$, which is a sheaf of algebras on $S$ given as a certain quotient of the tensor algebra $\mathrm{T}^\bullet(\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{L}^*)$. For the precise definition, see [1, Section 1.5] (cf. [9, Section 3.3]). We note that $\mathscr{C}\!\ell_0$ admits an $\mathcal{O}_S$-module filtration of length $\lfloor \frac{n}{2} \rfloor$ with associated graded pieces $\wedge^{2i}\mathcal{E} \otimes (\mathcal{L}^*)^i$.

In case the fibration $\pi : Q \to S$ is flat and $S$ is smooth over $k$, Kuznestov [9] established a semiorthogonal decomposition of $\mathrm{D^b}(Q)$ into a copy of $\mathrm{D^b}(S, \mathscr{C}\!\ell_0)$ and a number of copies of $\mathrm{D^b}(S)$. In fact, Kuznetsov stated his result under the assumption that $k$ is algebraically closed of characteristic 0, but as explained in [1, Theorem 2.11], the proof works without this hypothesis.

**Theorem 3.6** ([9, Theorem 4.2])**.** *Let* $\pi : Q \to S$ *be a flat quadric fibration of relative dimension* $n-2$ *over a smooth* $k$*-variety* $S$*. Let* $\mathcal{O}_Q(1)$ *denote the restriction of* $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ *to* $Q$*. Then the functor* $\pi^* : \mathrm{D^b}(S) \to \mathrm{D^b}(Q)$ *is fully faithful, and there is a fully faithful functor* $\Phi : \mathrm{D^b}(S, \mathscr{C}\!\ell_0) \to \mathrm{D^b}(Q)$ *such that there is a semiorthogonal decomposition*

$$\mathrm{D^b}(Q) = \langle \Phi(\mathrm{D^b}(S, \mathscr{C}\!\ell_0)), \pi^*\mathrm{D^b}(S) \otimes \mathcal{O}_Q(1), \dots, \pi^*\mathrm{D^b}(S) \otimes \mathcal{O}_Q(n-2) \rangle.$$

**Remark 3.7.** The functor $\Phi : \mathrm{D^b}(S, \mathscr{C}\!\ell_0) \to \mathrm{D^b}(Q)$ is given by an explicit Fourier–Mukai kernel, see [9, Section 4].

Now assume $\pi : Q \to S$ is a generically smooth quadric surface fibration over a smooth $k$-variety $S$, with smooth discriminant divisor and simple degeneration. As in the discussion at the end of Section 2.2, the double cover $f : X \to S$ ramified over $D$

is equipped with an Azumaya algebra $\mathcal{A}$. In terms of this data, we have the following alternative description of $\mathrm{D^b}(S, \mathcal{C}\ell_0)$, see [1, Proposition B.3] or [10, Lemma 4.2].

**Lemma 3.8.** *In the above situation, there is an isomorphism $f_*\mathcal{A} \cong \mathcal{C}\ell_0$. In particular, pushforward by $f$ induces an equivalence $f_* : \mathrm{D^b}(X, \mathcal{A}) \xrightarrow{\sim} \mathrm{D^b}(S, \mathcal{C}\ell_0)$.*

3.4. **Derived equivalence.** Let $\pi : Y \to \mathbf{P}(V_1) \times \mathbf{P}(V_2)$ be as in Section 2.2. Assume the discriminant divisors $D_i$ of the quadric fibrations $\pi_i : Y \to \mathbf{P}(V_i)$ are smooth, so that we get associated twisted K3 surfaces $(X_i, \mathcal{A}_i)$. Let $\mathcal{C}\ell_{0,i}$ denote the even Clifford algebra of the quadric fibration $\pi_i : Y \to \mathbf{P}(V_i)$. Then Lemma 3.8 gives an equivalence $f_{i*} : \mathrm{D^b}(X_i, \mathcal{A}_i) \xrightarrow{\sim} \mathrm{D^b}(\mathbf{P}(V_i), \mathcal{C}\ell_{0,i})$. Finally, let $\Phi_i : \mathrm{D^b}(\mathbf{P}(V_i), \mathcal{C}\ell_{0,i}) \to \mathrm{D^b}(Y)$ be the fully faithful functor from Theorem 3.6. In this setup, we prove the following result.

**Theorem 3.9.** *Assume $D_1$ and $D_2$ are smooth. Then there is an equivalence*

$$\mathrm{D^b}(X_1, \mathcal{A}_1) \simeq \mathrm{D^b}(X_2, \mathcal{A}_2)$$

*given by the composition*

$$f_{2*}^{-1} \circ \Phi_2^* \circ \mathrm{R}_{\mathcal{O}_Y(H_2)} \circ \mathrm{L}_{\mathcal{O}_Y(H_1)} \circ \Phi_1 \circ f_{1*} : \mathrm{D^b}(X_1, \mathcal{A}_1) \to \mathrm{D^b}(X_2, \mathcal{A}_2),$$

*where*

- *$\mathrm{L}_{\mathcal{O}_Y(H_1)}$ is the left mutation functor through $\langle \mathcal{O}_Y(H_1) \rangle \subset \mathrm{D^b}(Y)$,*
- *$\mathrm{R}_{\mathcal{O}_Y(H_2)}$ is the right mutation functor through $\langle \mathcal{O}_Y(H_2) \rangle \subset \mathrm{D^b}(Y)$,*
- *$\Phi_2^*$ is the left adjoint of $\Phi_2$,*
- *$f_{2*}^{-1}$ is the inverse of the equivalence $f_{2*} : \mathrm{D^b}(X_2, \mathcal{A}_2) \xrightarrow{\sim} \mathrm{D^b}(\mathbf{P}(V_2), \mathcal{C}\ell_{0,2})$.*

The theorem is an immediate consequence of the following proposition. We note that the proposition holds without assuming smoothness of the discriminant divisors $D_i$.

**Proposition 3.10.** *There is an equivalence*

$$\mathrm{D^b}(\mathbf{P}(V_1), \mathcal{C}\ell_{0,1}) \simeq \mathrm{D^b}(\mathbf{P}(V_2), \mathcal{C}\ell_{0,2})$$

*given by the composition*

$$\Phi_2^* \circ \mathrm{R}_{\mathcal{O}_Y(H_2)} \circ \mathrm{L}_{\mathcal{O}_Y(H_1)} \circ \Phi_1 : \mathrm{D^b}(\mathbf{P}(V_1), \mathcal{C}\ell_{0,1}) \to \mathrm{D^b}(\mathbf{P}(V_2), \mathcal{C}\ell_{0,2}).$$

*Proof.* Set $\mathcal{C}_i = \Phi_i(\mathrm{D^b}(\mathbf{P}(V_i), \mathcal{C}\ell_{0,i})) \subset \mathrm{D^b}(Y)$. Theorem 3.6 gives semiorthogonal decompositions

$$\mathrm{D^b}(Y) = \langle \mathcal{C}_1, \pi_1^*\mathrm{D^b}(\mathbf{P}(V_1)) \otimes \mathcal{O}(H_2), \pi_1^*\mathrm{D^b}(\mathbf{P}(V_1)) \otimes \mathcal{O}(2H_2) \rangle,$$
$$\mathrm{D^b}(Y) = \langle \mathcal{C}_2, \pi_2^*\mathrm{D^b}(\mathbf{P}(V_2)) \otimes \mathcal{O}(H_1), \pi_2^*\mathrm{D^b}(\mathbf{P}(V_2)) \otimes \mathcal{O}(2H_1) \rangle.$$

Recall Beilinson's decomposition $\mathrm{D^b}(\mathbf{P}(V_i)) = \langle \mathcal{O}, \mathcal{O}(H_i), \mathcal{O}(2H_i) \rangle$ (see Example 3.2). In each of the above decompositions of $\mathrm{D^b}(Y)$, we replace the first copy of $\mathrm{D^b}(\mathbf{P}(V_i))$ by Beilinson's decomposition and the second copy by the same decomposition twisted by $\mathcal{O}(H_i)$:

$$\begin{aligned}\mathrm{D^b}(Y) = \langle \mathcal{C}_1, \mathcal{O}(H_2), \mathcal{O}(H_1+H_2), \mathcal{O}(2H_1+H_2), \\ \mathcal{O}(H_1+2H_2), \mathcal{O}(2H_1+2H_2), \mathcal{O}(3H_1+2H_2)\rangle,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\mathrm{D^b}(Y) = \langle \mathcal{C}_2, \mathcal{O}(H_1), \mathcal{O}(H_1+H_2), \mathcal{O}(H_1+2H_2), \\ \mathcal{O}(2H_1+H_2), \mathcal{O}(2H_1+2H_2), \mathcal{O}(2H_1+3H_2)\rangle.\end{aligned} \tag{3.2}$$

We perform a sequence of mutations that identifies the categories generated by the exceptional objects in (3.1) and (3.2).

First consider (3.1). Mutate $\mathcal{O}(3H_1+2H_2)$ to the far left of the decomposition. Note that $Y$ is smooth with canonical class $K_Y = -2H_1 - 2H_2$, so by Lemma 3.5 the result of the mutation is

$$\begin{aligned}\mathrm{D^b}(Y) = \langle \mathcal{O}(H_1), \mathcal{C}_1, \mathcal{O}(H_2), \mathcal{O}(H_1+H_2), \mathcal{O}(2H_1+H_2), \\ \mathcal{O}(H_1+2H_2), \mathcal{O}(2H_1+2H_2)\rangle.\end{aligned}$$

Left mutating $\mathcal{C}_1$ through $\mathcal{O}(H_1)$ then gives a decomposition

$$\begin{aligned}\mathrm{D^b}(Y) = \langle \mathrm{L}_{\mathcal{O}(H_1)}\mathcal{C}_1, \mathcal{O}(H_1), \mathcal{O}(H_2), \mathcal{O}(H_1+H_2), \mathcal{O}(2H_1+H_2), \\ \mathcal{O}(H_1+2H_2), \mathcal{O}(2H_1+2H_2)\rangle.\end{aligned} \tag{3.3}$$

By the same argument, we obtain from (3.2) a similar decomposition

$$\begin{aligned}\mathrm{D^b}(Y) = \langle \mathrm{L}_{\mathcal{O}(H_2)}\mathcal{C}_2, \mathcal{O}(H_2), \mathcal{O}(H_1), \mathcal{O}(H_1+H_2), \mathcal{O}(H_1+2H_2), \\ \mathcal{O}(2H_1+H_2), \mathcal{O}(2H_1+2H_2)\rangle.\end{aligned} \tag{3.4}$$

Up to permutation, the exceptional objects in the decompositions (3.3) and (3.4) agree, hence they generate the same subcategory of $\mathrm{D^b}(Y)$. It follows that $\mathrm{L}_{\mathcal{O}(H_1)}\mathcal{C}_1$ and $\mathrm{L}_{\mathcal{O}(H_2)}\mathcal{C}_2$ coincide, as both are the right orthogonal to the same subcategory. Now the proposition follows since $\mathrm{R}_{\mathcal{O}(H_2)} \circ \mathrm{L}_{\mathcal{O}(H_2)} \cong \mathrm{id}$ on ${}^{\perp}\langle \mathcal{O}(H_2)\rangle$ by Lemma 3.3. □

**Remark 3.11.** The equivalence $\mathrm{D^b}(X_1, \alpha_1) \simeq \mathrm{D^b}(X_2, \alpha_2)$ of Theorem 3.9 implies other relations between $X_1$ and $X_2$. For instance, over $k = \mathbf{C}$ it implies the Picard numbers of $X_1$ and $X_2$ agree. Indeed, it suffices to note that the equivalence induces an isomorphism of twisted transcendental lattices $\mathrm{T}(X_1, \alpha_1) \cong \mathrm{T}(X_2, \alpha_2)$, whose ranks are the same as the usual transcendental lattices (see [8]).

## 4. Equations for the twisted K3 surfaces and local invariants

Let $k$ be a number field. Then for any place $v$ of $k$, class field theory provides an embedding $\mathrm{inv}_v : \mathrm{Br}(k_v) \to \mathbf{Q}/\mathbf{Z}$ (which is an isomorphism for nonarchimedian $v$). Now let $X$ be a smooth, projective, geometrically integral variety over $k$. Any subset $S \subset \mathrm{Br}(X)$ cuts out a subset $X(\mathbf{A}_k)^S \subset X(\mathbf{A}_k)$ of the adelic points of $X$, given by

$$X(\mathbf{A}_k)^S = \left\{ (x_v) \in X(\mathbf{A}_k) \mid \sum_v \mathrm{inv}_v\, \alpha(x_v) = 0 \text{ for all } \alpha \in S \right\}.$$

For fixed $(x_v) \in X(\mathbf{A}_k)$ and $\alpha \in \mathrm{Br}(X)$, the evaluation $\alpha(x_v) = 0$ for all but finitely many $v$, so the above sum is well-defined. Moreover, class field theory gives inclusions

$$X(k) \subset X(\mathbf{A}_k)^S \subset X(\mathbf{A}_k).$$

Hence, if $X(\mathbf{A}_k)^S$ is empty for some $S$, then $X$ has no $k$-points. We note that if $X(\mathbf{A}_k)^S$ is empty but $X(\mathbf{A}_k)$ is not, then $S$ is said to give a *Brauer–Manin obstruction* to the Hasse principle. See [12, 5.2] for more details.

In this section, we describe conditions on the $(2,2)$ divisor $Z \subset \mathbf{P}(V_1) \times \mathbf{P}(V_2)$ from Section 2.2, which allow us to control the local invariants $\mathrm{inv}_v\, \alpha_1(x_v)$ for any $v$-adic point $x_v \in X_1(k_v)$. In the end, we will see that if $k = \mathbf{Q}$ and enough conditions are met, then

$$\mathrm{inv}_v\, \alpha_1(x_v) = \begin{cases} 0 & \text{if } v \text{ is finite,} \\ \frac{1}{2} & \text{if } v \text{ is real,} \end{cases}$$

for all $x_v \in X_1(k_v)$. Hence $X_1(\mathbf{A}_k)^{\alpha_1}$ is empty and $X_1$ has no $k$-points. Our discussion follows [6] very closely, and differs only in the treatment of the 2-adic place (see Lemma 4.5).

4.1. **Equations for the twisted K3 surfaces.** Let the notation be as in Section 2.2 (in particular $k$ may be any field of characteristic not equal to 2).

Choose coordinates $x_0, x_1, x_2$ on $\mathbf{P}(V_1)$ and $y_0, y_1, y_2$ on $\mathbf{P}(V_2)$. The equation defining $Z$ can be written as

$$\begin{aligned} &A(x_0,x_1,x_2)y_0^2 + B(x_0,x_1,x_2)y_0y_1 + C(x_0,x_1,x_2)y_0y_2 + \\ &D(x_0,x_1,x_2)y_1^2 + E(x_0,x_1,x_2)y_1y_2 + F(x_0,x_1,x_2)y_2^2, \end{aligned} \tag{4.1}$$

where $A, \ldots, F$ are degree 2 homogeneous polynomials in the $x_i$, or as

$$\begin{aligned} &A'(y_0,y_1,y_2)x_0^2 + B'(y_0,y_1,y_2)x_0x_1 + C'(y_0,y_1,y_2)x_0x_2 + \\ &D'(y_0,y_1,y_2)x_1^2 + E'(y_0,y_1,y_2)x_1x_2 + F'(y_0,y_1,y_2)x_2^2. \end{aligned} \tag{4.2}$$

where $A', \ldots, F'$ are degree 2 homogeneous polynomials in the $y_i$. The first or second expression is useful depending on whether we regard $Y$ as a quadric fibration over $\mathbf{P}(V_1)$ or $\mathbf{P}(V_2)$. The following lemma summarizes the computations of [6, Section 3].

**Lemma 4.1.** (1) *Let*

$$M = \begin{pmatrix} 2A & B & C \\ B & 2D & E \\ C & E & 2F \end{pmatrix}$$

*Then the discriminant curve $D_1 \subset \mathbf{P}(V_1)$ is defined by* $\det(M) = 0$*, and $X_1$ is defined in the weighted projective space $\mathbf{P}(1,1,1,3)$ with coordinates $x_0, x_1, x_2, w$ by*

$$w^2 = -\frac{1}{2}\det(M).$$

*The analogous statements hold for* $D_2 \subset \mathbf{P}(V_2)$ *and* $X_2$ *with* $M$ *replaced by*

$$M' = \begin{pmatrix} 2A' & B' & C' \\ B' & 2D' & E' \\ C' & E' & 2F' \end{pmatrix}.$$

(2) *Define*

$$M_A = 4DF - E^2, \quad M_D = 4AF - C^2, \quad M_F = 4AD - B^2.$$

*Assume* $D_1 \subset \mathbf{P}(V_1)$ *is smooth, so that we have a twisted K3 surface* $(X_1, \alpha_1)$*. Then the image of* $\alpha_1$ *under the injection* $\mathrm{Br}(X_1) \to \mathrm{Br}(k(X_1))$ *(where* $k(X_1)$ *is the function field of* $X_1$*) can be represented by any of the following Hilbert symbols:*

$$(-M_F, A),\ (-M_D, A),\ (-M_F, D),\ (-M_A, D),\ (-M_D, F),\ (-M_A, F).$$

*Defining* $M'_{A'}, M'_{D'}, M'_{F'}$ *similarly, the analogous statement holds for* $\alpha_2 \in \mathrm{Br}(X_2)$*.*

From now on, we assume $D_1 \subset \mathbf{P}(V_1)$ is smooth, so that $(X_1, \alpha_1)$ is defined.

4.2. **Conditions controlling the local invariants.** The following result holds by Proposition 4.1 and Lemma 4.2 of [6]. It allows us to control local invariants at finite places of bad reduction, assuming the place is not 2-adic and the singularities are mild.

**Proposition 4.2.** *Let* $F$ *be a finite extension of* $\mathbf{Q}_p$ *for* $p \neq 2$*, and denote by* $\mathcal{O}_F$ *the ring of integers of* $F$*. Let* $X$ *be a K3 surface over* $F$*. Let* $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ *be a flat, proper morphism from a regular scheme* $\mathcal{X}$*, with generic fiber* $\mathcal{X}_\eta \cong X$*. Assume the singular locus of the geometric special fiber* $\mathcal{X}_{\bar{s}}$ *consists of less than* 8 *points, each of which is an ordinary double point. If* $X(F) \neq \emptyset$*, then for any* 2*-power torsion Brauer class* $\alpha \in \mathrm{Br}(X)[2^\infty]$*, the map* $X(F) \to \mathrm{Br}(F)$ *given by evaluation of* $\alpha$ *is constant. In particular,* $\alpha(x) = 0$ *for all* $x \in X(F)$ *if this holds for a single* $x$*.*

The next result is [6, Lemma 4.4]. It guarantees that the local invariants of $\alpha_1 \in \mathrm{Br}(X_1)$ vanish at finite places of good reduction, away from the prime 2.

**Lemma 4.3.** *Let* $k$ *be a number field. Let* $v$ *be a finite place of good reduction for* $X_1$ *which is not* 2*-adic. Then* $\mathrm{inv}_v\, \alpha_1(x) = 0$ *for all* $x \in X_1(k_v)$*.*

We are left to control the real and 2-adic invariants of $\alpha_1 \in \mathrm{Br}(X_1)$. The following result, which is [6, Corollary 4.6], gives conditions which guarantee $\alpha_1$ is nontrivial at any real point of $X_1$.

**Lemma 4.4.** *Let* $k = \mathbf{Q}$*. Assume the polynomials* $A, \ldots, F$ *from* (4.1)*, when regarded as quadratic forms, satisfy:*

(1) $A, D$*, and* $F$ *are negative definite,*
(2) $B, C$*, and* $E$ *are positive definite.*

*If* $\infty$ *denotes the real place, then* $\mathrm{inv}_\infty\, \alpha_1(x) = 1/2$ *for all* $x \in X_1(\mathbf{R})$*.*

The following lemma improves [6, Lemma 4.7], giving conditions such that $\alpha_1$ is trivial at every 2-adic point of $X_1$.

**Lemma 4.5.** *Let $k = \mathbf{Q}$. Write the polynomials $A, \dots, F \in \mathbf{Q}[x_0, x_1, x_2]$ from (4.1) as*

$$A = A_1x_0^2 + A_2x_0x_1 + A_3x_0x_2 + A_4x_1^2 + A_5x_1x_2 + A_6x_2^2,$$
$$B = B_1x_0^2 + B_2x_0x_1 + B_3x_0x_2 + B_4x_1^2 + B_5x_1x_2 + B_6x_2^2,$$
$$C = C_1x_0^2 + C_2x_0x_1 + C_3x_0x_2 + C_4x_1^2 + C_5x_1x_2 + C_6x_2^2,$$
$$D = D_1x_0^2 + D_2x_0x_1 + D_3x_0x_2 + D_4x_1^2 + D_5x_1x_2 + D_6x_2^2,$$
$$E = E_1x_0^2 + E_2x_0x_1 + E_3x_0x_2 + E_4x_1^2 + E_5x_1x_2 + E_6x_2^2,$$
$$F = F_1x_0^2 + F_2x_0x_1 + F_3x_0x_2 + F_4x_1^2 + F_5x_1x_2 + F_6x_2^2.$$

*Suppose the coefficients of $A, \dots, F$ satisfy:*

(1) *The 2-adic valuation of $A_1, B_1, C_6, D_4, E_4$, and $F_6$ is 0.*
(2) *The 2-adic valuation of all other coefficients is $> 0$.*

*Then* $\operatorname{inv}_2 \alpha_1(x) = 0$ *for all* $x \in X_1(\mathbf{Q}_2)$.

*Proof.* Let $x = [x_0, x_1, x_2, w]$ be a point of $X_1(\mathbf{Q}_2) \subset \mathbf{P}(1,1,1,3)(\mathbf{Q}_2)$. By scaling the coordinates, we may assume $x_0, x_1, x_2 \in \mathbf{Z}_2$ and at least one of the $x_i$ is a unit. By Lemma 4.1, the Hilbert symbols

$$(B^2 - 4AD, A), \quad (E^2 - 4DF, D), \quad (C^2 - 4AF, F)$$

all represent the image of $\alpha_1$ in $\operatorname{Br}(k(X))$. According to whether $x_0, x_1$, or $x_2$ is a 2-adic unit, the first, second, or third of these representatives can be used to see $\operatorname{inv}_2 \alpha_1(x) = 0$.

For instance, suppose $x_0$ is a 2-adic unit. Then by our assumptions on coefficients,

$$A(x) \quad \text{and} \quad B(x)^2 - 4A(x)D(x)$$

are also 2-adic units. In particular, they are nonzero, so

$$(B(x)^2 - 4A(x)D(x), A(x))_2$$

represents $\alpha_1(x) \in \operatorname{Br}(\mathbf{Q}_2)$. Recall (see for example [11, p. 20, Theorem 1]) that if $s, t \in \mathbf{Z}_2^\times$, then

$$(s,t)_2 = (-1)^{\frac{s-1}{2}\frac{t-1}{2}}.$$

But by our assumptions

$$B(x)^2 - 4A(x)D(x) \equiv 1 \pmod 4,$$

so the formula gives

$$(B(x)^2 - 4A(x)D(x), A(x))_2 = 1.$$

Thus $\alpha_1(x) = 0 \in \operatorname{Br}(\mathbf{Q}_2)$.

The same argument works when $x_1$ or $x_2$ is a 2-adic unit, using the other representatives for $\alpha_1$ from above. □

## 5. Proof of Theorem 1.2

Consider the following quadrics in $\mathbf{Z}[x_0, x_1, x_2]$:

$$\begin{aligned}
A &= -5x_0^2 + 4x_0x_2 - 4x_1^2 + 2x_1x_2 - 4x_2^2,\\
B &= 5x_0^2 + 2x_0x_1 - 2x_0x_2 + 2x_1^2 + 2x_1x_2 + 4x_2^2,\\
C &= 4x_0^2 + 2x_0x_1 - 4x_0x_2 + 2x_1^2 - 2x_1x_2 + 5x_2^2,\\
D &= -4x_0^2 - 2x_0x_1 - x_1^2 - 2x_1x_2 - 4x_2^2,\\
E &= 4x_0^2 + 3x_1^2 + 4x_2^2,\\
F &= -4x_0^2 + 4x_0x_1 + 2x_0x_2 - 2x_1^2 - 4x_1x_2 - 5x_2^2.
\end{aligned}$$

Inserting these polynomials in (4.1) gives the equation of a (2,2) divisor

$$Z \subset \mathbf{P}(V_1) \times \mathbf{P}(V_2),$$

which we regard as a variety over $\mathbf{Q}$. As in Section 2.2, $Z$ gives rise to a branched double cover $\pi : Y \to \mathbf{P}(V_1) \times \mathbf{P}(V_2)$, which is a quadric fibration via projection to each factor. Lemma 4.1 gives explicit equations for the discriminant curves $D_i \subset \mathbf{P}(V_i)$, and the Jacobian criterion can be used to check the $D_i$ are smooth. Hence, by Section 2.2, we get associated twisted K3 surfaces $(X_i, \alpha_i)$, which we will use to prove Theorem 1.2.

**Remark 5.1.** The quadrics $A, \dots, F$ above were found using the algorithm described in [6, Section 6], modified in two ways. First, we omitted the steps related to checking the geometric Picard number of $X_1$ is 1, since it was not our goal to produce an example with this property. Second, instead of using [6, Lemma 4.7] to constrain the quadrics, we used our Lemma 4.5, which results in much smaller coefficients. Indeed, the equations for $X_1$ and $X_2$ are:

$$\begin{aligned}
w^2 = &- 4x_0^6 - 308x_0^5x_1 - 190x_0^4x_1^2 - 278x_0^3x_1^3 - 203x_0^2x_1^4 - 40x_0x_1^5 - 28x_1^6\\
&+ 18x_0^5x_2 + 460x_0^4x_1x_2 + 276x_0^3x_1^2x_2 + 474x_0^2x_1^3x_2 + 40x_0x_1^4x_2\\
&+ 98x_1^5x_2 - 25x_0^4x_2^2 - 820x_0^3x_1x_2^2 - 247x_0^2x_1^2x_2^2 - 374x_0x_1^3x_2^2\\
&- 2x_1^4x_2^2 + 20x_0^3x_2^3 + 652x_0^2x_1x_2^3 + 14x_0x_1^2x_2^3 + 270x_1^3x_2^3\\
&- 20x_0^2x_2^4 - 562x_0x_1x_2^4 - 105x_1^2x_2^4 - 8x_0x_2^5 + 166x_1x_2^5 - 4x_2^6,\\
w^2 = &\ 236y_0^6 - 740y_0^5y + 1268y_0^4y_1^2 - 1092y_0^3y_1^3 + 624y_0^2y_1^4 - 164y_0y_1^5\\
&+ 32y_1^6 - 616y_0^5y_2 + 416y_0^4y_1y_2 - 96y_0^3y_1^2y_2 - 976y_0^2y_1^3y_2 + 548y_0y_1^4y_2\\
&- 288y_1^5y_2 + 1236y_0^4y_2^2 - 456y_0^3y_1y_2^2 + 1484y_0^2y_1^2y_2^2 - 356y_0y_1^3y_2^2\\
&+ 676y_1^4y_2^2 - 1332y_0^3y_2^3 - 804y_0^2y_1y_2^3 - 372y_0y_1^2y_2^3 - 1024y_1^3y_2^3 + 1036y_0^2y_2^4\\
&+ 768y_0y_1y_2^4 + 812y_1^2y_2^4 - 472y_0y_2^5 - 388y_1y_2^5 + 40y_2^6.
\end{aligned}$$

In contrast, most of the coefficients appearing in the corresponding equations in [6] have 5 or 6 digits. Smaller coefficients are crucial in making a computer search for points of $X_2$ feasible.

The following proposition is reduced to a series of computations by the results in Section 4.2. We postpone its proof to the end of the section.

**Proposition 5.2.** (1) *$X_1(\mathbf{Q}_v) \neq \emptyset$ for all places $v$, or equivalently $X_1(\mathbf{A}_\mathbf{Q}) \neq \emptyset$.*

(2) *The local invariants of the class $\alpha_1 \in \mathrm{Br}(X_1)$ satisfy*

$$\mathrm{inv}_v\, \alpha_1(x_v) = \begin{cases} 0 & \text{if } v \text{ is finite,} \\ \frac{1}{2} & \text{if } v \text{ is real,} \end{cases}$$

*for all $x_v \in X_1(\mathbf{Q}_v)$. In particular, $\alpha_1$ obstructs the existence of $\mathbf{Q}$-points on $X_1$, and hence gives a Brauer–Manin obstruction to the Hasse principle.*

Using the proposition, we construct the examples which prove Theorem 1.2. The necessary computations that appear below were carried out using Magma[1].

5.1. **Example over Q.** Using the equation for $X_2$ given by Lemma 4.1, it can be checked that $x = [1, 1, 1, 0] \in \mathbf{P}(1, 1, 1, 3)$ is a $\mathbf{Q}$-point of $X_2$. Let $\beta = \alpha_2(x) \in \mathrm{Br}(\mathbf{Q})$, let $\beta_i$ be the constant class given by the image of $\beta$ under $\mathrm{Br}(\mathbf{Q}) \to \mathrm{Br}(X_i)$, and set $\alpha'_i = \beta_i^{-1}\alpha_i$.

Then there is a $\mathbf{Q}$-linear equivalence $\mathrm{D^b}(X_1, \alpha'_1) \simeq \mathrm{D^b}(X_2, \alpha'_2)$ induced by the equivalence $\mathrm{D^b}(X_1, \alpha_1) \simeq \mathrm{D^b}(X_2, \alpha_2)$ of Theorem 3.9. By Proposition 5.2, $X_1$ has no $\mathbf{Q}$-point, so a fortiori the pair $(X_1, \alpha'_1)$ has no $\mathbf{Q}$-point. On the other hand, by construction $x$ is a $\mathbf{Q}$-point of $(X_2, \alpha'_2)$.

5.2. **Example over $\mathbf{Q}_2$.** Replace the pairs $(X_i, \alpha_i)$ defined above over $\mathbf{Q}$ by their base changes to $\mathbf{Q}_2$. It can be checked that $x = [-3, -1, 1, \sqrt{357008}] \in \mathbf{P}(1, 1, 1, 3)$ is a $\mathbf{Q}_2$-point of $X_2$ (note that Hensel's lemma can be used to see 357008 is a 2-adic square). One then checks that $\beta = \alpha_2(x) = (B(x)^2 - 4A(x)D(x), A(x))_2$ is nontrivial. Let $\beta_i$ be the constant class given by the image of $\beta$ under $\mathrm{Br}(\mathbf{Q}) \to \mathrm{Br}(X_i)$, and set $\alpha'_i = \beta_i^{-1}\alpha_i$.

Then there is a $\mathbf{Q}_2$-linear equivalence $\mathrm{D^b}(X_1, \alpha'_1) \simeq \mathrm{D^b}(X_2, \alpha'_2)$ induced by the equivalence of Theorem 3.9. By Proposition 5.2, $\alpha_1(y)$ is trivial for any $y \in X_1(\mathbf{Q}_2)$, and hence $\alpha'_1(y) = \alpha_2^{-1}(x)\alpha_1(y)$ is nontrivial (since $\alpha_2(x)$ is). Thus $(X_1, \alpha'_1)$ has no $\mathbf{Q}_2$-points. On the other hand, by design $x$ is a point of $(X_2, \alpha'_2)$.

5.3. **Example over R.** Replace the pairs $(X_i, \alpha_i)$ by their base changes to $\mathbf{R}$. Then Theorem 3.9 still gives an $\mathbf{R}$-linear equivalence $\mathrm{D^b}(X_1, \alpha_1) \simeq \mathrm{D^b}(X_2, \alpha_2)$. Moreover, Proposition 5.2 shows $\alpha_1(x)$ is nontrivial for any $x \in X_1(\mathbf{R})$, so $(X_1, \alpha_1)$ has no $\mathbf{R}$-points. On the other hand, using Lemma 4.1, it can be checked that the point

$$x = [4, 3, 3, \sqrt{5204}] \in \mathbf{P}(1, 1, 1, 3)$$

lies on $X_2$ and that $\alpha_2(x) = (B(x)^2 - 4A(x)D(x), A(x))_\infty$ is trivial. Hence $x$ is an $\mathbf{R}$-point of $(X_2, \alpha_2)$.

5.4. **Proof of Proposition 5.2.**

[1]Code available at `http://www.math.brown.edu/~kenascher/magma/magma.html`.

5.4.1. *Local points.* We first check that $X_1(\mathbf{Q}_v) \neq \emptyset$ for all $v$. This is obvious when $v = \infty$. Let $v = p$ be a finite prime of good reduction with $p > 22$. Then if $(X_1)_p$ is a smooth reduction of $X_1$ at $p$, there is an $\mathbf{F}_p$-point of $(X_1)_p$ by the Weil conjectures. This lifts to a $\mathbf{Q}_p$-point of $X_1$ by Hensel's lemma.

It therefore suffices to check that $X_1(\mathbf{Q}_p) \neq \emptyset$ for primes $p$ of bad reduction for $X_1$ and for all primes $p < 22$. A Gröbner basis calculation as in [6, Section 5.1] can be used to show the primes of bad reduction for $X_1$ are:

$$2,\ 5,\ 7,\ 307,\ 4591,\ 27077,\ 371857,\ 6902849,\ 104388233,$$
$$541264119547919951,\ 609786360964131092114 9279,$$
$$261667838892628639800286446901484281709500931284479047 9$$

In the table below, we list for each prime $p$ of bad reduction and each $p < 22$ the $(x_0, x_1, x_2)$ coordinates of a $\mathbf{Q}_p$-point of $X_1$. (By Lemma 4.1, $(x_0, x_1, x_2)$ gives a $\mathbf{Q}_p$-point if $-\frac{1}{2}\det(M)(x_0, x_1, x_2)$ is a square in $\mathbf{Q}_p$, which can be checked using Hensel's lemma).

| $p$ | $(x_0, x_1, x_2)$ |
|---|---|
| 2 | (-1,0,-1) |
| 3 | (-1,-1,1) |
| 5 | (-1,-1,0) |
| 7 | (-1,-1,1) |
| 11 | (-1,-1,0) |
| 13 | (-1,-1,1) |
| 17 | (-1,-1,-1) |
| 19 | (-1,-1,-1) |
| 307 | (-1,-1,-1) |
| 4591 | (-1,-1,0) |
| 27077 | (-1,-1,-1) |
| 371857 | (-1,-1,-1) |
| 6902849 | (-1,0,0) |
| 104388233 | (-1,-1,-1) |
| 541264119547919951 | (-1,-1,1) |
| 6097863609641310921149279 | (-1,1,-1) |
| 2616678388926286398002864469014842817095009312844790479 | (-1,-1,0) |

5.4.2. *Local invariants.* One computes that for each prime $p \neq 2$ of bad reduction, $X_{1,\mathbf{Q}_p}$ satisfies the assumptions of Proposition 4.2. Moreover, using the representatives for $\alpha_1$ given in Lemma 4.1, it can be computed that $\alpha_1$ is trivial when evaluated at the $\mathbf{Q}_p$-points specified in the table above. We conclude by Proposition 4.2 that $\mathrm{inv}_v\, \alpha_1(x_v) = 0$ at the non-2-adic finite places $v$ of bad reduction. On the other hand, at the non-2-adic finite places of good reduction, we also have $\mathrm{inv}_v\, \alpha_1(x_v) = 0$ by Lemma 4.3.

Finally, it is straightforward to check the quadrics $A, \dots, F$ satisfy the hypotheses of Lemmas 4.4 and 4.5. The conclusions of these lemmas give Proposition 5.2(2) at the real and 2-adic places. □

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