ON THE CONNECTED COMPONENTS OF AFFINE DELIGNE-LUSZTIG VARIETIES

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Abstract. We study the set of connected components of certain unions of affine Deligne-Lusztig varieties arising from the study of Shimura varieties. We determine the set of connected components for basic σ-conjugacy classes. As an application, we verify the Axioms in [20] for certain PEL type Shimura varieties. We also show that for any nonbasic σ-conjugacy class in a residually split group, the set of connected components is "controlled" by the set of straight elements associated to the σ-conjugacy class, together with the obstruction from the corresponding Levi subgroup. Combined with [47], this allows one to verify in the residually split case, the description of the mod-p isogeny classes on Shimura varieties conjectured by Langland and Rapoport. Along the way, we determine the Picard group of the Witt vector affine Grassmannian of [3] and [49] which is of independent interest.

Introduction

0.1. Let F be a non-archimedean local field with valuation ring $\mathcal{O}_F$ and residue field $\mathbb{F}_q$, and $\breve{F}$ be the completion of the maximal unramified extension of F. Let G be a connected reductive group over F and let σ be the Frobenius automorphism on $G(\breve{F})$. Let $\{\mu\}$ be a geometric conjugacy class of cocharacters of G. Let $b \in G(\breve{F})$ and $\breve{\mathcal{K}}$ be a standard σ-invariant parahoric subgroup of $G(\breve{F})$. Let K be the set of simple reflections corresponding to $\breve{\mathcal{K}}$. The (union of) affine Deligne-Lusztig varieties associated to $(G, \{\mu\}, b, \breve{\mathcal{K}})$ is defined to be

$$X(\{\mu\}, b)_K = \{g \in G(\breve{F})/\breve{\mathcal{K}}; g^{-1} b\sigma(g) \in \breve{\mathcal{K}} \operatorname{Adm}(\{\mu\})\breve{\mathcal{K}}\},$$

where $\operatorname{Adm}(\{\mu\})$ is the admissible set associated to $\{\mu\}$.

The motivation to study $X(\{\mu\}, b)_K$ comes from the theory of Shimura varieties. If $(\mathbf{G}, X)$ is a Shimura datum with $\{\mu\}$ the conjugacy class of the inverse of the Hodge cocharacter and $F = \mathbb{Q}_p$, the set $X(\{\mu\}, b)_K$ is closely related to the $\overline{\mathbb{F}}_q$-points of an integral model of the corresponding Shimura variety with parahoric level structure at p, and in special cases to the $\overline{\mathbb{F}}_q$ points of a moduli space of p-divisible groups defined by Rapoport and Zink [42].

In the equal characteristic case, $X(\{\mu\}, b)_K$ is the set of $\overline{\mathbb{F}}_q$-valued points of a locally closed, locally of finite type subscheme of the partial affine flag variety $G(\breve{F})/\breve{\mathcal{K}}$. Thanks to the recent work of Zhu [49] and Bhatt-Scholze [3], we may, even in the mixed characteristic case, endow $X(\{\mu\}, b)_K$ with the structure of a perfect scheme. More precisely, it is a locally closed subscheme of the Witt vector partial affine flag variety $Gr_{\mathcal{G}}$ (here $\mathcal{G}$ is the group scheme associated to $\breve{\mathcal{K}}$). Therefore topological notions such as irreducible and connected components make sense.

If the group G is unramified, μ is minuscule and $\breve{\mathcal{K}}$ is a hyperspecial parahoric subgroup, $\breve{\mathcal{K}} \operatorname{Adm}(\{\mu\})\breve{\mathcal{K}}$ is a single double coset of $\breve{\mathcal{K}}$. The set of connected components of $X(\{\mu\}, b)_K$ for split group G, hyperspecial parahoric $\breve{\mathcal{K}}$ and arbitrary cocharacter μ is determined by Viehmann [45]. For other unramified groups with hyperspecial parahoric $\breve{\mathcal{K}}$, Chen, Kisin and Viehmann [5] determined the set of connected components under the assumption that μ is minuscule, and Nie [37] extended the result to non-minuscule cocharacters.

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0.2. The aim of this paper is to study the set of connected components of $X(\{\mu\}, b)_K$ for any reductive group G and any parahoric subgroup $\breve{\mathcal{K}}$.

Let us first recall the nonemptiness pattern of $X(\{\mu\}, b)_K$. Let $B(G, \{\mu\})$ be the set of neutral acceptable elements, a certain subset of the σ-conjugacy classes of $G(\breve{F})$ (see §2 for the precise definitions). Then it is known that $X(\{\mu\}, b)_K$ is nonempty if and only if the σ-conjugacy class of b lies in $B(G, \{\mu\})$. This was conjectured by Kottwitz and Rapoport in [28] and proved for quasi-split groups by Wintenberger [46] and in the general case by the first author in [16].

Note that the set of connected components of the partial affine flag variety $G(\breve{F})/\breve{\mathcal{K}}$ is isomorphic to $\pi_1(G)_{\Gamma_0}$, where $\pi_1(G)_{\Gamma_0}$ is the set of coinvariants of the fundamental group of G under the action of the Galois group $\Gamma_0 = \mathrm{Gal}(\bar{\breve{F}}/\breve{F})$. This gives the first obstruction to connecting points of $X(\{\mu\}, b)_K \subset G(\breve{F})/\breve{\mathcal{K}}$. Our first main result is that for basic b, this is the only obstruction except in trivial cases. See Theorem 6.3.

Theorem 0.1. *Assume that G_{ad} is simple. Let $[b] \in B(G, \{\mu\})$ be a basic σ-conjugacy class of $G(\breve{F})$.*

(1) If μ is central, then

$$X(\{\mu\}, b)_K = G(F)/\mathcal{K},$$

where $\mathcal{K}$ is the group of $\mathcal{O}_F$-points of the parahoric group scheme corresponding to K.

(2) If μ is noncentral, then

$$\pi_0(X(\{\mu\}, b)_K) \cong \pi_1(G)^{\sigma}_{\Gamma_0}.$$

0.3. Now we discuss some applications. In joint work [20] of the first author with Rapoport, there is the formulation of five axioms on Shimura varieties with parahoric level structure. These axioms allow us to apply group-theoretic techniques to study many questions on certain characteristic subsets (Newton strata, the Ekedahl-Oort strata and the Kottwitz-Rapoport strata, etc.) in the reduction modulo p of a general Shimura variety with parahoric level structure. It is shown in [20] that if the axioms are satisfied, then the Newton strata are nonempty in their natural range, and that the Grothendieck conjecture on the closure relations of Newton strata is true. We refer to [20] for more details and for other consequences of the axioms. These axioms are also used in an essential way in the work [12] on a family of Shimura varieties with nice geometric properties on both the basic locus and on all the non-basic Newton strata and the forthcoming work [19] on the density problem of the μ-ordinary locus of Shimura varieties.

One major difficulty in verifying the axioms is to show that for Shimura varieties with Iwahori level structure, the minimal Kottwitz-Rapoport stratum intersects every connected component of it. As an application of Theorem 0.1, we show in Proposition 9.2 that for certain PEL type Shimura varieties (unramified of type A and C and odd ramified unitary groups), every point in the basic locus is connected to a point in the minimal KR stratum. In §9, we verify the axioms on these Shimura varieties. In particular, combined with [20], we finish the proof of the nonemptiness pattern and closure relations on the Newton strata of these Shimura varieties.

0.4. Now we come to the second main result. Let $\breve{W}$ be the Iwahori-Weyl group of $G(\breve{F})$. The Frobenius morphism σ on $G(\breve{F})$ induces a group automorphism σ on $\breve{W}$. There is a natural map from the set of σ-conjugacy classes of $\breve{W}$ to the set of σ-conjugacy classes of $G(\breve{F})$. This map is not injective. However, it is proved by the first author that the map, when restricted to the set of straight σ-conjugacy classes of $\breve{W}$, becomes a bijection (see [15]), and for any $[b] \in B(G, \{\mu\})$, there exists a straight representative in the admissible set of μ (see [16]).

For basic $[b]$, there is only one straight element in the associated straight σ-conjugacy class in $\breve{W}$ which lies in $\mathrm{Adm}(\{\mu\})$. For nonbasic $[b]$, the straight elements of $\mathrm{Adm}(\{\mu\})$ in the associated straight σ-conjugacy class in $\breve{W}$ are not unique anymore. However, if G is residually split, then for $[b] \in B(G, \{\mu\})$, all the straight representatives of $[b]$ lie in the admissible set for μ.

Assume that μ is noncentral on each simple factor of G_{ad}. As mentioned above, the set of connected components of $G(\breve{F})/\breve{\mathcal{K}}$ give the first obstruction to the connectedness of $X(\{\mu\}, b)_K$. For basic b, as we have proved in Theorem 0.1, this is the only obstruction in general. For nonbasic $[b] \in B(G, \{\mu\})$, there is another obstruction, coming from the non-uniqueness of the straight representatives of $[b]$. If there is more than one straight representative of $[b]$, then one cannot expect in general that the points associated to different straight elements are connected. For example, in the case of split G and minuscule μ, when b is a translation element, the only non-empty strata in $X(\{\mu\}, b)_K$ are the ones corresponding to translation elements. In particular these are all open in $X(\{\mu\}, b)_K$ and one cannot connect points in different strata.

Our second main result is that in the residually split case, these are all the obstructions to the connectedness of $X(\{\mu\}, b)_K$.

Theorem 0.2. *Let G be residually split and $[b] \in B(G, \{\mu\})$. Assume that $[b]$ is strongly noncentral in the associated Levi subgroup. Then we have a surjection*

$$\coprod_{w \in \breve{W} \text{ is a straight element with } \dot{w} \in [b]} \pi_1(M_{\nu_w})_{\Gamma_0} \twoheadrightarrow \pi_0(X(\{\mu\}, b))_K.$$

We refer to §7 for the definitions and notations in the theorem and the more general cases where the "strongly noncentral" assumption is dropped.

In fact, if G is unramified and $\breve{\mathcal{K}}$ is a hyperspecial parahoric subgroup, it is shown in [45], [5] and [37] that the precise description of the connected components for nonbasic elements is given by the Hodge-Newton decomposability. It is a challenging problem to determine explicitly the connected components for general G and $\breve{\mathcal{K}}$. However, the above result is enough for some important applications, which we discuss in the next subsection.

0.5. In [23], Kisin showed that the mod-p points on a Shimura variety of abelian type with hyperspecial level structure agrees with the conjectural description given in the Langlands-Rapoport conjecture.[1] The key part of the proof is to define a certain map from $X(\{\mu\}, b)_K$ into a mod-p isogeny class. His strategy consists of a local and a global part: first use a deformation theoretic argument to show that a map from the affine Deligne-Lusztig variety to the Shimura variety is well-defined on a connected component once it is defined on a point; then show that every connected component of the affine Deligne-Lusztig variety contains a point where the map is well-defined using isogenies which lift to characteristic 0. The last part uses in a crucial way the explicit description of the set of connected components of $X(\{\mu\}, b)_K$ determined in [5].

For hyperspecial level structure, only a single affine Deligne-Lusztig variety occurs; for other parahoric level structures, one has to consider a union of affine Deligne-Lusztig varieties. This is one of the major new difficulties in the study of arbitrary parahoric level structure. In [47], the second author generalizes the deformation theoretic argument to the integral models of Shimura varieties of Hodge type with arbitrary parahoric level structure constructed in [25] (these models are constructed under a tameness hypothesis on the group at p). One innovative result is that one may move between different Levi subgroups using isogenies which lift to characteristic 0. In other words, for the application to the Langlands-Rapoport conjecture for residually split groups, one does not need to know whether any two straight elements in Theorem 0.2 are connected. Thus our Theorem 0.2 provides enough local information for this purpose. Combining the global argument [47] together with the local result here, one deduces that the mod-p isogeny classes have the form predicted in [31] for those Shimura varieties associated to groups which are residually split and have arbitrary parahoric level at p.

0.6. Our proof of Theorem 0.1 and Theorem 0.2 is different from the proof of [45], [5] and [37] in the case of hyperspecial level structure. At the time, the scheme structure on the mixed characteristic affine Deligne Lusztig varieties was not known and the authors worked with an ad hoc definition of connected components. A consequence of this was that their proofs were completely combinatorial. By contrast, we work in the Zariski topology, and our proofs use both geometry and combinatorics.

More precisely, we rely on the following key ingredients:

- The relation between the straight elements in the admissible set $\mathrm{Adm}(\{\mu\})$ and the σ-conjugacy classes in the neutral acceptable set $B(G, \{\mu\})$ established in [16] and the reduction method introduced in [15];

[1] More precisely, Kisin shows the description of the isogeny classes agrees with the conjecture up to conjugation by a certain group element.

- The line bundles on the affine flag varieties and the quasi-affineness of irreducible components of affine Deligne-Lusztig varieties;
- The structure of the σ-centralizer J_b of b and the construction of explicit curves in $X(\{\mu\}, b)$ for each affine root subgroup of J_b.

0.7. Now let us provide more details for the Iwahori case. We let $\breve{\mathcal{I}}$ be an Iwahori subgroup and we simply write $X(\{\mu\}, b)$ instead of $X(\{\mu\}, b)_K$ in this case.

Note that $X(\{\mu\}, b) = \sqcup_{w \in \mathrm{Adm}(\{\mu\})} X_w(b)$ is a union of affine Deligne-Lusztig varieties, where $X_w(b) = \{g \in G(\breve{F})/\breve{\mathcal{I}}; g^{-1} b \sigma(g) \in \breve{\mathcal{I}} \dot{w} \breve{\mathcal{I}}\}$ (here $\dot{w}$ is a representative of w in $G(L)$). In general, the structure of $X_w(b)$ is very complicated. However, it is proved in [15] that if w is σ-straight, then $X_w(b)$ is either empty or discrete with transitive action of J_b.

Our first major step is to show that every point in $X(\{\mu\}, b)$ is connected to a point in $X_w(b)$ for some σ-straight element $w \in \mathrm{Adm}(\{\mu\})$. This is the first reduction theorem, established in §4. It is based on the reduction method in [15] and the quasi-affineness of irreducible components of affine Deligne-Lusztig varieties.

The quasi-affineness is obtained by constructing certain ample line bundles on affine Deligne-Lusztig varieties. This requires knowledge of the Picard group and ample line bundles on affine flag varieties.

We temporarily change our notation to let F be an arbitrary complete discrete valuation ring with perfect residue field k of characteristic p, and let k denote an algebraic closure of $\mathbf{k}$. Then in the equal characteristic case, the description of the Picard group when G is tamely ramified is due to Pappas and Rapoport in [38]. In the mixed characteristic case, we work with the Witt vector affine flag variety $\mathcal{FL} = LG/L^+\breve{\mathcal{I}}$ of Zhu [49] and Bhatt-Scholze [3]. In this case we have the following result which is of independent interest and is true even for wildly ramified groups in equal characteristic upon taking perfections.

Theorem 0.3. *Assume G is simply connected and $k = \mathbf{k}$. There is an isomorphism*

$$Pic\mathcal{FL} \cong \bigoplus_{\iota \in \breve{\mathbb{S}}} \mathbb{Z}[\frac{1}{p}]$$

given by taking a line bundle to the degrees of its restriction to the projective lines corresponding to affine simple roots. Moreover, a line bundle $\mathscr{L}$ corresponding to $\oplus_{\iota \in \breve{\mathbb{S}}} \lambda_\iota$ is ample if and only if $\lambda_\iota > 0$ for all $\iota \in \breve{\mathbb{S}}$.

Here $\breve{\mathbb{S}}$ is an indexing set for the set of simple affine reflections $\{s_\iota; \iota \in \breve{\mathbb{S}}\}$ corresponding to $\breve{\mathcal{I}}$. We refer to §3 for the definitions and other notations. This is analogous to the result of [38, Theorem 10.1] in equal characteristic, but it is proved in a different way by using the method of h-descent developed in [3]. The idea is to descend line bundles from a suitable Demazure resolution of $\mathcal{FL}$, whose Picard group has a simple description. By analyzing this resolution and using [3, Theorem 6.8], one shows that the correct line bundles descend.

Using the above description of $\mathrm{Pic}\,\mathcal{FL}$ and applying the descent result to the fibration $LG/L^+\breve{\mathcal{I}} \to LG/L^+\breve{\mathcal{K}}$, we also get a description of the Picard group of the partial affine flag varieties. In particular, we show that if $\breve{\mathcal{K}}$ is hyperspecial, then $\mathrm{Pic}(LG/\breve{L}^+\mathcal{K}) \cong \mathbb{Z}[\frac{1}{p}]$. This answers a question of Bhatt and Scholze in [3].

0.8. We return to our assumption that F is a non-archimedean local field. Our next major step is to connect certain points in $X_w(b)$ for a σ-straight element w inside $X(\{\mu\}, b)$. As J_b acts transitively on $X_w(b)$, we only need to connect certain elements of J_b. To do this we find an explicit set of generators for J_b; this is Theorem 5.5. Roughly speaking, J_b is generated by a certain subgroup of $\breve{\mathcal{I}}$, and some elements u_{-J} defined in §5.4.

We then construct a curve inside $X(\{\mu\}, b)$ that connects p and $u_{-J} \cdot p$, where p is a point in $X_w(b)$. This is based on the comparison of the admissible sets of the (non-standard) Levi subgroups and the whole reductive group, and on Görtz's result on the connectedness of classical Deligne-Lusztig varieties.

Finally in the Appendix we show that the notion of connected components as defined in [5] agrees with the notion in the Zariski topology. Thus there is no ambiguity when we talk about connected components. Moreover the notion in [5] is useful for applications to Shimura varieties and Rapoport-Zink spaces, so it is useful to know the two notions coincide.

Apart from the introduction, we will work only in the mixed characteristic setting, only mentioning the equal characteristic setting to make some comparisons between the two cases. The proofs of all the results in §4-§8 also work in the equal characteristic setting under the assumption that the group is tamely ramified. Moreover, we note that §3-§8 also holds in the equal characteristic setting for possible wildly ramified groups, upon taking perfections of all the objects involved.

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1. Preliminaries

1.1. We let k be a perfect field of characteristic $p > 0$, and F a finite totally ramified extension of $W(k)$, where $W(k)$ is the Witt vectors of k. We let $\mathcal{O}_F$ denote its ring of integers. We fix $\mathbf{k}$ an algebraic closure of k and we let $\breve{F}$ (resp. $\mathcal{O}_{\breve{F}}$) denote the base change $F \otimes_{W(k)} W(\mathbf{k})$ (resp. $\mathcal{O}_F \otimes_{W(k)} W(\mathbf{k})$). We fix a uniformizer π of F and let Γ_0 denote the Galois group $\mathrm{Gal}(\overline{\breve{F}}/\breve{F})$.

Let G be a connected reductive group over F. Let $\breve{S}$ be a maximal $\breve{F}$-split torus of $G_{\breve{F}}$ and let $\breve{T}$ be its centralizer. By Steinberg's theorem, G is quasi split over $\breve{F}$. Hence T is a maximal torus. Let $\breve{\mathcal{I}}$ be the Iwahori subgroup fixing an alcove $\breve{\mathfrak{a}}$ in the apartment V attached to $\breve{S}$. The relative Weyl group $\breve{W}_0$ and the Iwahori-Weyl group $\breve{W}$ are defined by

$$\breve{W}_0 = \breve{N}(\breve{F})/\breve{T}(\breve{F}), \qquad \breve{W} = N(\breve{F})/\breve{T}(\breve{F}) \cap \breve{\mathcal{I}},$$

where $\breve{N}$ denotes the normalizer of $\breve{S}$ in G. For any $w \in \breve{W}$, we choose a representative $\dot{w}$ of w in $\breve{N}(\breve{F})$.

We follow [14]. We fix a special vertex $\breve{\mathfrak{s}}$ of the base alcove $\breve{\mathfrak{a}}$. The Iwahori-Weyl group $\breve{W}$ is a split extension of $\breve{W}_0$ by the subgroup $\breve{T}(\breve{F})/\breve{T}(\breve{F}) \cap \breve{\mathcal{I}} \cong X_*(\breve{T})_{\Gamma_0}$. When considering an element $\lambda \in X_*(\breve{T})_{\Gamma_0}$ as an element of $\breve{W}$, we write t^λ. We may also identify $V \cong X_*(\breve{T})_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R}$. Let $\mathrm{Aff}(V)$ be the group of affine transformations on the apartment V. Then we have the decomposition $\mathrm{Aff}(V) = V \rtimes GL(V)$.

Let $\breve{W}_a$ be the associated affine Weyl group. Let $\breve{\mathbb{S}}$ denote an indexing set for the of simple affine reflections $\{s_\iota; \iota \in \breve{\mathbb{S}}\}$ corresponding to $\breve{\mathfrak{a}}$. The Iwahori-Weyl group $\breve{W}$ contains the affine Weyl group $\breve{W}_a$ as a normal subgroup and we have a natural splitting

$$\breve{W} = \breve{W}_a \rtimes \breve{\Omega},$$

where $\breve{\Omega}$ is the normalizer of $\breve{\mathfrak{a}}$ and is isomorphic to $\pi_1(G)_{\Gamma_0}$.

The length function ℓ and the Bruhat order $\leqslant$ on the Coxeter group $\breve{W}_a$ extend in a natural way to $\breve{W}$.

For $K \subset \breve{\mathbb{S}}$, we let $\breve{W}_K \subset \breve{W}$ denote the subgroup generated by the set $\{s_\iota; \iota \in K\}$. If $\breve{W}_K$ is finite, we let $\breve{\mathcal{K}}$ denote the standard parahoric subgroup corresponding to K. Here standard means $\breve{\mathcal{I}} \subset \breve{\mathcal{K}}$.

1.2. Apart from section §3, we are only interested in the following special case of the above. We let $k = \mathbb{F}_q$, where $q = p^r$. Then F is a non-archimedean local field and $\breve{F}$ is the completion of the maximal unramified extension of F. We let $\Gamma = \mathrm{Gal}(\bar{F}/F)$ be the absolute Galois group; then we may identify Γ_0 with the inertia subgroup $\mathrm{Gal}(\bar{F}/F^{un})$ of Γ. We let σ be the Frobenius of $\breve{F}$ over F.

Let G be a connected reductive group over F as above. In this case we will make the extra assumption that the maximal $\breve{F}$-split torus $\breve{S}$ is defined over F, and also that $\breve{\mathcal{I}}$ is the Iwahori subgroup corresponding to fixed *σ-invariant* alcove $\breve{\mathfrak{a}}$ in the apartment V attached to $\breve{S}$. Then $\breve{W}$ and $\breve{W}_a$ are both equipped with an action of σ, and this action preserves $\breve{\Omega}$ and $\{s_\iota : \iota \in \breve{\mathbb{S}}\}$.

For the rest of this section we will assume that F is a non-archimedean local field as above.

1.3. For any $b \in G(\breve{F})$, we denote by $[b] = \{g^{-1} b \sigma(g); g \in G(\breve{F})\}$ its σ-conjugacy class. Let $B(G)$ be the set of σ-conjugacy classes of $G(\breve{F})$. The σ-conjugacy classes are classified by Kottwitz in [26] and [27]. We denote by $\bar{\nu}$ the Newton map

$$\bar{\nu} : B(G) \longrightarrow (X_*(\breve{T})^+_{\Gamma_0, \mathbb{Q}})^\sigma,$$

where $X_*(\breve{T})^+_{\Gamma_0,\mathbb{Q}}$ is the intersection of $X_*(\breve{T})_{\Gamma_0} \otimes \mathbb{Q} = X_*(\breve{T})^{\Gamma_0} \otimes \mathbb{Q}$ with the set of dominant (rational) coweights in $X_*(\breve{T}) \otimes \mathbb{Q}$. Here the dominant chamber is chosen such that the base alcove $\breve{\mathfrak{a}}$ is contained in the anti-dominant chamber under the identification $V \cong X_*(\breve{T})_{\Gamma_0} \otimes_{\mathbb{Z}} \mathbb{R}$. In [27] Kottwitz defined a map $\tilde{\kappa} : G(\breve{F}) \to \pi_1(G)_{\Gamma_0}$. We denote by κ the map

$$\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$$

obtained by composing $\tilde{\kappa}$ with the projection $\pi_1(G)_{\Gamma_0} \to \pi_1(G)_\Gamma$. By [27, §4.13], the map

$$(\bar{\nu}, \kappa) : B(G) \longrightarrow (X_*(\breve{T})^+_{\Gamma_0,\mathbb{Q}})^\sigma \times \pi_1(G)_\Gamma$$

is injective. A σ-conjugacy class $[b]$ is called *basic* if $\bar{\nu}(b)$ is central.

The maps $\bar{\nu}$ and κ can be described in an explicit way as follows. We denote by $B(\breve{W}, \sigma)$ the set of σ-conjugacy classes of $\breve{W}$. For any $w \in \breve{W}$, we associate the σ-conjugacy class $[\dot{w}]$ of $G(\breve{F})$. By Lang's theorem on $\breve{T}(\breve{F}) \cap \breve{I}$, $[\dot{w}]$ does not depend on the choice of the representative $\dot{w}$. The map $w \mapsto [\dot{w}]$ induces a natural map $\Psi : B(\breve{W}, \sigma) \to B(G)$.

On the other hand, we denote by $\tilde{\kappa}(w)$ the image of w under the natural projection map $\breve{W} \to \breve{W}/\breve{W}_a \cong \pi_1(G)_{\Gamma_0}$, and by $\kappa(w)$ the image of w under the map $\breve{W} \xrightarrow{\tilde{\kappa}} \pi_1(G)_{\Gamma_0} \to \pi_1(G)_\Gamma$. For any $w \in \breve{W}$, there exists a positive integer n such that σ^n acts trivially on $\breve{W}$ and such that $w\sigma(w) \cdots \sigma^{n-1}(w) = t^\lambda$ for some $\lambda \in X_*(\breve{T})_{\Gamma_0}$. We set $\nu_w = \frac{\lambda}{n} \in X_*(\breve{T})_{\Gamma_0,\mathbb{Q}}$. We denote by $\bar{\nu}_w$ the unique dominant rational coweight in the $\breve{W}_0$-orbit of ν_w. We have a commutative diagram

$$\begin{array}{ccc} B(\breve{W}, \sigma) & \xrightarrow{\quad\Psi\quad} & B(G) \\ & \searrow \qquad \swarrow & \\ & (X_*(\breve{T})^+_{\Gamma_0,\mathbb{Q}})^\sigma \times \pi_1(G)_\Gamma & \end{array}$$

By [15], the map Ψ is surjective. The map $B(\breve{W}, \sigma) \to B(G)$ is not injective. However, its restriction to the set of straight σ-conjugacy classes is bijective.

By definition, an element $w \in \breve{W}$ is *σ-straight* if for any $n \in \mathbb{N}$,

$$\ell(w\sigma(w) \cdots \sigma^{n-1}(w)) = n\ell(w).$$

It is equivalent to the condition that $\ell(w) = \langle \bar{\nu}_w, 2\rho \rangle$, where ρ is the half sum of all positive roots in the reduced root system associated to $\breve{W}_a$. A σ-conjugacy class is *straight* if it contains a σ-straight element. It is easy to see that the minimal length elements in a given straight σ-conjugacy class are exactly the σ-straight elements. When the action of σ on $\breve{W}$ is trivial, we will call these straight elements instead of σ-straight.

It is proved in [15, Theorem 3.7] that

Theorem 1.1. *The restriction of $\Psi : B(\breve{W}, \sigma) \to B(G)$ gives a bijection from the set of straight σ-conjugacy classes of $\breve{W}$ to $B(G)$.*

1.4. Now we recall some remarkable properties on the minimal length elements in a conjugacy class of $\breve{W}$.

For $w, w' \in \breve{W}$ and a simple reflection s, we write $w \xrightarrow{s}_\sigma w'$ if $w' = sw\sigma(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \to_\sigma w'$ if there is a sequence $w = w_0, w_1, \ldots, w_n = w'$ of elements in $\breve{W}$ such that for any k, $w_{k-1} \xrightarrow{s_k}_\sigma w_k$ for some simple reflection s_k. Note that if moreover, $\ell(w') < \ell(w)$, then there exists i such that $\ell(w) = \ell(w_i)$ and $s_{i+1} w_i \sigma(s_{i+1}) < w_i$.

We write $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $w' \to_\sigma w$. It is easy to see that $w \approx_\sigma w'$ if $w \to_\sigma w'$ and $\ell(w) = \ell(w')$. We write $w \tilde{\approx}_\sigma w'$ if there exists $\tau \in \breve{\Omega}$ such that $w \approx_\sigma \tau w' \sigma(\tau)^{-1}$. If the action of σ on $\breve{W}$ is trivial, then we will omit σ in the subscript.

It is proved in [17, Theorem 2.9 & Theorem 3.8] that

Theorem 1.2. *Let $\mathcal{O}$ be a σ-conjugacy class of $\breve{W}$. Then*
(1) For any $w \in \mathcal{O}$, there exists a minimal length element w' of $\mathcal{O}$ such that $w \to_\sigma w'$.
(2) If $\mathcal{O}$ is straight, then all the σ-straight elements in $\mathcal{O}$ form a single $\tilde{\approx}_\sigma$-equivalence class.

2. The affine Deligne-Lusztig variety $X(\{\mu\}, b)_K$

2.1. Let $\breve{\mathcal{K}}$ be a σ-invariant standard parahoric subgroup of $G(\breve{F})$. Recall that $\{s_\iota; \iota \in K\}$ is the corresponding set of simple reflections and $\breve{W}_K \subset \breve{W}$ is the subgroup generated by the simple reflections associated to K. We have

$$G(\breve{F}) = \sqcup_{w \in \breve{W}_K \backslash \breve{W} / \breve{W}_K} \breve{\mathcal{K}} \dot{w} \breve{\mathcal{K}}.$$

The affine Deligne-Lusztig variety was introduced by Rapoport in [40]. For any $w \in \breve{W}_K \backslash \breve{W} / \breve{W}_K$ and $b \in G(\breve{F})$, we set

$$X_{K,w}(b) = \{g\breve{\mathcal{K}} \in G(\breve{F})/\breve{\mathcal{K}}; g^{-1} b \sigma(g) \in \breve{\mathcal{K}} \dot{w} \breve{\mathcal{K}}\}.$$

If $\breve{\mathcal{K}} = \breve{\mathcal{I}}$, we simply write the corresponding affine Deligne-Lusztig variety as $X_w(b)$.

By [3, Theorem 9.6] (see also [49]), we may endow $X_{K,w}(b)$ with the structure of a perfect scheme over $\overline{\mathbb{F}}_q$, i.e. a scheme over $\overline{\mathbb{F}}_q$ such that the absolute Frobenius is an isomorphism. In general, it is not of finite type over $\overline{\mathbb{F}}_q$. However all topological notions are well-defined using the Zariski topology, and therefore we have notions of dimension, irreducible components and connected components for $X_{K,w}(b)$. In this paper we are interested in studying the connected components of this object.

Remark 2.1. Given a scheme X over $\mathbb{F}_q$, the perfection X^{perf} of X is defined to be the inverse limit $\varprojlim_\varphi X$ where φ is the absolute Frobenius. The canonical map $X^{perf} \to X$ is a universal homeomorphism, hence to study any topological properties of a scheme, it suffices to study its perfection.

Remark 2.2. In what follows, we will sometimes abuse notation and identify $\breve{\mathcal{K}}$ with the corresponding parahoric group scheme over $\mathcal{O}_{\breve{F}}$. The usage will usually be clear from the context. However when there is the possibility of confusion we sometimes introduce an auxiliary $\mathcal{G}$ to denote the group scheme.

2.2. Let $\{\mu\}$ be a geometric conjugacy class of cocharacters of G and $\underline{\mu}$ be the image in $X_*(\breve{T})_{\Gamma_0}$ of a dominant representative μ in $X_*(\breve{T})$ of the conjugacy class $\{\mu\}$. The admissible set is defined by

$$\mathrm{Adm}(\{\mu\}) = \{w \in \breve{W}; w \leqslant t^{x(\underline{\mu})} \text{ for some } x \in \breve{W}_0\}.$$

Note that $\mathrm{Adm}(\{\mu\})$ has a unique minimal element with respect to the Bruhat order $\leqslant$, i.e., the unique element $\tau_{\{\mu\}}$ in $\breve{\Omega}$ with $\tau_{\{\mu\}} \in t^{\underline{\mu}} \breve{W}_a$.

More generally, for any standard parahoric subgroup $\breve{\mathcal{K}}$, we set

$$\mathrm{Adm}(\{\mu\})^K = \breve{W}_K \,\mathrm{Adm}(\{\mu\}) \breve{W}_K \subset \breve{W},$$
$$\mathrm{Adm}(\{\mu\})_K = \breve{W}_K \backslash \mathrm{Adm}(\{\mu\})^K / \breve{W}_K \subset \breve{W}_K \backslash \breve{W} / \breve{W}_K.$$

Let

$$\begin{aligned} X(\{\mu\}, b)_K &= \{g \in G(\breve{F})/\breve{\mathcal{K}}; g^{-1} b \sigma(g) \in \cup_{w \in \mathrm{Adm}(\{\mu\})_K} \breve{\mathcal{K}} \dot{w} \breve{\mathcal{K}}\} \\ &= \sqcup_{w \in \mathrm{Adm}(\{\mu\})_K} X_{K,w}(b). \end{aligned}$$

This is a union of affine Deligne-Lusztig varieties in $G(\breve{F})/\breve{\mathcal{K}}$. If $\breve{\mathcal{K}} = \breve{\mathcal{I}}$, we simply write $X(\{\mu\}, b)$ instead of $X(\{\mu\}, b)_K$.

In this paper, we are mainly interested in these unions of affine Deligne-Lusztig varieties. They play an important role in the study of the reduction of Shimura varieties.

2.3. We recall the nonemptiness pattern of $X(\{\mu\}, b)_K$.

To do this, we recall the definition of neutral acceptable set $B(G, \{\mu\})$ in [41]. The Frobenius morphism σ on $G(\breve{F})$ induces an affine morphism on V (see [44, 1.10]), which we still denote by σ. We denote by $\varsigma \in GL(V)$ the linear part of σ with respect to the decomposition $\mathrm{Aff}(V) = V \rtimes GL(V)$. Let $\mu^\natural$ be the common image of $\mu \in \{\mu\}$ in $\pi_1(G)_\Gamma$, and $\mu^\diamond \in X_*^+(\breve{T})_{\Gamma_0,\mathbb{Q}}$ be the average of the ς-orbit of a dominant representative of the image of an element of $\{\mu\}$ in $X_*(\breve{T})_{\Gamma_0}$.

Note that there is a partial order on the set of dominant elements in $X_*(\breve{T}) \otimes \mathbb{Q}$ (namely, the *dominance order*) defined as follows. For $\lambda, \lambda' \in X_*(\breve{T}) \otimes \mathbb{Q}$, we write $\lambda \leq \lambda'$ if $\lambda' - \lambda$ is a non-negative rational linear combination of positive coroots. Set

$$B(G, \{\mu\}) = \{[b] \in B(G); \kappa([b]) = \mu^\natural, \bar{\nu}_b \leq \mu^\diamond\}.$$

The set $B(G, \{\mu\})$ inherits a partial order from $X_*(\breve{T}) \otimes \mathbb{Q}$. It has a unique minimal element, the σ-conjugacy class $[\dot{\tau}_{\{\mu\}}]$. It is proved in [18] that $B(G, \{\mu\})$ has a unique maximal element.

The following result is conjectured by Kottwitz and Rapoport in [28] and [40] and proved by the first author in [16].

Theorem 2.3. *Let $\breve{\mathcal{K}}$ be a σ-stable standard parahoric subgroup of $G(\breve{F})$ and $b \in G(\breve{F})$. Then*
(1) The set $X(\{\mu\}, b)_K \neq \emptyset$ if and only if $[b] \in B(G, \{\mu\})$.
(2) The natural projection $G(\breve{F})/\breve{\mathcal{I}} \to G(\breve{F})/\breve{\mathcal{K}}$ induces a surjection

$$X(\{\mu\}, b) \twoheadrightarrow X(\{\mu\}, b)_K.$$

3. Affine flag varieties in mixed characteristics and line bundles

3.1. The aim of this section is to give a description of the Picard group of the mixed characteristic affine flag variety of [3]. In the equal characteristic setting and under a tameness hypothesis, the description of the Picard group is well-known, see, e.g. [38, Proposition 10.1]. An essential component of the proof there is the existence of the big cell corresponding to the open $G(\mathbf{k}[t^{-1}])$-orbit on the affine flag variety. It is pointed out by Bhatt and Scholze in [3, Question 10.6 (iv)] that such an approach seems to break down in mixed characteristic. Here we develop a different strategy to study the Picard group, based on the relation between the Picard groups of Demazure varieties and of the affine flag variety and the method of h-descent developed in [3]. This approach works for both mixed characteristic and equal characteristic after taking perfections.

3.2. **Line bundles in equal characteristic.** We briefly recall the construction of the affine flag variety in the equal characteristic setting.

Let $\mathbf{k}$ be an algebraically closed field of characteristic p. Let G be a reductive group over $\mathbf{k}((t))$, which splits over a tamely ramified extension, and $\breve{\mathcal{I}}$ be an Iwahori subgroup of G. Pappas and Rapoport in [38] constructed an associated ind-projective ind-scheme over $\mathbf{k}$ called the affine flag variety. This is defined to be the fpqc quotient

$$\mathcal{FL} = LG/L^+\breve{\mathcal{I}}.$$

For simplicity we assume that G is simply connected. Then $\mathcal{FL}$ is connected. By [38, Proposition 10.1], we have an isomorphism

$$\mathrm{Pic}(\mathcal{FL}) \cong \bigoplus_{\iota \in \breve{\mathbb{S}}} \mathbb{Z}.$$

The line bundles are constructed by identifying $\mathcal{FL}$ as an inductive limit of the Schubert varieties appearing in Kac-Moody theory [38, 9.27]. Under this identification, to each affine weight λ, [35, XVIII, Proposition 28] constructs a line bundle $\mathscr{L}(\lambda)$ on $\mathcal{FL}$. If $\breve{\mathcal{K}}_\iota$ is the parahoric subgroup corresponding to $\iota \in \breve{\mathbb{S}}$, the degree of the restriction of $\mathscr{L}(\lambda)$ to $\mathbb{P}^1 \cong L^+\breve{\mathcal{K}}_\iota/L^+\breve{\mathcal{I}}$ is given by the coefficient of the fundamental weight ϵ_ι in λ, and these degrees suffice to characterize $\mathscr{L}(\lambda)$.

3.3. We now revert to the mixed characteristic setting. For the rest of §3 only, we will use slightly different notation to the rest of the paper. We let k be a perfect field of characteristic $p>0$ and F a finite totally ramified extension of $W(k)$, in particular F is not necessarily local non-archimdean. The reason for this is because although we are mainly interested in the case F is local non-archimedean, the proofs in this section require us to check properties over arbitrary geometric points. We therefore would like to consider more general F as above; we still use the notations from §1.1. In particular $\mathbf{k}$ denotes an algebraic closure of k.

We recall the Witt vector affine Grassmannian of [3], see also [49]. We refer to [3, §3] for generalities on perfect schemes.

For a k-algebra R, we define the relative Witt vectors

$$W_{\mathcal{O}_F}(R)=W(R)\otimes_{W(k)}\mathcal{O}_F.$$

For $x\in R$, we write $[x]\in W(R)$ for its Teichmüller representative. Let $\mathcal{Z}$ be a finite type affine scheme over $\mathcal{O}_F$ and Z its generic fiber. We consider the two functors on perfect k-algebras R given by

$$LZ(R)=Z(W_{\mathcal{O}_F}(R)[\frac{1}{p}]),\quad L^+\mathcal{Z}(R)=\mathcal{Z}(W_{\mathcal{O}_F}(R)).$$

We are interested in the following special case. We let $\mathcal{G}$ denote a smooth affine group scheme over $\mathcal{O}_{\breve{F}}$. The p-adic loop group is the functor on perfect k-algebras R given by

$$LG(R)=G(W_{\mathcal{O}_F}(R)[\frac{1}{p}]).$$

The positive p-adic loop group is the functor on perfect k-algebras R given by

$$L^+\mathcal{G}(R)=\mathcal{G}(W_{\mathcal{O}_F}(R)).$$

Clearly these are functors valued in groups, and it is known that $L^+\mathcal{G}$ is representable by a perfect affine scheme and LG is a strict ind-perfect affine scheme. In fact the functor $L^+\mathcal{G}$ is the perfection of the scheme $L^+_p\mathcal{G}=\varprojlim L^h_p\mathcal{G}$ where $L^h_p\mathcal{G}$ is the finite type scheme over k given by the Greenberg realization of $\mathcal{G}$ over $\mathcal{O}_{\breve{F}}/\pi^h$.

Let Perf be the category of perfect schemes over k. The affine Grassmannian associated to $\mathcal{G}$ is the functor on Perf given by the fpqc quotient

$$Gr_{\mathcal{G}}=LG/L^+\mathcal{G}.$$

It is proved in [3, Corollary 10.6] that $Gr_{\mathcal{G}}$ is an increasing union of perfections of quasi-projective schemes over k.

We are particularly interested in $Gr_{\mathcal{G}}$ when $\mathcal{G}$ is a (connected) parahoric group scheme. In this case it is known that $Gr_{\mathcal{G}}$ is representable by an inductive limit of perfections of projective schemes, cf. [3, Corollary 10.6].

We assume $k=\mathbf{k}$, so that $\breve{F}=F$. Then we also know that in this case, the Kottwitz homomorphism $\tilde{\kappa}$ induces bijections

$$\pi_0(LG)\cong\pi_0(Gr_{\mathcal{G}})\cong\pi_1(G)_{\Gamma_0},$$

cf. [49, Proposition 1.21].

When $\mathcal{G}$ is the connected smooth affine group scheme corresponding to an Iwahori subgroup, we will call $LG/L^+\mathcal{G}$ the *affine flag variety* for $\mathcal{G}$ and denote it by $\mathcal{FL}$. By abuse of notation we will identify parahoric group schemes with their $\mathcal{O}_{\breve{F}}$-points and write $L^+\breve{\mathcal{K}}$ for the respective positive loop group. For $\iota\in\breve{\mathbb{S}}$, we write $\breve{\mathcal{K}}_\iota$ for the corresponding parahoric subgroup.

The main result of this section is the following:

Theorem 3.1. *Assume that G is simply connected and $k=\mathbf{k}$.*

(1) There is an isomorphism

$$Pic\mathcal{FL}\cong\bigoplus_{\iota\in\breve{\mathbb{S}}}\mathbb{Z}[\frac{1}{p}],$$

where the isomorphism is given by taking $\mathscr{L}$ to the degree of its restriction to $L^+\breve{\mathcal{K}}_\iota/L^+\breve{\mathcal{I}}\cong\mathbb{P}^{1,p^{-\infty}}$.

(2) A line bundle $\mathscr{L}$ corresponding to $(\lambda_\iota)_{\iota\in\breve{\mathbb{S}}}$ is ample if and only if $\lambda_\iota > 0$ for all $\iota \in \breve{\mathbb{S}}$.

Remark 3.2. Recall that the Picard group of a perfect scheme is always a $\mathbb{Z}[\frac{1}{p}]$-module. This follows since the Frobenius is an isomorphism and it induces multiplication by p on the Picard group.

3.4. We first explain how this theorem can be used to describe the Picard group of the affine flag variety for non-simply connected G. Recall that the Kottwitz homomorphism $G(\breve{F}) \to \pi_1(G)_{\Gamma_0}$ is surjective. For an element $\tau \in \pi_1(G)_{\Gamma_0}$, we may take an element $g \in LG(\mathbf{k})$ whose image in $\pi_1(G)_{\Gamma_0}$ is τ. Then multiplication by g induces an isomorphism over $\mathbf{k}$ of the connected component of $\mathcal{FL}$ corresponding to τ and the neutral component $\mathcal{FL}^0$ corresponding to 0.

Let G_{der} be the derived group of G and $\tilde{G}$ the simply connected covering of G_{der}. Let $\widetilde{\mathcal{FL}}$ be flag variety for $\tilde{G}$ and the corresponding Iwahori subgroup of $\tilde{G}$.

Proposition 3.3. *The map $L\tilde{G} \to LG$ identifies $\widetilde{\mathcal{FL}}$ with $\mathcal{FL}^0$.*

Proof. The same argument as in the discussion in [38, §6.2] shows that $\widetilde{\mathcal{FL}} \to \mathcal{FL}^0$ is a universally injective (radicial) and surjective morphism. Since the map is also proper, it is a universal homeomorphism. That this is an isomorphism follows from [49, Corollary A.16]. □

It follows that $\operatorname{Pic}\mathcal{FL} = (\operatorname{Pic}\widetilde{\mathcal{FL}})^{\oplus|\pi_1(G)_{\Gamma_0}|}$, thus in order to describe the Picard group of $\mathcal{FL}$, we may and do assume that G is simply connected.

3.5. We now introduce the Demazure resolutions for the affine flag variety $\mathcal{FL}$.

Recall that G is simply connected. So $\breve{W} = \breve{W}_a$ is a Coxeter group. For $w \in \breve{W}$, let S_w be the closure of the $L^+\breve{\mathcal{I}}$ orbit of $\dot{w}$, where we consider $\dot{w}$ as a point in $\mathcal{FL}$. This is the Schubert variety corresponding to w. For $\iota \in \breve{\mathbb{S}}$, the Schubert variety S_{s_ι} is isomorphic to $L^+\breve{\mathcal{K}}_\iota/L^+\breve{\mathcal{I}}$.

Let $\underline{w} = (s_{j_1}, \dots, s_{j_n})$ be a reduced expression of w. Let $\operatorname{supp}(w) = \{s_{j_1}, \dots, s_{j_n}\}$ be the support of w. It is known that $\operatorname{supp}(w)$ is independent of the reduced word decomposition for w. For $i = 0, \dots, n$, define $\underline{w}_i = (s_{j_1}, \dots, s_{j_i})$. We form the Demazure variety

$$D_{\underline{w}} = L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_2} \times^{L^+\breve{\mathcal{I}}} \cdots \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_n}/L^+\breve{\mathcal{I}}$$

as an object in Perf.

For any $j \in \breve{\mathbb{S}}$, the quotient $L^+\breve{\mathcal{K}}_j/L^+\breve{\mathcal{I}}$ is isomorphic to $\mathbb{P}^{1,p^{-\infty}}$. Thus we have a sequence of morphisms

$$D_{\underline{w}} = D_{\underline{w}_n} \to D_{\underline{w}_{n-1}} \to \cdots \to D_{\underline{w}_0} = \operatorname{Spec}\mathbf{k},$$

given by forgetting the last coordinate. Each morphism is a locally trivial fibration with fiber isomorphic to $\mathbb{P}^{1,p^{-\infty}}$, see for example [38, Prop. 8.8], [49, 1.4.2]. In particular $D_{\underline{w}}$ is perfectly finitely presented over $\mathbf{k}$, cf. [49, Corollary A.23].

For each i, we have maps

$$\phi_i : S_{s_{j_i}} = L^+\breve{\mathcal{K}}_{j_i}/L^+\breve{\mathcal{I}} \to D_{\underline{w}}$$

given by $\phi_i(x) = (1, \dots, x, \dots, 1)$ where the x appears in the i^{th} position. For each $h \gg 0$, we have an isomorphism $L_p^h\breve{\mathcal{K}}_{j_i}/L_p^h\breve{\mathcal{I}} \cong \mathbb{P}^1$ and taking perfection gives an isomorphism $S_{j_i} \cong \mathbb{P}^{1,p^{-\infty}}$. This isomorphism is well-defined up to the action of $\operatorname{Aut}(\mathbb{P}^1)$, and hence gives a canonical identification of $\operatorname{Pic} S_{s_{j_i}} \cong \mathbb{Z}[\frac{1}{p}]$; from now on we fix this isomorphism.

Proposition 3.4. *The Demazure variety $D_{\underline{w}}$ is the perfection of a smooth projective $\mathbf{k}$-scheme and there is an isomorphism*

$$Pic(D_{\underline{w}}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}[\frac{1}{p}],$$

given by taking a line bundle $\mathscr{L}$ to the degree of its restriction $\phi_i^*\mathscr{L}$ to $S_{s_{j_i}}$.

Proof. We prove the statement for $D_{\underline{w}_i}$ by induction on i.

For any scheme S over $\mathbf{k}$ and $m \in \mathbb{N}$, we denote by $S^{(m)}$ the scheme S with $\mathbf{k}$-structure given by $S \xrightarrow{\mathrm{Fr}^m} S \to \operatorname{Spec}\mathbf{k}$, where Fr is the absolute Frobenius. Suppose $D_{\underline{w}_{i-1}}$ is the perfection of a smooth projective scheme X. By [49, Lemma A.9], $D_{\underline{w}_i} \to D_{\underline{w}_{i-1}}$ arises as the perfection of a $\mathbb{P}^1$-bundle $Y' \to D_{\underline{w}_i}$. Since $D_{\underline{w}_{i-1}}$ is quasi-compact, we can find a covering $\{U_i\}$ of $D_{\underline{w}_{i-1}}$ consisting of finitely many Zariski open subsets such that $Y'|_{U_i} \cong \mathbb{P}^1 \times U_i$. Then we obtain a Cech cocycle $f_{ij} \in PGL_2(U_{ij})$, where $U_{ij} = U_i \cap U_j$. Since $D_{\underline{w}_{i-1}} \to X$ is a universal homeomorphism, the U_i descend to a Zariski open subset $U_i' \subset X$. Let U_{ij}' be the intersection $U_i' \cap U_j'$. The f_{ij} arises from $f_{ij}' \in PGL_2(U_{ij}^{(m)})$ for some sufficiently large m. Increasing m if necessary, we may assume furthermore that $f_{ij}' f_{jk}' f_{ki}' = 1 \in PGL_2(U_{ijk}')$, i.e. f_{ij}' is a cocycle. This defines a $\mathbb{P}^1$-bundle $Y \to X^{(m)}$ whose perfection is $D_{\underline{w}} \to D_{\underline{w}_{i-1}}$.

Since $X^{(m)}$ is smooth, Y is smooth projective. By [34, Theorem 5], $\mathrm{Pic}(Y) = \mathrm{Pic}(X^{(m)}) \oplus \mathbb{Z}$. Here the $\mathbb{Z}$ factor is given by the degree of the restriction of a line bundle to a fiber of $Y \to X^{(m)}$. Taking perfections and using [3, Lemma 3.5], it follows that $\mathrm{Pic}(D_{\underline{w}_i}) = \mathrm{Pic}(D_{\underline{w}_{i-1}}) \oplus \mathbb{Z}[\frac{1}{p}]$. The statement follows by induction. □

Remark 3.5. We would like to draw attention to the following subtlety. In general, the identification $\mathrm{Pic}(S_{s_{j_i}}) \cong \mathbb{Z}[\frac{1}{p}]$ depends on the choice of an isomorphism $S_{s_{j_i}} \cong \mathbb{P}^{1,p^{-\infty}}_{\mathbf{k}}$. Composing such an isomorphism by the absolute Frobenius changes the identification on Picard groups by multiplication by p. Thus to canonify the identification $\mathrm{Pic}(S_{s_{j_i}}) \cong \mathbb{Z}[\frac{1}{p}]$, one must choose an isomorphism $S_{s_{j_i}} \cong \mathbb{P}^{1,p^{-\infty}}_{\mathbf{k}}$, i.e. a deperfection of $S_{s_{j_i}}$. The choice of deperfection we made above was suggested to us by Görtz.

3.6. As in [38], there is a proper perfectly finitely presented surjective map

$$\Psi : D_{\underline{w}} \to S_w.$$

Since both S_w and $S_{\underline{w}}$ are perfectly finitely presented over $\mathrm{Spec}\,\mathbf{k}$, so is the map Ψ. It follows that

$$S_w = \bigcup_{v \le w} L^+\breve{\mathcal{I}} v L^+\breve{\mathcal{I}}/L^+\breve{\mathcal{I}}. \tag{3.1}$$

In this situation, we may apply the results of [3] to descend line bundles from $D_{\underline{w}}$ to S_w. The general strategy is as follows.

Theorem 3.6 ([3] Theorem 6.8)**.** *Let $f : X \to Y$ be a proper surjective perfectly finitely presented map in Perf such that $Rf_*\mathcal{O}_X = \mathcal{O}_Y$; in particular, all geometric fibers of f are connected. Let $\mathscr{L}$ be a vector bundle on X. Then $\mathscr{L}$ descends to Y if and only if for all geometric points $\overline{y}$ of Y, $\mathscr{L}_{\overline{y}}$ is trivial on the fiber $X_{\overline{y}}$.*

Now we show that $D_{\underline{w}} \to S_w$ satisfies the conditions in Theorem 3.6.

Proposition 3.7. *Let $w \in \breve{W}$ and $\underline{w}$ be a reduced expression of w. Then*

$$R\Psi_*\mathcal{O}_{D_{\underline{w}}} = \mathcal{O}_{S_w}.$$

Proof. We use induction on the length of w; for $\ell(w) = 0$, Ψ is an isomorphism so this holds.

Suppose that $\underline{w} = (s_{j_1}, \dots, s_{j_n})$ with $n > 0$. Let $w = s_{j_1} w'$ and $\underline{w}' = (s_{j_2}, \dots, s_{j_n})$. Then $\underline{w}'$ is a reduced expression of w' and Ψ factors as

$$D_{\underline{w}} = L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} D_{\underline{w}'} \to L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w'} \to S_w.$$

The first map satisfies the condition on direct images by induction. This follows from either the base change result [3, Lemma 3.18], or alternatively, it can be checked using the fiberwise criterion [3, Lemma 7.8]. The second map is a proper perfectly finitely presented surjective map, and so by [3, Lemma 7.8], it suffices to check for all geometric points $\overline{y}$ of S_w with residue field $k(\overline{y})$ and fiber $D_{\underline{w},\overline{y}}$, that $R\Gamma(D_{\underline{w},\overline{y}}, \mathcal{O}_{D_{\underline{w},\overline{y}}}) = k(\overline{y})$.

Let $Z = \bigcup_{w''} S_{w''} \subset S_w$ where the union runs over $w'' < w'$, with $s_{j_1} w'' < w''$. Then the second map is an isomorphism away from Z and the fiber over a point $\overline{y}$ in Z is given by $\mathbb{P}^{1,p^{-\infty}}_{k(\overline{y})}$. The result then follows. □

Note that $\Psi \circ \phi_{j_i}$ is the natural embedding of

$$S_{s_{j_i}} = L^+\breve{\mathcal{K}}_{j_i}/L^+\breve{\mathcal{I}} \hookrightarrow S_w.$$

Then

Lemma 3.8. *Let $\mathscr{L}$ be a line bundle on S_w and suppose $\Psi^*\mathscr{L}$ corresponds to (λ_i) under the isomorphism in Proposition 3.4. Then $\lambda_\iota = \deg(\mathscr{L}|_{S_{s_{j_\iota}}})$.* □

Proposition 3.9. *We have an isomorphism*

$$\mathrm{Pic}(S_w) \xrightarrow{\cong} \bigoplus_{s_\iota \in \mathrm{supp}(w)} \mathbb{Z}[\frac{1}{p}], \qquad \mathscr{L} \longmapsto (\deg(\mathscr{L}|_{S_{s_\iota}}))_{s_\iota \in \mathrm{supp}(w)}.$$

Proof. Let $\underline{w} = (s_{j_1}, \dots, s_{j_n})$ be a reduced expression of w. The map $\Psi : D_{\underline{w}} \to S_w$ induces a map $\Psi^* : \mathrm{Pic}(S_w) \to \mathrm{Pic}(D_{\underline{w}}) \cong \oplus_{i=1}^n \mathbb{Z}[\frac{1}{p}]$. By [3, Proposition 7.1], Ψ^* is injective. By Lemma 3.8, the map is given by $\mathscr{L} \mapsto (\deg(\mathscr{L}|_{S_{j_i}}))_{i=1,\dots,n}$. Therefore the map in the statement of the proposition is an injection.

It remains to show that the map is surjective. In other words, let $\lambda_i \in \mathbb{Z}[\frac{1}{p}]$ for $i = 1, \dots, n$ such that $\lambda_r = \lambda_s$ for $j_r = j_s$ and $\tilde{\mathscr{L}} \in \mathrm{Pic}(D_{\underline{w}})$ that corresponds to $(\lambda_1, \dots, \lambda_n)$, we need to show that $\tilde{\mathscr{L}}$ descends to a line bundle on S_w.

We argue by induction on $\ell(w)$. It is clearly true when $\ell(w) = 1$, i.e. w is a simple reflection. Suppose it is surjective for all w' with $\ell(w') < w$. For $i = 0, 1, \dots, n$, define $X_i = D_{\underline{w}_i} \times^{L^+\breve{\mathcal{I}}} S_{s_{j_{i+1}} \cdots s_{j_n}}$. We have $X_n = D_{\underline{w}}$ and $X_0 = S_w$ and there is a sequence of morphisms

$$X_n \to X_{n-1} \to \cdots \to X_0$$

induced by multiplication $L^+\breve{\mathcal{K}}_{j_i} \times^{L^+\breve{\mathcal{I}}} S_{s_{j_{i+1}} \cdots s_{j_n}} \to S_{s_{j_i} \cdots s_{j_n}}$.

We show that $\tilde{\mathscr{L}}$ descends to a line bundle $\mathscr{L}_i$ on X_i for each i by descending induction. This is tautologically true for $i = n$. Suppose $\tilde{\mathscr{L}}$ descends to X_{i+1}. To show $\tilde{\mathscr{L}}$ descends to X_i, it suffices by Theorem 3.6 to check the restriction of $\mathscr{L}_{i+1}$ to any fiber of $X_{i+1} \to X_i$ is trivial.

Let $\mathbf{k}'$ be an algebraically closed field containing $\mathbf{k}$ and let

$$(p_1, \dots, p_i) \in L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} \cdots \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_i}(\mathbf{k}').$$

We define a map

$$\alpha : S_{s_{j_{i+1}} \cdots s_{j_n}, \mathbf{k}'} \to X_{i,\mathbf{k}'}, \qquad s \longmapsto (p_1, \dots, p_i, s).$$

Here the subscript $\mathbf{k}'$ denotes the base change to $\mathbf{k}'$. Upon pulling back $X_{i+1,\mathbf{k}'} \to X_{i,\mathbf{k}'}$ along α, we obtain a Cartesian diagram:

$$\begin{array}{ccc} Y & \xrightarrow{\gamma} & S_{s_{j_{i+1}} \cdots s_{j_n}, \mathbf{k}'} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ X_{i+1,\mathbf{k}'} & \longrightarrow & X_{i,\mathbf{k}'}, \end{array}$$

where γ can be identified with the multiplication map $L^+\breve{\mathcal{K}}_{j_{i+1}} \times^{L^+\breve{\mathcal{I}}} S_{s_{j_{i+2}} \cdots s_{j_n}, \mathbf{k}'} \to S_{s_{j_{i+1}} \cdots s_{j_n}, \mathbf{k}'}$.

Since all geometric points of X_i arise as the image of α for some choice of $\mathbf{k}'$ and $(p_1, \dots, p_i)$, it suffices to prove that $\beta^*\mathscr{L}_{i+1}$ descends along γ. Upon replacing $\mathbf{k}'$ by $\mathbf{k}$ and relabeling, we reduce to the case

$$f : X_1 = L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w'} \to S_w = X_0$$

where $s_{j_1} w' = w$ and $\ell(w) > \ell(w')$. The projection $X_1 \to L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} = S_{j_1} \cong \mathbb{P}^{1,p^{-\infty}}$ exhibits X_1 as an $S_{w'}$-bundle over $\mathbb{P}^{1,p^{-\infty}}$, hence we have an isomorphism $\mathrm{Pic}(X_1) \cong \mathrm{Pic}(S_{w'}) \oplus \mathbb{Z}[\frac{1}{p}]$, see [34, Theorem 5]. By the induction hypothesis on $l(w')$, we have

$$\mathrm{Pic}(X_1) \cong \left(\bigoplus_{s_\iota \in \mathrm{supp}(w')} \mathbb{Z}[\frac{1}{p}] \right) \oplus \mathbb{Z}[\frac{1}{p}].$$

Note that if $j_1 = j_i$ for some $i > 1$, then there are two maps

$$g_1, g_i : L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} = L^+\breve{\mathcal{K}}_{j_i}/L^+\breve{\mathcal{I}} \to X_1$$

given by $g_1(x) = (x, e)$ and $g_i(x) = (1, x)$ where e is the base point of $S_{w'}$ and we consider $L^+\breve{\mathcal{K}}_{j_i}/L^+\breve{\mathcal{I}} \subset S_{w'}$ by 3.1. By our assumption on $\mathscr{L}_1$, we have

$$\deg(g_1^*\mathscr{L}_1) = \lambda_1 = \lambda_i = \deg(g_i^*\mathscr{L}_1).$$

The map $f : X_1 \to X_0$ is proper perfectly finitely presented and by the proof of Proposition 3.7 we have

$$Rf_*\mathcal{O}_{X_1} \cong \mathcal{O}_{X_0}.$$

Therefore by Theorem 3.6, to show that $\mathscr{L}_1$ descends, it suffices to check that the restriction of $\mathscr{L}_1$ to each geometric fiber is trivial. As in Proposition 3.7, f is an isomorphism away from $S_{w''}$ where $w'' < w'$ and $s_{j_1}w'' < w''$, so the condition is satisfied away from this locus. We have the following fiber diagram:

$$\begin{array}{ccc} L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w''} & \xrightarrow{h} & S_{w''} \\ \downarrow g & & \downarrow i \\ L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w'} & \xrightarrow{f} & S_w. \end{array}$$

We will show that $g^*\mathscr{L}_1$ descends to a line bundle on $S_{w''}$.

By induction we know

$$\operatorname{Pic} S_{w''} \cong \bigoplus_{s_\iota \in \operatorname{supp}(w'')} \mathbb{Z}[\frac{1}{p}].$$

Furthermore since $h : L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w''} \to S_{w''}$ is a $\mathbb{P}^{1,p^{-\infty}}$-bundle, it follows that

$$\operatorname{Pic}(L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w''}) \cong \left(\bigoplus_{s_\iota \in \operatorname{supp}(w'')} \mathbb{Z}[\frac{1}{p}]\right) \oplus \mathbb{Z}[\frac{1}{p}].$$

Here the isomorphism is determined by the section $S_{w''} \to L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w''}$ given by $x \mapsto (1, x)$.

By our construction, the map $h^* : \operatorname{Pic} S_{w''} \to \operatorname{Pic} L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w''}$ is given by

$$\bigoplus_{s_\iota \in \operatorname{supp}(w'')} \mathbb{Z}[\frac{1}{p}] \to \left(\bigoplus_{s_\iota \in \operatorname{supp}(w'')} \mathbb{Z}[\frac{1}{p}]\right) \oplus \mathbb{Z}[\frac{1}{p}], \quad (\lambda_\iota) \longmapsto ((\lambda_\iota), 0).$$

We need to check that $g^*\mathscr{L}_1$ is of this form, i.e. its restriction to a fiber of h is trivial. The fiber of h over the base point $e \in S_{w''}$ is contained in $L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} \subset L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} S_{w''}$. Indeed it is given by the image of the map

$$\alpha : L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} \to L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}}$$

given by $p \mapsto (p, p^{-1})$.

Thus it suffices to show the restriction of $g^*\mathscr{L}_1$ to $L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}}$ somes from pullback along the multiplication map. As above we have an isomorphism

$$\operatorname{Pic} L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} \cong \mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}].$$

The isomorphism is given by the degrees of the restriction to the two copies of $L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}}$ embedded into $L^+\breve{\mathcal{K}}_{j_1} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}}$ by $x \mapsto (1, x)$ and $x \mapsto (x, e)$. By our assumption $g^*\mathscr{L}_1$ corresponds to $(\lambda_1, \lambda_1) \in \mathbb{Z}[\frac{1}{p}] \oplus \mathbb{Z}[\frac{1}{p}]$ under the above isomorphism.

Now if we consider the line bundle $\mathcal{O}(\lambda_1) \in \operatorname{Pic}(\mathbb{P}^{1,p^\infty}) \cong \operatorname{Pic}(L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}})$, then this pulls back to $g^*\mathscr{L}_1$ under the multiplication map

$$m : L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}} \to L^+\breve{\mathcal{K}}_{j_1}/L^+\breve{\mathcal{I}}.$$

It follows that the restrictions of $\mathscr{L}_1$ to all geometric fibers are trivial, and hence $\mathscr{L}_1$ descends to S_w. The surjectivity of the map in the Proposition is proved. □

3.7. **Proof of Theorem 3.1 (1).** Since $\mathcal{FL}$ is an increasing union of S_w as w ranges over $\breve{W}$, in order to prove Theorem 3.1 it suffices to show that the description of the Picard groups of S_w in Proposition 3.9 is compatible with the natural inclusions $S_{w'} \to S_w$ for $w' < w$. This follows from the following commutative diagram

$$\begin{array}{ccc} \coprod\limits_{s_\iota \in \mathrm{supp}(w')} S_{s_\iota} & \longrightarrow & S_{w'} \ , \\ {\scriptstyle h}\big\downarrow & & \big\downarrow \\ \coprod\limits_{s_\iota \in \mathrm{supp}(w)} S_{s_\iota} & \longrightarrow & S_w \end{array}$$

where the horizontal maps induce the isomorphisms of Picard groups and h^* is the natural map

$$\bigoplus_{s_\iota \in \mathrm{supp}(w)} \mathbb{Z}[\frac{1}{p}] \to \bigoplus_{s_\iota \in \mathrm{supp}(w')} \mathbb{Z}[\frac{1}{p}].$$

3.8. For the proof of Theorem 3.1 (2), we follow the strategy of [3, §8.4]. The proofs of loc. cit. go through in our situation with some minor changes using the Demazure resolution $D_{\underline{w}} \to S_w$.

We denote by $(\mathscr{L}(\epsilon_\iota))_{\iota \in \breve{\mathbb{S}}}$ the basis elements of $\mathrm{Pic}(\mathcal{FL})$ obtained by the isomorphism in Theorem 3.1 (1). Similarly, for any $w \in \breve{W}$ and a reduced expression $\underline{w}$, we denote by $(\mathscr{L}(a_i))_{i=1,\dots,n}$ the basis element of $\mathrm{Pic}(D_{\underline{w}})$ obtained by the isomorphism in Proposition 3.4.

For $w \in \breve{W}$, let O_w be the $L^+\breve{\mathcal{I}}$ orbit of the point $\dot{w}L^+\breve{\mathcal{I}}$ in $\mathcal{FL}$. For $i = 1, \dots, n$, let C_i be the closed subscheme of $D_{\underline{w}}$ where the i^{th} term is in $L^+\breve{\mathcal{I}}$ and i'^{th} term for $i' \neq i$ is in $L^+\breve{\mathcal{K}}_{j_{i'}}$. Let $\partial D_{\underline{w}} = \bigcup_{i=1}^n C_i$ be the boundary of $D_{\underline{w}}$. By [29, Theorem 5.1.3 (i)], the map $D_{\underline{w}} \smallsetminus \partial D_{\underline{w}} \to O_w$ is an isomorphism.

Lemma 3.10. *(1) The line bundle $\otimes_{i=1}^n \mathscr{L}(a_i)^{\lambda_i}$ is ample if $\lambda_1 \gg \lambda_2 \gg \cdots \gg \lambda_n \gg 0$.*

(2) For $x \in D_{\underline{w}} \smallsetminus \partial D_{\underline{w}}$, there exists a section $s \in H^0(D_{\underline{w}}, \mathscr{L}(a_i))$ such that $s(x) \neq 0$.

Proof. (1) We prove this inductively using the sequence of $\mathbb{P}^{1,p^{-\infty}}$ fibrations

$$D_{\underline{w}_n} \to D_{\underline{w}_{n-1}} \to \cdots \to D_{\underline{w_0}}.$$

More precisely, we will prove by induction that for $m \leq n$, the line bundle $\otimes_{i=1}^m \mathscr{L}(a_i)^{\mu_i}$ is ample on $D_{\underline{w}_m}$ for $\mu_1 \gg \mu_2 \gg \cdots \gg \mu_m \gg 0$. For $m = 1$, $D_{\underline{w}_1} \cong \mathbb{P}^{1,p^{-\infty}}$, and so the statement is clear. Assume inductively there exists $\mu'_1 \gg \mu'_2 \gg \cdots \gg \mu'_{m-1} \gg 0$ such that $\otimes_{i=1}^{m-1} \mathscr{L}(a_i)^{\mu'_i}$ is ample on $D_{\underline{w}_{m-1}}$. Since $D_{\underline{w}_m} \to D_{\underline{w}_{m-1}}$ is a $\mathbb{P}^{1,p^{-\infty}}$ fibration and $\mathscr{L}(a_m)$ is relatively ample for this morphism, we have $\mathscr{L}(a_m) \otimes (\otimes_{i=1}^{m-1} \mathscr{L}(a_i)^{\mu'_i})^N$ is ample for $N \gg 0$. This proves the induction step and hence the result.

(2) We have a $\mathbb{P}^{1,p^{-\infty}}$ fibration $D_{\underline{w}_i} \to D_{\underline{w}_{i-1}}$ and hence a line bundle $\mathcal{O}_i(1)$ on $D_{\underline{w}_i}$. The line bundle $\mathscr{L}(a_i)$ is given by the pullback of $\mathcal{O}_i(1)$ along the map $D_{\underline{w}_n} \to D_{\underline{w}_i}$.

We have a section of $\mathcal{O}_i(1)$ corresponding to the section $D_{\underline{w}_{i-1}} \to D_{\underline{w}_i}$ defined by

$$(x_1, \dots, x_{i-1}) \longmapsto (x_1, \dots, x_{i-1}, 1).$$

The pullback of this section along $D_{\underline{w}_n} \to D_{\underline{w}_i}$ gives a section of $\mathscr{L}(a_i)$ which does not vanish on $D_{\underline{w}} \smallsetminus \partial D_{\underline{w}}$. □

3.9. **Proof of Theorem 3.1 (2).** Let $\mathscr{L} = \otimes \mathscr{L}(\epsilon_\iota)^{\delta_\iota}$ be an ample line bundle on S_w. Then for $s_\iota \in \mathrm{supp}(w)$, the restriction of $\mathscr{L}$ to $S_{s_\iota} \cong \mathbb{P}^{1,p^{-\infty}}$ is ample and hence has positive degree. Thus $\delta_\iota > 0$.

For the converse, let $\mathscr{L} = \otimes \mathscr{L}(\epsilon_\iota)^{\lambda_\iota}$ be a line bundle on S_w with $\lambda_\iota > 0$ for all ι and let $\mathscr{M} = \otimes \mathscr{L}(a_i)^{\lambda_i}$ be its pullback to $D_{\underline{w}}$ so that $\lambda_i = \lambda_\iota$ for $s_{j_i} = s_\iota$. We prove that $\mathscr{L} \mid_{S_w}$ is ample by induction on $\ell(w)$.

We show that

(a) $\mathscr{L}$ is strictly nef.

This can be proved in the same way as [3, Lemma 9.10]. Indeed let $C_{\rm perf}$ be the perfection of a smooth projective curve over k and let $f : C_{\rm perf} \to S_w$ be a non-constant map. Without loss of generality, we may assume f maps the generic point of $C_{\rm perf}$ into O_w. As $D_{\underline{w}} \to S_w$ is surjective, the map $C_{\rm perf} \to S_w$ lifts generically to $D_{\underline{w}}$, and since $D_{\underline{w}} \to S_w$ is proper, f lifts to $\tilde{f} : C_{\rm perf} \to D_{\underline{w}}$.

Since $C_{\rm perf}$ meets $D_{\underline{w}} \setminus \partial D_{\underline{w}}$, the pullback of $\mathscr{L}(a_i)^{\lambda_i}$ to $C_{\rm perf}$ has a non-vanishing section, hence it is effective and therefore has non-negative degree. Since $f^*\mathscr{L} = \otimes_{i=1}^n \tilde{f}^*\mathscr{L}(a_i)^{\lambda_i}$, if this has degree 0, then $\tilde{f}^*\mathscr{L}(a_i)^{\lambda_i}$ has degree 0 for all i, hence it is trivial. However by Lemma 3.10 a weighted product of $\mathscr{L}(a_i)^{\lambda_i}$ is ample on $D_{\underline{w}}$, hence has positive degree on $C_{\rm perf}$. Therefore the pullback of $\mathscr{L}$ to $C_{\rm perf}$ has positive degree. This proves (a).

We show that

(b) $\mathscr{L}$ is semiample.

As in the proof of [3, Lemma 9.11] we have that $\mathscr{M}$ is big with exceptional locus $E(\mathscr{M})$ contained in the boundary $\partial D_{\underline{w}} = \Psi^{-1}(\bigcup_{w'<w} S_{w'})$. Since $\mathscr{M}$ is nef, by [21, Theorem 1.9] it suffices to show $\mathscr{M}|_{E(\mathscr{M})}$ semiample. By induction we have that $\mathscr{L}|_{S_{w'}}$ is ample for all $w' < w$. By [21, Lemma 1.8], we have that $\mathscr{L}|_{\bigcup_{w'<w} S_{w'}}$ is ample and hence $\mathscr{M}|_{\partial D_{\underline{w}}}$ is semiample. Since $D_{\underline{w}} \to S_w$ has connected fibers, $\mathscr{L}$ is semiample. This proves (b).

Combining (a) and (b), we have that $\mathscr{L}$ is ample. □

3.10. **Line bundles on partial affine flag varieties.** Let $K \subset \breve{\mathbb{S}}$ be any subset and let $\breve{\mathcal{K}}$ denote the corresponding parahoric subgroup. We can use the above results to get a description of the Picard groups of the partial affine flag varieties $Gr_{\breve{\mathcal{K}}}$ corresponding to $\breve{\mathcal{K}}$. We continue to assume G is simply connected and that $k = \mathbf{k}$.

As in [38, Proposition 8.7 a)] we have a closed immersion $L^+\breve{\mathcal{I}} \to L^+\breve{\mathcal{K}}$. The identity morphism on $G(\breve{F})$ induces a surjection:

$$\Phi : \mathcal{FL} \to Gr_{\breve{\mathcal{K}}}.$$

Proposition 3.11. *The map Φ is a fibration with fiber isomorphic to $L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{I}}$ which admits sections locally for the étale topology.*

Proof. The proof is the same as [38, 8.5.1]. Indeed this can be deduced from the fact that the natural map

$$LG \to LG/L^+\breve{\mathcal{K}}$$

admits sections locally for the étale topology. The proof of this last fact in equal characteristic, cf. [38, Theorem 1.4], extends verbatim to our situation. □

Let $\breve{W}^K$ be the set of minimal length representatives of $\breve{W}/\breve{W}_K$. For $w \in \breve{W}^K$, let $S_w^{\breve{\mathcal{K}}} \subset Gr_{\breve{\mathcal{K}}}$ denote the closure of $O_w^{\breve{\mathcal{K}}} = L^+\breve{\mathcal{I}}\dot{w}L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{K}}$ in $Gr_{\breve{\mathcal{K}}}$. Since $S_w^{\breve{\mathcal{K}}}$ is the image of S_w under Φ, we have

$$S_w^{\breve{\mathcal{K}}} = \bigcup_{w' \in \breve{W}^K, w' \leq w} L^+\breve{\mathcal{I}}\dot{w}L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{K}}$$

and $Gr_{\breve{\mathcal{K}}}$ is the rising union of the $S_w^{\breve{\mathcal{K}}}$. For $w \in \breve{W}^K$, we have $\Phi^{-1}(S_w^{\breve{\mathcal{K}}}) = \bigcup_{u \in \breve{W}_K} S_{wu}$ and we write h for the map

$$h : \bigcup_{u \in \breve{W}_K} S_{wu} \to S_w^{\breve{\mathcal{K}}}.$$

induced by Φ.

Let $\overline{\breve{\mathcal{K}}}^{red}$ be the reductive quotient of the special fiber of $\breve{\mathcal{K}}$. As in [38, Proposition 8.7 b)], we have that $L_p^+\breve{\mathcal{I}}$ is the preimage in $L_p^+\breve{\mathcal{K}}$ of a perfection of a Borel subgroup in $\overline{B}$ in $\overline{\breve{\mathcal{K}}}^{red}$. Thus the quotient $L_p^+\breve{\mathcal{K}}/L_p^+\breve{\mathcal{I}}$ can be identified with $\overline{\breve{\mathcal{K}}}^{red}/\overline{B}$, a finite type flag variety. Taking perfections, we obtain an identification of

$$L^+\breve{\mathcal{K}}/L^+\breve{\mathcal{I}}$$

with the perfection of a finite type flag variety.

Proposition 3.12. *(1) We have an isomorphism:*

$$\mathrm{Pic}(Gr_{\breve{\mathcal{K}}}) \cong \bigoplus_{\iota \in \breve{\mathbb{S}} \smallsetminus K} \mathbb{Z}[\frac{1}{p}],$$

where the isomorphism is given taking $\mathscr{L}$ to the degree of its restriction along $\mathbb{P}^{1,p^{-\infty}} \cong S_{s_\iota} \cong S^{\breve{\mathcal{K}}}_{s_\iota} \to Gr_{\breve{\mathcal{K}}}$.

(2) A line bundle $\mathscr{L} = \otimes_{\iota \in \breve{\mathbb{S}} - K} \mathscr{L}(\epsilon_\iota)^{\lambda_\iota}$ is ample if and only if $\lambda_\iota > 0$ for all $\iota \in \breve{\mathbb{S}} \smallsetminus K$.

Proof. (1) We show $\mathscr{L} = \otimes_{\iota \in \breve{\mathbb{S}}} \mathscr{L}(\epsilon)_\iota)^{\lambda_\iota} \in \mathrm{Pic}(\mathcal{FL})$ descends to $Gr_{\breve{\mathcal{K}}}$ if and only if $\lambda_\iota = 0$ for $\iota \in K$.

Let $w \in \breve{W}^K$. We apply the fibral criterion for descent of vector bundles, Theorem 3.6, to the fibration $\bigcup_{u \in \breve{W}_K} S_{wu} \to S^{\breve{\mathcal{K}}}_w$. The map is proper surjective perfectly finitely presented since both $\bigcup_{\breve{W}_K} S_{wu}$ and $S^{\breve{\mathcal{K}}}_w$ are, and the condition $Rh_* \mathcal{O}_{\bigcup_{u \in \breve{W}_K} S_{wu}} = \mathcal{O}_{S^{\breve{\mathcal{K}}}_w}$ can be checked fiberwise by [3, Theorem 7.8]. We need to check that $R\Gamma(U_{\overline{y}}, \mathcal{O}_{U_{\overline{y}}}) = k(\overline{y})$, where $\overline{y}$ is a geometric point of $S^{\breve{\mathcal{K}}}_w$ and $U_{\overline{y}}$ is the fiber of h over $\overline{y}$. Since $U_{\overline{y}}$ is the perfection of a finite type flag variety, this follows from the Borel-Weil-Bott Theorem.

Let $\mathscr{L} = \otimes_{\iota \in \breve{\mathbb{S}}} \mathscr{L}(\epsilon_\iota)^{\lambda_\iota}$. By Theorem 3.6, $\mathscr{L}$ descends to $Gr_{\breve{\mathcal{K}}}$ if and only if $\mathscr{L}|_{U_{\overline{y}}}$ is trivial. Thus if $\mathscr{L}|_{U_{\overline{y}}}$ descends to $Gr_{\breve{\mathcal{K}}}$, we have $\lambda_\iota = 0$ for $\iota \in K$ since S_{s_ι} maps to a point in $Gr_{\breve{\mathcal{K}}}$.

For the converse, suppose $\lambda_\iota = 0$ for all $\iota \in K$. We show that $\mathscr{L}|_{U_{\overline{y}}}$ is trivial for all geometric points $\overline{y}$ of $S^{\breve{\mathcal{K}}}_w$.

Let $\pi : V \to S^{\breve{\mathcal{K}}}_w$ be an étale covering such that the fiber bundle $\bigcup_{u \in \breve{W}_K} S_{wu} \to S^{\breve{\mathcal{K}}}_w$ splits. We have a pullback diagram:

$$\begin{array}{ccc} X \times_{\mathbf{k}} V & \longrightarrow & V \\ \downarrow q & & \downarrow \alpha \\ \bigcup_{u \in \breve{W}_K} S_{wu} & \longrightarrow & S^{\breve{\mathcal{K}}}_w, \end{array}$$

where X is isomorphic to the perfection of a finite type flag variety. By the Borel-Weil-Bott Theorem, we have $H^1(X, \mathcal{O}_X) = 0$. Hence $\mathrm{Pic}(X \times_{\mathbf{k}} V) \cong \mathrm{Pic}(X) \times \mathrm{Pic}(V)$, where the isomorphism is given by restricting a line bundle to the fibers of the projections $p_1 : X \times_{\mathbf{k}} V \to X$ and $p_2 : X \times_{\mathbf{k}} V \to V$. Let $\overline{x}$ be the base point of $S^{\breve{\mathcal{K}}}_w$, i.e. the image of 1 in $S^{\breve{\mathcal{K}}}_w$. By the description of the Picard group of finite type flag varieties, the Picard group of $U_{\overline{x}}$ isomorphic to $\oplus_{\iota \in K} \mathbb{Z}[\frac{1}{p}]$, where the isomorphism is given by the degrees of the restriction a line bundle to $L^+\breve{\mathcal{K}}_\iota / L^+\breve{\mathcal{I}}$. Thus by our assumptions on $\mathscr{L}$, we have $\mathscr{L}|_{U_{\overline{x}}}$ is trivial. Thus by the above, the restriction of $q^*\mathscr{L}$ to any fiber of $X \times_{\mathbf{k}} V \to V$ is trivial. Since $V \to S^{\breve{\mathcal{K}}}_w$ is surjective, $\mathscr{L}|_{U_{\overline{y}}}$ is trivial for all geometric points $\overline{y}$ of $S^{\breve{\mathcal{K}}}_w$.

(2) Let $\mathscr{M}$ be any ample line bundle on $S^{\breve{\mathcal{K}}}_w$. Note that $\mathscr{M} = \otimes_{\iota \in \breve{\mathbb{S}} \smallsetminus K} \mathscr{L}(\epsilon_\iota)^{\delta_\iota}$ with $\delta_\iota > 0$. Indeed $S^{\breve{\mathcal{K}}}_{s_\iota} \cong S_{s_\iota} \cong \mathbb{P}^{1,p^{-\infty}}$, and so the restriction of $\mathscr{M}$ to $S^{\breve{\mathcal{K}}}_{s_\iota}$ has positive degree, but this degree is just δ_ι.

Now let $\mathscr{L} = \otimes_{\iota \in \breve{\mathbb{S}} \smallsetminus K} \mathscr{L}(\epsilon_\iota)^{\lambda_\iota}$ where $\lambda_\iota > 0$, we will show $\mathscr{L}$ is ample using the same strategy as the proof of Theorem 3.1 (2).

(a) $\mathscr{L}$ is strictly nef.

Let C_{perf} be the perfection of a smooth projective curve and $C_{\mathrm{perf}} \to S^{\breve{\mathcal{K}}}_w$ any map. As before we may assume C_{perf} intersects $O^{\breve{\mathcal{K}}}_w$. Again $\mathscr{L}(\epsilon_\iota)$ has a non-vanishing section on $O^{\breve{\mathcal{K}}}_w$ since its pullback to S_w has a non-vanishing section on O_w, hence $\mathscr{L}(\epsilon_\iota)|_{C_{\mathrm{perf}}}$ is effective. Since a positive combination of the $\mathscr{L}(\epsilon_\iota)$ is ample, $\mathscr{L} |_{C_{\mathrm{perf}}}$ has positive degree.

(b) $\mathscr{L}$ is semiample.

We prove that $\mathscr{L}|S^{\breve{\mathcal{K}}}_w$ is semiample for $w \in \breve{W}^K$ by induction on $\ell(w)$. Indeed this is true for $l(w) = 1$, i.e. $w = s_\iota \in \breve{\mathbb{S}} \smallsetminus K$. Now assume it's true for all $w' \in \breve{W}^K$, with $l(w') < l(w)$.

Let $\mathscr{M} = \otimes \mathscr{L}(\epsilon)^{\delta_\iota}$ be an ample line bundle on $Gr_{\breve{\mathcal{K}}}$. Upon raising $\mathscr{L}$ to a sufficiently large power, we may assume $\lambda_\iota > \delta_\iota$ and hence $\mathscr{L}|_{S_w^{\breve{\mathcal{K}}}}$ is the tensor product of an ample line bundle with an effective one, i.e. $\mathscr{L}\,|_{S_w^{\breve{\mathcal{K}}}}$ is big. Moreover, since $\mathscr{L}$ has a non-vanishing section on $O_w^{\breve{\mathcal{K}}}$, the exceptional locus lies in $S_w^{\breve{\mathcal{K}}} \smallsetminus O_w^{\breve{\mathcal{K}}} = \bigcup_{w' \in \breve{W}^K, w' < w} S_{w'}^{\breve{\mathcal{K}}}$. By induction $\mathscr{L}$ is semiample on $S_{w'}^{\breve{\mathcal{K}}}$ for all $w' \in \breve{W}^K$ with $w' < w$, hence by [21, Lemma 1.8], $\mathscr{L}\,|_{\bigcup_{w' \in \breve{W}^K, w' < w} S_{w'}^{\breve{\mathcal{K}}}}$ is semiample. Thus $\mathscr{L}$ is semiample.

The result now follows from (a) and (b). □

Remark 3.13. In the case when $G = SL_n$ and $\breve{\mathcal{K}}$ is a hyperspecial parahoric subgroup, the proposition gives an isomorphism $\operatorname{Pic} Gr_{\breve{\mathcal{K}}} \cong \mathbb{Z}[\frac{1}{p}]$. This answers a question of Bhatt and Scholze [3, Question 10.6 (iii)].

4. First reduction theorem

In this section we study the connected components of $X(\{\mu\}, b)$. From now on, we revert back to the notations in §1.2; in particular F is now a non-archimedean local field. We also change our convention slightly from §3 on the affine flag varieties so that from now $\mathcal{FL}$ will always denote the affine flag variety over $\mathbf{k} = \overline{\mathbb{F}}_q$.

4.1. For $w \in \breve{W}$ we write

$$X_{\leq w}(b) = \bigcup_{w' \leq w} X_{w'}(b).$$

In general, for any finite subset C of $\breve{W}$ we write

$$X_C(b) = \bigcup_{w \in C} X_w(b).$$

If moreover C is closed under the Bruhat order, then $X_C(b)$ is a closed subscheme of the affine flag variety, in particular it is closed and hence projective (in the sense of perfect schemes). The main theorem of this section is the following:

Theorem 4.1. *Let C be a finite subset of $\breve{W}$ that is closed under the Bruhat order and Y be a connected component of $X_C(b)$. Then $Y \cap X_x(b) \neq \emptyset$ for some σ-straight element x in C.*

In the rest of this paper, we will mainly use the following weaker statement of Theorem 4.1.

Corollary 4.2. *Every point in $X(\{\mu\}, b)$ lies in the same Zariski connected component as a point in $X_x(b)$ for some σ-straight element $x \in \operatorname{Adm}(\{\mu\})$.* □

4.2. **Reduction to adjoint groups.** For $b \in G(\breve{F})$ and $\tau \in \pi_1(G)_{\Gamma_0}$, we let b_{ad} and τ_{ad} be their images in $G_{\text{ad}}(\breve{F})$ and $\pi_1(G_{\text{ad}})_{\Gamma_0}$ respectively. Choosing maximal tori in G and G_{ad} compatibly, we obtain a map $\breve{W} \to \breve{W}_{\text{ad}}$ denoted $x \mapsto x_{\text{ad}}$. The alcove $\breve{\mathbf{a}}$ determines an alcove $\breve{\mathbf{a}}_{\text{ad}}$ in the building of G_{ad} and we let $\breve{\mathcal{I}}_{\text{ad}}$ be the corresponding Iwahori subgroup.

Let $\mathcal{FL}_{\text{ad}}$ be the affine flag variety for G_{ad}. For $\tau \in \pi_1(G)_{\Gamma_0}$ let $\mathcal{FL}^\tau$ be the connected component of $\mathcal{FL}$ corresponding to τ and similarly for τ_{ad}. We have the following proposition which can be proved in the same way as [11, Proposition 2.2.1].

Proposition 4.3. *The map $G \to G_{ad}$ induces an isomorphism*

$$\mathcal{FL}^\tau \cong \mathcal{FL}_{ad}^{\tau_{ad}},$$

For an affine Deligne Lusztig variety $X_w(b)$ we let $X_w(b)^\tau = X_w(b) \cap \mathcal{FL}^\tau$, and similarly for G_{ad}. The previous proposition implies

Corollary 4.4. *The isomorphism $\mathcal{FL}^\tau \cong \mathcal{FL}_{ad}^{\tau_{ad}}$ induces an isomorphism*

$$X_w(b)^\tau \cong X_{w_{ad}}(b_{ad})^{\tau_{ad}}.$$ □

Remark 4.5. In fact, for any $K \subset \breve{\mathbb{S}}$ corresponding to a σ-invariant parahoric $\breve{\mathcal{K}}$, we have an isomorphism $X_{K,w}(b)^\tau \cong X_{K_{\text{ad}}, w_{\text{ad}}}(b_{\text{ad}})^{\tau_{\text{ad}}}$. Here K_{ad} is the corresponding subset of $\breve{\mathbb{S}}_{\text{ad}}$, and $X_{K,w}(b)^\tau$ and $X_{K_{\text{ad}}, w_{\text{ad}}}(b_{\text{ad}})^{\tau_{\text{ad}}}$ are defined analogously to the above.

Since $w \mapsto w_{\text{ad}}$ is compatible with the property of σ-straightness, it suffices to prove Theorem 4.1 for G_{ad}.

4.3. **Reduction to simply connected groups.** Let $\breve{\Omega}_{\rm ad} \subset \breve{W}_{\rm ad}$ be the normalizer of $\breve{\mathfrak{a}}_{\rm ad}$. We may write $w_{\rm ad}$ as $w_{\rm ad} = w'\tau$ for $w' \in \breve{W}_a$ and $\tau \in \breve{\Omega}_{\rm ad}$. Let $H_{\rm ad}$ be the inner form of $G_{\rm ad}$ with $G_{\rm ad}(\breve{F}) = H_{\rm ad}(\breve{F})$ and the Frobenius on $H_{\rm ad}$ given by $\sigma' = {\rm int}(\dot{\tau}) \circ \sigma$. Then we have an identification of buildings $B(G_{\rm ad}, \breve{F}) \cong B(H_{\rm ad}, \breve{F})$. Since $\tau \in \breve{\Omega}_{\rm ad}$, the alcove $\breve{\mathbf{a}}_{\rm ad}$ corresponds to a σ'-invariant alcove in $B(H_{\rm ad}, \breve{F})$, and its corresponding Iwahori subgroup can be identified with $\breve{\mathcal{I}}_{\rm ad}$. It is then straightforward to check that the natural identification $G_{\rm ad}(\breve{F}) = H_{\rm ad}(\breve{F})$ induces an isomorphism

$$X^{G_{\rm ad}}_{w_{\rm ad}}(b) \cong X^{H_{\rm ad}}_{w'}(b\dot{\tau}^{-1}).$$

Let H be the simply connected cover of $H_{\rm ad}$ and $\pi : H \to H_{\rm ad}$ be the projection. Since $w' \in \breve{W}_a$, $X^{H_{\rm ad}}_{w'}(b\dot{\tau}^{-1}) = \emptyset$ unless $\kappa_{H_{\rm ad}}(b\dot{\tau}^{-1}) = 0$. In the latter case, there exists $b' \in H(\breve{F})$ such that $b\dot{\tau}^{-1}$ is σ-conjugate to $\pi(b')$.

Recall that $\breve{\Omega}_{\rm ad} \cong \pi_1(G_{\rm ad})_{\Gamma_0}$. We have $X^{H_{\rm ad}}_{w'}(\pi(b')) = \sqcup_{\gamma \in \breve{\Omega}_{\rm ad}} X^{H_{\rm ad}}_{w'}(\pi(b'))^\gamma$. Since H is simply connected, $\tilde{\kappa}_{H_{\rm ad}}(\pi(b')) = 0 = \tilde{\kappa}_{H_{\rm ad}}(w')$ and hence $X^{H_{\rm ad}}_{w'}(\pi(b'))^\gamma = \emptyset$ unless $\gamma \in \breve{\Omega}^\sigma_{\rm ad}$. Moreover, in this case we an isomorphism $X^H_{\gamma w' \gamma^{-1}}(b') \cong X^{H_{\rm ad}}_{w'}(\pi(b'))^\gamma$ given by $h \mapsto \pi(h)\dot{\gamma}$.

As a summary, we have the identification $X^{G_{\rm ad}}_{w_{\rm ad}}(b) \cong \sqcup_{\gamma \in \breve{\Omega}^\sigma_{\rm ad}} X^H_{\gamma w' \gamma^{-1}}(b')$. It is easy to check that w is σ-straight if and only if $\gamma w' \gamma^{-1}$ is σ'-straight for some (or equivalently, any) $\gamma \in \breve{\Omega}^\sigma_{\rm ad}$.

Thus to prove Theorem 4.1, we only need to consider simply connected groups. Therefore, for the rest of this section, we will assume G is a simply connected group.

We have the following result on the dimension of irreducible components of affine Deligne-Lusztig varieties.

Proposition 4.6. *Let $w \in \breve{W}$ and $b \in G(\breve{F})$ such that $X_w(b) \neq \emptyset$. If w is not σ-straight, then* $\dim Y > 0$ *for any irreducible component Y of $X_w(b)$.*

Proof. We follow the strategy of [15, Theorem 6.1]. If w is not minimal length in its σ-conjugacy class, then by Theorem 1.2 (1), there exists $w' \in \breve{W}$ and $s \in \breve{W}$ a simple affine reflection such that $w \approx_\sigma w'$ and $sw'\sigma(s) < w'$. By the reduction method à la Deligne and Lusztig ([15, Proposition 4.2]) we have:

- $X_{w'}(b)$ is universally homeomorphic to $X_w(b)$.
- For any irreducible component Y_1 of $X_{w'}(b)$, there exists an irreducible component Y_2 of $X_{sw'}(b)$ or $X_{sw'\sigma(s)}(b)$, such that $\dim Y_1 > \dim Y_2$.

Therefore $\dim Y > 0$ for any irreducible component Y of $X_w(b)$.

Now suppose w is of minimal length in its σ-conjugacy class. By [15, Theorem 3.5], we have $X_w(b) \neq \emptyset$ if and only if $\bar{\nu}_b = \overline{\nu}_w$ and $\kappa(b) = \kappa(w)$. By [15, Theorem 4.8], every irreducible component of $X_w(b)$ is of dimension $\ell(w) - \langle \overline{\nu}_b, 2\rho \rangle$. In particular, the dimension is 0 if and only if $\ell(w) = \langle \overline{\nu}_b, 2\rho \rangle$, i.e. w is σ-straight. □

4.4. Following [7, §9.6], we view $X_w(b)$ as the intersection in $\mathcal{FL} \times \mathcal{FL}$ of the graph of the map $b\sigma$ and the LG-orbit $O(w)$ of the point $(w, 1)$. We write $p_1, p_2 : \mathcal{FL} \times \mathcal{FL} \to \mathcal{FL}$ for the projection maps. Note that $O(w)$ is an O_w-bundle under the restriction of the second projection, and an $O_{w^{-1}}$-bundle under the restriction of the first projection.

The Iwahori-Weyl group $\breve{W}$ acts on ${\rm Pic}\,\mathcal{FL}$. In the equal characteristic setting, this follows from the identification of ${\rm Pic}\,\mathcal{FL}$ with the set of fundamental affine weights for the Kac-Moody algebra. In mixed characteristic, there is no such identification, however one may still define such an action using the Cartan matrix $(A_{ij})_{i,j \in \breve{\mathbb{S}}}$ of the Iwahori-Weyl group. Explicitly, for $i \in \breve{\mathbb{S}}$, we define

$$s_i \mathscr{L}(\epsilon_j) = \begin{cases} \mathscr{L}(\epsilon_i) - \sum_{k \in \breve{\mathbb{S}}} A_{ik} \mathscr{L}(\epsilon_k), & \text{if } j = i; \\ \mathscr{L}(\epsilon_j), & \text{if } j \neq i. \end{cases}$$

We then extend it linearly to an action on ${\rm Pic}\,\mathcal{FL}$. From this description, one checks that the translation element t^μ acts on ${\rm Pic}\,\mathcal{FL}$ via

$$t^\mu : \mathscr{L} \longmapsto \mathscr{L} + m(\mathscr{L}) \sum_{\iota \in \breve{\mathbb{S}}} \langle \mu, \overline{\alpha}_\iota \rangle \mathscr{L}(\epsilon_\iota) \qquad (4.1)$$

where $\overline{\alpha}_\iota$ denotes the finite relative root associated to the affine root α_ι and for $\mathscr{L} = \otimes_{\iota\in\breve{\mathbb{S}}}\mathscr{L}(\epsilon_\iota)^{\lambda_\iota}$, $m(\mathscr{L}) = \sum_{\iota\in\breve{\mathbb{S}}}\lambda_\iota$.

We have the following result.

Proposition 4.7. *Let $\mathscr{L}$ be a line bundle on $\mathcal{FL}$ and $w \in \breve{W}$. Then the restriction to $O(w)$ of the line bundle $p_1^*(w\mathscr{L}) \otimes p_2^*\mathscr{L}^{-1}$ on $\mathcal{FL} \times \mathcal{FL}$ is trivial.*

Proof. Let $s_{j_1} \cdots s_{j_n}$ be a reduced word decomposition for w. We use the interpretation of $O(w)$ as the product $O(s_{j_1})\times_{\mathcal{FL}} \cdots \times_{\mathcal{FL}} O(s_{j_n})$. Here the Cartesian product $O(s_{j_k}) \times_{\mathcal{FL}} O(s_{j_{k+1}})$ is taken with respect to the maps $p_1 : O(s_{j_k}) \to \mathcal{FL}$ and $p_2 : O(s_{j_{k+1}}) \to \mathcal{FL}$.

Consider the line bundle

$$(\mathscr{L}_1, -\mathscr{L}_2)\times_{\mathcal{FL}} \cdots \times_{\mathcal{FL}}(\mathscr{L}_n, -\mathscr{L}_{n+1})$$

on $O(s_{j_1})\times_{\mathcal{FL}} \cdots \times_{\mathcal{FL}} O(s_{j_n})$. Here $(\mathscr{L}_k, -\mathscr{L}_{k+1})$ denotes the line bundle $p_1^*\mathscr{L}_k \otimes p_2^*\mathscr{L}_{k+1}^{-1}$ on $O(s_{j_k})$. This corresponds to the line bundle $(\mathscr{L}_1, -\mathscr{L}_{n+1}) = p_1^*\mathscr{L}_1 \otimes p_2^*\mathscr{L}_{n+1}^{-1}$ on $O(w)$. Setting $\mathscr{L}_k = s_{j_k}\mathscr{L}_{k+1}$ and $\mathscr{L}_{n+1} = \mathscr{L}$ we obtain the line bundle $p_1^*(w\mathscr{L}) \otimes p_2^*\mathscr{L}^{-1}$ on $O(w)$. Thus it suffices to consider the case where $w = s_i$ is a simple reflection.

Since $O(s_i) \to \mathcal{FL}$ is a fibration with fiber $\mathbb{A}^{1,p^{-\infty}}$, we have an isomorphism $p_2^* : \operatorname{Pic}\mathcal{FL} \cong \operatorname{Pic} O(s_i)$. Let $O(s_i)^{s_j}$ be the pullback of $p_2 : O(s_i) \to \mathcal{FL}$ along $S_{s_j} \to \mathcal{FL}$. Then since a line bundle on $\mathcal{FL}$ is determined by its restrictions to the S_{s_j}, it suffices to check that for all $j \in \breve{\mathbb{S}}$, the restrictions of $p_1^*s_i\mathscr{L}$ and $p_2^*\mathscr{L}$ to $O(s_i)^{s_j}$ are isomorphic. When s_j and s_i generate a finite subgroup of $\breve{W}$, this follows from [7, §9.6]. Indeed in this case all computations take place within the perfection of $\overline{\breve{\mathcal{K}}}^{red}/\overline{B} \times \overline{\breve{\mathcal{K}}}^{red}/\overline{B} \subset \mathcal{FL} \times \mathcal{FL}$. Here $\overline{\breve{\mathcal{K}}}^{red}$ is the reductive quotient of the special fiber of the parahoric corresponding to $\{s_i, s_j\}$ and $\overline{B}$ is the image of $\breve{\mathcal{I}}$, so that $\overline{\breve{\mathcal{K}}}^{red}/\overline{B}$ is a finite type flag variety.

Then we consider the case where $\{s_i, s_j\}$ generate an infinite subgroup of $\breve{W}$. In this case the affine root system generated by s_i, s_j is of type A_1 or C-BC_1 (here we use the notation of Tits' table [44, §4]). We only do the calculation for type A_1; the case of type C-BC_1 is completely analogous.

We may therefore reduce to the case $G = SL_2$. Let $\breve{\mathcal{I}}, \breve{\mathcal{K}}_i, \breve{\mathcal{K}}_j$ be the subgroups of $SL_2(\breve{F})$ consisting of matrices of the form

$$\breve{\mathcal{I}} = \begin{pmatrix} * & * \\ \pi * & * \end{pmatrix}, \quad \breve{\mathcal{K}}_j = \begin{pmatrix} * & * \\ * & * \end{pmatrix}, \quad \breve{\mathcal{K}}_i = \begin{pmatrix} * & \pi^{-1} * \\ \pi * & * \end{pmatrix}$$

where $*$ denotes an element of $\mathcal{O}_{\breve{F}}$. All calculations take place within the product of Schubert varieties $S_{s_j s_i} \times S_{s_j}$. We have an isomorphism $S_{s_j s_i} \cong L^+\breve{\mathcal{K}}_j \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_i/L^+\breve{\mathcal{I}}$ and the projection onto $L^+\breve{\mathcal{K}}_j/L^+\breve{\mathcal{I}}$ presents $L^+\breve{\mathcal{K}}_j \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_i/L^+\breve{\mathcal{I}}$ as a $\mathbb{P}^{1,p^{-\infty}}$-bundle over $\mathbb{P}^{1,p^{-\infty}}$. We prove that

(a) $L^+\breve{\mathcal{K}}_j \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_i/L^+\breve{\mathcal{I}}$ is isomorphic to the perfection of the Hirzebruch surface F_2. The section $L^+\breve{\mathcal{K}}_j/L^+\breve{\mathcal{I}} \to L^+\breve{\mathcal{K}}_j \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_i/L^+\breve{\mathcal{I}}$ is the directrix.

We verify (a) by direct computation using coordinates. Indeed we identify $L^+\breve{\mathcal{K}}_j/L^+\breve{\mathcal{I}}$ with $\mathbb{P}^{1,p^{-\infty}}$ by using the coordinates:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto [a \bmod \pi : c \bmod \pi].$$

We consider $L^+\breve{\mathcal{K}}_j \times^{L^+\breve{\mathcal{I}}} L^+\breve{\mathcal{K}}_i/L^+\breve{\mathcal{I}} \subset LG/L^+\breve{\mathcal{I}}$. For a matrix $B \in SL_2$, we let x, y denote the coordinates $B_{1,2}$ and $B_{2,2}$ and we consider these as morphisms $x, y : LG \to L\mathbb{A}^1$. One checks that over the open affine subset $c \neq 0$, the coordinates

$$\frac{a}{c} \bmod \pi \times [\pi y \bmod \pi, \frac{a}{c}y - x \bmod \pi]$$

induce a well-defined trivialization of the above $\mathbb{P}^{1,p^{-\infty}}$-bundle. Similarly over $a \neq 0$, we have a trivialization given by

$$\frac{c}{a} \bmod \pi \times [\pi x \bmod \pi, \frac{c}{a}x - y \bmod \pi].$$

The gluing isomorphism over the fiber of a point $[a : c]$ is given by $[s : t] \mapsto [s : \frac{a^2}{c^2} t]$, hence this is the perfection of the Hirzebruch surface F_2. The second part of the claim follows easily from the description using these coordinates.

Now (a) is proved.

As before, we write $\mathscr{L}(\epsilon_i)$ (resp. $\mathscr{L}(\epsilon_j)$) for the line bundle which has degree 1 (resp. 0) on S_{s_i} and degree 0 (resp. 1) on S_{s_j}.

We first check the case $\mathscr{L} = \mathscr{L}(\epsilon_i)$. Then $p_2^* \mathscr{L}|_{O(s_i)^{s_j}}$ is trivial. It remains to show that $p_1^* s_i \mathscr{L}(\epsilon_i)$ is trivial on $O(s_i)^{s_j}$. We have the two divisors S_{s_i}, S_{s_j} inside the Hirzebruch surface $S_{s_j s_i}$ which are generators for the divisor class group. The intersection pairing for this basis has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}.$$

We have $s_i \mathscr{L}(\epsilon_i) = -\mathscr{L}(\epsilon_i) + 2\mathscr{L}(\epsilon_j)$. Since $\deg s_i \mathscr{L}(\epsilon_i)|_{S_{s_i}} = -1$ and $\deg s_i \mathscr{L}(\epsilon_i)|_{S_{s_j}} = 2$, we have $s_i \mathscr{L}(\epsilon_i) = \mathcal{O}_{S_{s_j s_i}}(-S_{s_j})$. Thus $s_i \mathscr{L}(\epsilon_i)$ has a non-vanishing section on $S_{s_j s_i} \smallsetminus S_{s_j}$. Since $O(s_i)^{s_j}$ does not intersect $S_{s_j} \times S_{s_j}$, we have $p_1^* s_i \mathscr{L}(\epsilon_i)$ is trivial on $O(s_i)^{s_j}$.

We then check the case $\mathscr{L} = \mathscr{L}(\epsilon_j)$, then $\mathscr{L}(\epsilon_j)$ is the pullback of $\mathscr{L}(\epsilon_j)$ along $S_{s_j s_i} \to S_{s_j}$. We have $s_i \mathscr{L}(\epsilon_j) = \mathscr{L}(\epsilon_j)$ and both $p_1^* \mathscr{L}(\epsilon_j)_{O(s_i)^{s_j}}$ and $p_2^* \mathscr{L}(\epsilon_j)_{O(s_i)^{s_j}}$ have sections vanishing on the divisor $O_{s_i} \times 1 \subset O(s_i)^{s_j}$. Hence these line bundles are isomorphic.

Since $\mathscr{L}(\epsilon_i)$ and $\mathscr{L}(\epsilon_j)$ form a basis for $\operatorname{Pic} \mathcal{FL}$, this proves the result.

□

Remark 4.8. 1) To be precise, in order to carry out the calculation using intersection theory in the previous Lemma, one must work with a deperfection of the Schubert varieties $S_{s_j s_i}$. It can be checked that the deperfections corresponding to the choice of coordinates in a) are compatible with the deperfections of S_{s_i} given by the isomorphism $S_{s_i} \cong \mathbb{P}^{1, p^{-\infty}}$ in Remark 3.5. In particular the line bundles $\mathscr{L}(\epsilon_i)$ and $\mathscr{L}(\epsilon_j)$ come from pullback from the deperfection F_2 of $S_{s_j s_i}$. One may then perform the calculations over F_2 to obtain the result.

2) As mentioned, the calculations for type C-BC_1 case are completely analogous. We may take the group to be the quasi-split special unitary group in dimension 3 associated to a split Hermitian form for a ramified quadratic extension $\breve{F}'/\breve{F}$. See [44, §1.5]. In this case the Schubert varieties $S_{s_j s_i}$ are the perfections of Hirzebruch surfaces F_4 or F_1 depending on whether the s_i corresponds to an affine root whose finite part is a long or short relative coroot. The corresponding intersection matrices are

$$\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

The same argument then applies.

Proposition 4.9. *Let $w \in \breve{W}$ and $b \in G(\breve{F})$ such that $X_w(b) \neq \emptyset$. Let Y be an irreducible component of $X_w(b)$. Then Y is the perfection of a quasi-affine variety.*

Proof. By §4.2 and §4.3, it suffices to consider the case where G is simply connected. We will construct a line bundle on $X_w(b)$ which is both ample and trivial. By Theorem 3.1, we have

$$\operatorname{Pic}(\mathcal{FL}) = \bigoplus_{\iota \in \breve{\mathbb{S}}} \mathbb{Z}[\frac{1}{p}].$$

The Frobenius σ induces an action on $\operatorname{Pic} \mathcal{FL}$ which can be identified with the action of σ on $\breve{\mathbb{S}}$ multiplied by a factor of q. Furthermore, since $\operatorname{Pic}(\mathcal{FL})$ is discrete, the induced action of LG on $\operatorname{Pic}(\mathcal{FL})$ is trivial.

For sufficiently divisible n, we have

$$(\sigma w^{-1})^n = t^{n\sigma(\nu_{w^{-1}})} \sigma^n$$

in the group $\breve{W} \ltimes \langle \sigma \rangle$. In particular, upon increasing n we may assume σ^n acts trivially on $\breve{W}$ and hence $(\sigma w^{-1})^n$ acts on $\operatorname{Pic}(\mathcal{FL})$ as $t^{n\sigma(\nu_{w^{-1}})} \circ q^n$. By the description of this action in (4.1), we see that $(\sigma w^{-1})^n$, and hence σw^{-1}, has no eigenvalue 1 on $\operatorname{Pic}(\mathcal{FL})_{\mathbb{Q}}$. Therefore $\sigma w^{-1} - 1$ is invertible on $\operatorname{Pic}(\mathcal{FL})_{\mathbb{Q}}$.

In particular, there exists $\mathscr{L} \in \mathrm{Pic}(\mathcal{FL})$ such that $(\sigma w^{-1})(\mathscr{L}) - \mathscr{L}$ is dominant and regular (i.e. it is of the form $\oplus \lambda_\iota \mathscr{L}(\epsilon_\iota)$ with $\lambda_\iota > 0$ for all ι). The restriction to $X_w(b)$ of the corresponding line bundle $p_1^* \mathscr{L} \otimes p_2^*(w^{-1}\mathscr{L})^{-1}$ is therefore ample. The statement then follows from the Lemma 4.7. Restricting this line bundle to Y shows that Y is the perfection of a quasi-affine variety. □

4.5. **Proof of Theorem 4.1.** Let x be a minimal length element in $\{w \in C; Y \cap X_w(b) \neq \emptyset\}$. Then Y contains an irreducible component Y_1 of $X_x(b)$.

Let $\overline{Y}_1$ be the closure of Y_1 in $X_C(b)$. By minimality of x, we have $\overline{Y}_1 \cap X_{<x} = \emptyset$, hence $\overline{Y}_1 = Y_1$ and Y_1 is projective. By Proposition 4.9, Y_1 is also quasi-affine. Thus $\dim Y = 0$. It then follows from proposition 4.6 that x is a σ-straight element.

5. Structure of the σ-centralizer

5.1. Let $b \in G(\breve{F})$ and $J_b = \{g \in G(\breve{F}); g^{-1} b \sigma(g) = b\}$ be the σ-centralizer group of b. Then J_b acts by left multiplication on $X_w(b)$ for any $w \in \breve{W}$. It is proved in [15, Theorem 3.5 & Theorem 4.8] that

Theorem 5.1. *Let $w \in \breve{W}$ be a σ-straight element and $b \in G(\breve{F})$. Then $X_w(b) \neq \emptyset$ if and only if $\bar{\nu}_b = \bar{\nu}_w$ and $\kappa(b) = \kappa(w)$. In this case, J_b acts transitively on $X_w(b)$.*

5.2. In this section we study the structure of the σ-centralizer group J_b.

We first describe the standard Levi subgroup associated to a given σ-conjugacy class. It is based on the notion of P-alcove elements introduced by Görtz, Haines, Kottwitz and Reuman in [10] for split groups and generalized in [11] for quasi-split groups. We use here the reformulation given by Nie in [36]. It is stated for unramified (and hence quasi-split groups), but works for arbitrary reductive groups as well.

Let Φ be the set of (relative) roots of G over $\breve{F}$ with respect to $\breve{S}$ and Φ_a the set of affine roots. The roots in Φ determine hyperplanes in V and the relative Weyl group $\breve{W}_0$ can be identified with the subgroup of $\mathrm{Aff}(V)$ generated by the reflections through these hyperplanes. For $a \in \Phi$, we denote by $U_a \subset G$ the corresponding root subgroup.

For any $v \in V$, we set $\Phi_{v,0} = \{a \in \Phi; \langle a, v\rangle = 0\}$ and $\Phi_{v,+} = \{a \in \Phi; \langle a, v\rangle > 0\}$. Let $M_v \subset G(\breve{F})$ be the Levi subgroup generated by $\breve{T}$ and $U_a(\breve{F})$ for $a \in \Phi_{v,0}$ and $N_v \subset G(\breve{F})$ be the unipotent subgroup generated by $U_a(\breve{F})$ for $a \in \Phi_{v,+}$. Set $P_v = M_v N_v$. Then P_v is a semistandard parabolic subgroup and M_v is a Levi subgroup of P_v. Here semistandard means that the parabolic subgroup contains $\breve{T}$.

Now we give several definitions. Let $w \in \breve{W}$.

- We say that w is *fundamental* if the σ-conjugation action of $\breve{\mathcal{I}}$ on $\breve{\mathcal{I}} \dot{w} \breve{\mathcal{I}}$ is transitive.
- We say that w is a *(v, σ)-alcove element* if
  (1) the linear part of $w \circ \sigma \in \mathrm{Aff}(V)$ fixes v;
  (2) $N_v \cap \dot{w} \breve{\mathcal{I}} \dot{w}^{-1} \subset N_v \cap \breve{\mathcal{I}}$.[2]
- We say that w is *(v, σ)-fundamental* if w is a (v, σ)-alcove element and $\dot{w}\sigma(\breve{\mathcal{I}} \cap M_v)\dot{w}^{-1} = \breve{\mathcal{I}} \cap M_v$.

As explained in [11, §3.3], the condition (1) implies that $\mathrm{Ad}(\dot{w}) \circ \sigma$ stabilizes M_v. Thus if w is (v, σ)-fundamental, then $\mathrm{Ad}(\dot{w}) \circ \sigma$ gives a group automorphism on $\breve{W}_v$ that preserves the set of simple reflections of $\breve{W}_v$. Here $\breve{W}_v = \{w \in \breve{W}; w(v) = v\}$ is the Iwahori-Weyl group of M_v. It is worth mentioning that the set of simple affine reflections $\{s_\iota; \iota \in \breve{\mathbb{S}}_v\}$ for $\breve{W}_v$ is determined by the Iwahori subgroup $\breve{\mathcal{I}} \cap M_v$ of M_v, and in general $\{s_\iota; \iota \in \breve{\mathbb{S}}_v\} \not\subset \{s_j; j \in \breve{\mathbb{S}}\}$.

We have the following result.

Theorem 5.2. *Let $w \in \breve{W}$. The following conditions are equivalent:*
(1) The element w is σ-straight;
(2) The element w is fundamental;
(3) The element w is a (v, σ)-fundamental element for some $v \in V$;

[2]The second condition stated in [36] is $w\breve{a} \geqslant_a \breve{a}$ for $a \in \Phi_{v,+}$. The equivalence of that condition with our condition (2) above is explained in [11, §4.1].

(4) The element w is a (ν_w, σ)-fundamental element.

The result is proved by Nie in [36] for unramified groups and the general case follows from the same argument. The equivalence among the conditions (1), (2) and (3) are stated in [36, Theorem 1.3] while the terminology we use here is slightly different. The Condition (4) automatically implies the condition (3). And the implication (1) $\Rightarrow$ (4) is obtained in [36, Page 494].

5.3. Let $[b] \in B(G)$. By Theorem 1.1, there exists a σ-straight element $w \in \breve{W}$ with $\dot{w} \in [b]$. We may take $b = \dot{w}$. Set $M = M_{\nu_w}$ and $\breve{\mathcal{I}}_M = \breve{\mathcal{I}} \cap M(\breve{F})$. By Theorem 5.2, w is (ν_w, σ)-fundamental. Set $\tau = \mathrm{Ad}(\dot{w}) \circ \sigma$. Then τ induces a group automorphism on M and a length-preserving action on $\breve{W}_{\nu_w}$. In particular, $\breve{\mathcal{I}}_M^\tau$ is an Iwahori subgroup of $M(\breve{F})^\tau$.

By [26, Remark 6.5] and [27, §4.3], $J_b = M(\breve{F})^\tau$. By [43, Lemma 1.6], the Iwahori-Weyl group (over F) of J_b is $\breve{W}_{\nu_w}^\tau$. For any $x \in \breve{W}_{\nu_w}^\tau$, we denote by $n_x \in G(\breve{F})$ a τ-stable representative of x.

We have $\breve{W}_{\nu_w} = \breve{W}_{a,\nu_w} \rtimes \breve{\Omega}_{\nu_w}$, where $\breve{W}_{a,\nu_w}$ is the affine Weyl group associated to M and $\breve{\Omega}_{\nu_w} \subset \breve{W}_{\nu_w}$ is the subgroup of length-zero elements. Since τ is a length-preserving group automorphism on $\breve{W}_{a,\nu_w}$, we have $\tau(\breve{\Omega}_{\nu_w}) = \breve{\Omega}_{\nu_w}$. In particular,

(a) $\breve{W}_{\nu_w}^\tau = \breve{W}_{a,\nu_w}^\tau \rtimes \breve{\Omega}_{\nu_w}^\tau$.

The following result is proved in [32, Theorem A.8].

Theorem 5.3. *Let $\tilde{W}$ be a Coxeter group and $\tilde{\mathbb{S}}$ be the set of simple reflections. Let $\varphi : \tilde{W} \to \tilde{W}$ be a group automorphism such that $\varphi(\tilde{\mathbb{S}}) = \tilde{\mathbb{S}}$. Let $\mathbb{S}$ be the set of φ-orbits I on $\tilde{\mathbb{S}}$ such that W_I is finite. For any $I \in \mathbb{S}$, let w_0^I be the longest element of the finite Coxeter group W_I. Then $\tilde{W}^\varphi$ is a Coxeter group generated by simple reflections w_0^I for $I \in \mathbb{S}$.*

Remark 5.4. The Braid relations between the simple reflections w_0^I are given in [32, §A.7]. In this paper, we only need the fact that $\tilde{W}^\varphi$ is generated by the elements w_0^I.

Let $\breve{\mathbb{S}}_{\nu_w}(\tau)$ be the set of τ-orbits J on $\breve{\mathbb{S}}_{\nu_w}$ with W_J finite. By Theorem 5.3, we have that

(b) The group $\breve{W}_{a,\nu_w}^\tau$ is an affine Weyl group generated by simple reflections w_0^J with $J \in \breve{\mathbb{S}}_{\nu_w}(\tau)$.

5.4. For any affine root α of M, the corresponding affine root subgroup $\mathcal{U}_\alpha$[3] is a group scheme over $\mathcal{O}_{\breve{F}}$. By [4, §4.3.2, §4.3.5,§4.3.7], $\mathcal{U}_\alpha(\mathcal{O}_{\breve{F}})$ is a finite free $\mathcal{O}_{\breve{F}}$-module. Similarly, for $\epsilon > 0$ we write $\mathcal{U}_{\alpha+\epsilon}(\mathcal{O}_{\breve{F}})$ for the subgroup of $\mathcal{U}_\alpha(\mathcal{O}_{\breve{F}})$ corresponding to the affine function $\alpha + \epsilon$ as in [4, §4.3.2, §4.3.5, §4.3.7]. We let $\mathcal{U}_{\alpha+}(\mathcal{O}_{\breve{F}})$ be the union of subgroups $\mathcal{U}_{\alpha+\epsilon}(\mathcal{O}_{\breve{F}})$ for all $\epsilon > 0$ (note that $\mathcal{U}_{\alpha+}(\mathcal{O}_{\breve{F}}) = \mathcal{U}_{\alpha+\epsilon}(\mathcal{O}_{\breve{F}})$ for sufficiently small ϵ). As the notation suggests, $\mathcal{U}_{\alpha+\epsilon}(\mathcal{O}_{\breve{F}})$ and $\mathcal{U}_{\alpha+}(\mathcal{O}_{\breve{F}})$ arise as the $\mathcal{O}_{\breve{F}}$-points of group schemes $\mathcal{U}_{\alpha+\epsilon}$ and $\mathcal{U}_{\alpha+}$ over $\mathcal{O}_{\breve{F}}$. We have that $\mathcal{U}_\alpha(\mathcal{O}_{\breve{F}})/\mathcal{U}_{\alpha+}(\mathcal{O}_{\breve{F}})$ is a 1-dimensional vector space over $\overline{\mathbb{F}}_q$.

Let $J \in \breve{\mathbb{S}}_{\nu_w}(\tau)$. Now we construct a certain element u_{-J} in J_b.

We regard J as a subset of the set of vertices in the affine Dynkin diagram for $\breve{\mathbb{S}}_{\nu_w}$. We first consider the case where J does not contain any adjacent vertices.

We have $J = \{\alpha, \tau(\alpha), \dots, \tau^{n-1}(\alpha)\}$ for some $n \in \mathbb{N}$ and α a simple affine root for M_{ν_w}. Then τ^n induces σ^n-linear automorphisms of $\mathcal{U}_{-\alpha}(\mathcal{O}_{\breve{F}})$ and $\mathcal{U}_{-\alpha+}(\mathcal{O}_{\breve{F}})$, and induces a σ^n-linear automorphism of the quotient $\mathcal{U}_{-\alpha}(\mathcal{O}_{\breve{F}})/\mathcal{U}_{-\alpha+}(\mathcal{O}_{\breve{F}})$. Let F_n be the σ^n-fixed field of $\breve{F}$. Then by [4, 5.1.17], $\mathcal{U}_{-\alpha}$ (resp. $\mathcal{U}_{-\alpha+}$) arises from a group scheme $\mathcal{U}_{-\alpha}^{\tau^n}$ (resp. $\mathcal{U}_{-\alpha+}^{\tau^n}$) over $\mathcal{O}_{F_n}$ and moreover these are group schemes associated to a finite free $\mathcal{O}_{F_n}$ module. In particular $\mathcal{U}_{-\alpha}^{\tau^n}$ and $\mathcal{U}_{-\alpha}^{\tau^n}$ are connected.

By [4, 5.1.18], the projection map $\mathcal{U}_{-\alpha}^{\tau^n}(\mathcal{O}_{F_n}) \to [\mathcal{U}_{-\alpha}(\mathcal{O}_{\breve{F}})/\mathcal{U}_{-\alpha+}(\mathcal{O}_{\breve{F}})]^{\tau^n}$ is surjective. We choose $u \in \mathcal{U}_{-\alpha}^{\tau^n}(\mathcal{O}_{F_n})$ that maps to a non-zero element in $[\mathcal{U}_{-\alpha}(\mathcal{O}_{\breve{F}})/\mathcal{U}_{-\alpha+}(\mathcal{O}_{\breve{F}})]^{\tau^n}$. Define

$$u_{-J} = u\tau(u) \dots \tau^{n-1}(u).$$

Since J does not contain adjacent vertices, $\tau^i(u)$ and $\tau^j(u)$ commute for all i and j. Hence u_{-J} is fixed by τ, i.e. $u_{-J} \in J_{\dot{w}}$.

Next we consider the case where J contains some adjacent vertices of $\breve{\mathbb{S}}_{\nu_w}$.

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[3]In the notation of [4], these are the groups $\mathfrak{U}_{a,k}$, where a is linear part of α and $\alpha = a + k$.

We pick $\alpha \in J$. Let m be the smallest positive integer such that α and $\tau^m(\alpha)$ are contained in the same connected component of the affine Dynkin diagram of $\breve{\mathbb{S}}_{\nu_w}$. We denote this connected component by C. Then τ^m induces a diagram automorphism on C and $\{\tau^{ml}(\alpha); l \in \mathbb{Z}\}$ contains some adjacent vertices of C. By the classification of the affine Dynkin diagrams, this is possible only when C is of type $\tilde{A}, \tilde{C}$ or $\tilde{D}$. Moreover, if C is type $\tilde{C}_n$ or $\tilde{D}_n$, then n is odd and $\{\tau^{ml}(\alpha); l \in \mathbb{Z}\} = \{\frac{n-1}{2}, \frac{n+1}{2}\}$. If C is of type $\tilde{A}$, we put the vertices of C evenly on a circle in counterclockwise order. A diagram automorphism of C is either a rotation or a reflection of the circle. However, if τ^m is a rotation, then as $\{\tau^{ml}(\alpha); l \in \mathbb{Z}\}$ contains some adjacent vertices of C, $\{\tau^{ml}(\alpha); l \in \mathbb{Z}\} = C$. This is not possible since W_J is finite. Thus τ is a reflection and hence $\{\tau^{ml}(\alpha); l \in \mathbb{Z}\} = \{\alpha, \alpha'\}$, where α' is a vertex in C that is adjacent to α.

As a summary, in all the cases above, $\{\tau^{ml}(\alpha); l \in \mathbb{Z}\} = \{\alpha, \alpha'\}$, where α' is a vertex of C that is adjacent to α and the vertices α, α' are connected by a single edge in the affine Dynkin diagram of C. Hence $J = \{\alpha, \tau(\alpha), \dots, \tau^{m-1}(\alpha), \alpha', \tau(\alpha'), \dots, \tau^{m-1}(\alpha')\}$. Then $\alpha + \tau^m(\alpha)$ is an affine root of M.

By [4, 5.1.18], the projection map $\mathcal{U}^{\tau^m}_{-\alpha-\tau^m(\alpha)}(\mathcal{O}_{F_m}) \to [\mathcal{U}_{-\alpha-\tau^m(\alpha)}(\mathcal{O}_{\breve{F}})/\mathcal{U}_{-(\alpha+\tau^m(\alpha))+}(\mathcal{O}_{\breve{F}})]^{\tau^m}$ is surjective. We then choose $u \in \mathcal{U}^{\tau^m}_{-\alpha-\tau^m(\alpha)}(\mathcal{O}_{F_m})$ that maps to a non-zero element in $[\mathcal{U}_{-\alpha-\tau^m(\alpha)}(\mathcal{O}_{\breve{F}})/\mathcal{U}_{-(\alpha+\tau^m(\alpha))+}(\mathcal{O}_{\breve{F}})]^{\tau^m}$. Define

$$u_{-J} = u\tau(u)\cdots\tau^{m-1}(u).$$

Note that for $0 \leqslant i < j \leqslant m-1$, $\tau^i(\alpha)$ and $\tau^j(\alpha)$ are in different connected components of the affine Dynkin diagram of $\breve{\mathbb{S}}_{\nu_w}$. Hence $\tau^i(u)$ and $\tau^j(u)$ commute for $0 \leqslant i < j \leqslant m-1$. Therefore u_{-J} is fixed by τ, i.e. $u_{-J} \in J_{\dot{w}}$.

Now we describe the structure of $J_{\dot{w}}$.

Theorem 5.5. *Let w be a σ-straight element in $\breve{W}$ and $\tau = \mathrm{Ad}(\dot{w}) \circ \sigma$ be the corresponding group automorphism on $M_{\nu_w}(\breve{F})$. Then the σ-centralizer $J_{\dot{w}}$ is generated by $(\breve{\mathcal{I}} \cap M_{\nu_w}(\breve{F}))^\tau$, u_{-J} for $J \in \breve{\mathbb{S}}_{\nu_w}(\tau)$ and n_x for $x \in \breve{\Omega}^\tau_{\nu_w}$.*

Proof. Set $M = M_{\nu_w}$. We first obtain the Bruhat decomposition of $J_{\dot{w}}$.

(a) $J_{\dot{w}} = \sqcup_{x \in \breve{W}^\tau_M} \breve{\mathcal{I}}^\tau_M n_x \breve{\mathcal{I}}^\tau_M$.

Recall that $M(\breve{F}) = \sqcup_{x \in \breve{W}_M} \breve{\mathcal{I}}_M \dot{x} \breve{\mathcal{I}}_M$. For any $x \in \breve{W}_M$, $\tau(\breve{\mathcal{I}}_M \dot{x} \breve{\mathcal{I}}_M) = \breve{\mathcal{I}}_M \tau(\dot{x}) \breve{\mathcal{I}}_M$. Thus

$$J_{\dot{w}} = M(\breve{F})^\tau = \sqcup_{x \in \breve{W}^\tau_M} (\breve{\mathcal{I}}_M \dot{x} \breve{\mathcal{I}}_M)^\tau.$$

Let $x \in \breve{W}^\tau_M$ and $j \in (\breve{\mathcal{I}}_M n_x \breve{\mathcal{I}}_M)^\tau$. We write j as $j = h_1 n_x h_2$ for some $h_1, h_2 \in \breve{\mathcal{I}}_M$.

Let $\breve{\mathcal{I}}_x = n_x^{-1} \breve{\mathcal{I}}_M n_x \cap \breve{\mathcal{I}}_M$. Then h_2 is determined up to multiplication by an element of $\breve{\mathcal{I}}_x$, hence its image in $\breve{\mathcal{I}}_x \backslash \breve{\mathcal{I}}_M$ is well defined. Thus $\tau(h_2)h_2^{-1} \in \breve{\mathcal{I}}_x$. Note that $\breve{\mathcal{I}}_x$ is the intersection of $\ker \tilde{\kappa}_M$ and the stabilizer of a bounded subset of the building of M over $\breve{F}$. Thus by [14, Prop. 3 and Remark 4], $\breve{\mathcal{I}}_x$ arises as the $\mathcal{O}_{\breve{F}}$-points of a smooth connected group scheme over $\mathcal{O}_{\breve{F}}$ and τ induces a Lang isogeny of $\breve{\mathcal{I}}_x$. By Lang's theorem, upon modifying h_2 by an element of $\breve{\mathcal{I}}_x$, we may assume h_2 is fixed by τ. It follows that h_1 is also fixed by τ. Thus $(\breve{\mathcal{I}}_M n_x \breve{\mathcal{I}}_M)^\tau = \breve{\mathcal{I}}^\tau_M n_x \breve{\mathcal{I}}^\tau_M$. (a) is proved.

By (a), $J_{\dot{w}}$ is generated by $\breve{\mathcal{I}}^\tau_M$ and n_x for $x \in \breve{W}^\tau_M$. By our construction,

$$u_{-J} \in \sqcup_{x \in W_J} \breve{\mathcal{I}}_M \dot{x} \breve{\mathcal{I}}_M \cap M(\breve{F})^\tau = \sqcup_{x \in W^\tau_J} \breve{\mathcal{I}}^\tau_M n_x \breve{\mathcal{I}}^\tau_M.$$

Since $\breve{W}^\tau_J = \{1, w^J_0\}$ and $u_{-J} \notin \breve{\mathcal{I}}_M$, we have that $u_{-J} \in \breve{\mathcal{I}}^\tau_M n_{w^J_0} \breve{\mathcal{I}}^\tau_M$ Therefore $n_{w^J_0}$ is contained in the subgroup of $J_{\dot{w}}$ generated by $\breve{\mathcal{I}}^\tau_M$ and u_{-J}. The statement then follows from §5.3(a) and §5.3(b). □

6. Connected components for the basic σ-conjugacy class

The first obstruction to connecting points in $X(\{\mu\}, b)$ is given by the Kottwitz homomorphism which parametrizes the connected components of $\mathcal{FL}$.

Let $b \in G(\breve{F})$ such that $[b] \in B(G, \{\mu\})$. Then $\kappa([b]) = \mu^\natural$ in $\pi_1(G)_\Gamma$. Hence there exists $c_{b,\{\mu\}} \in \pi_1(G)_{\Gamma_0}$ such that $c_{b,\{\mu\}} - \sigma(c_{b,\{\mu\}}) = \tilde{\kappa}(\mu) - \tilde{\kappa}(b)$, where $\tilde{\kappa}(\mu)$ is the class of $\{\mu\}$ in $\pi_1(G)_{\Gamma_0}$. Note that the choice of $c_{b,\{\mu\}}$ is determined up to addition by an element of $\pi_1(G)^\sigma_{\Gamma_0}$.

Lemma 6.1. *Let $b \in G(\breve{F})$ such that $[b] \in B(G, \{\mu\})$. Then the image of the map $\tilde{\kappa}$: $X(\{\mu\}, b) \to \pi_1(G)_{\Gamma_0}$ equals $c_{b,\{\mu\}} + \pi_1(G)^{\sigma}_{\Gamma_0}$.*

Remark 6.2. Recall from Theorem 2.3 that $X(\{\mu\}, b) \neq \emptyset$ if and only if $[b] \in B(G, \{\mu\})$.

Proof. For $g\breve{\mathcal{I}} \in X(\{\mu\}, b)$, we have $\tilde{\kappa}(g) - \sigma(\tilde{\kappa}(g)) = \tilde{\kappa}(\mu) - \tilde{\kappa}(b) = c_{b,\{\mu\}} - \sigma(c_{b,\{\mu\}})$. Thus $\tilde{\kappa}(X(\{\mu\}, b)) \subset c_{b,\{\mu\}} + \pi_1(G)^{\sigma}_{\Gamma_0}$.

On the other hand, for $\gamma \in \breve{\Omega}$ with $\sigma(\gamma) = \gamma$ and $g\breve{\mathcal{I}} \in X_w(b)$, we have $g\dot{\gamma}\breve{\mathcal{I}} \in X_{\gamma^{-1}w\gamma}(b)$. Since $\mathrm{Adm}(\{\mu\})$ is stable under the conjugation action of $\breve{\Omega}$, the right multiplication of $\breve{\Omega}^{\sigma}$ on $\mathcal{FL}$ stabilizes $X(\{\mu\}, b)$. Since $X(\{\mu\}, b) \neq \emptyset$, we have $\tilde{\kappa}(X(\{\mu\}, b)) = c_{b,\{\mu\}} + \pi_1(G)^{\sigma}_{\Gamma_0}$. □

6.1. Recall that the set $B(G, \{\mu\})$ contains a unique minimal element, the basic σ-conjugacy class $[\dot{\tau}_{\{\mu\}}]$. In this section we give a description of the connected components of $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$. We show that, except in trivial cases, the first obstruction (given in Lemma 6.1) is the only obstruction to connecting points in $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$. Note that we have $\tilde{\kappa}(\dot{\tau}_{\{\mu\}}) = \tilde{\kappa}(\mu) \in \pi_1(G)_{\Gamma_0}$ so in what follows we may take $c_{\dot{\tau}_{\{\mu\}},\{\mu\}} = 0$. The main theorem in this section is the following.

Theorem 6.3. *Assume that G_{ad} is simple.*

(1) If μ is central, then

$$X(\{\mu\}, \dot{\tau}_{\{\mu\}}) = G(F)/\mathcal{I},$$

where $\mathcal{I}$ is the group of $\mathcal{O}_F$-points of the parahoric group scheme (over $\mathcal{O}_F$) corresponding to $\breve{\mathcal{I}}$.

(2) If μ is noncentral, then $\tilde{\kappa}$ induces an isomorphism

$$\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})) \cong \pi_1(G)^{\sigma}_{\Gamma_0}.$$

Before we begin the proof, let us explain how the above can be used to get a description of $\pi_0(X(\{\mu\}, b))$ for any G. We may write G_{ad} as $G_{\mathrm{ad}} = G_1 \times \cdots \times G_n$, where G_i are the simple factors of G_{ad}. In other words, the action of σ on the set of connected components of the affine Dynkin diagram of G_i is transitive for each i. Let μ be an element of the geometric conjugacy class $\{\mu\}$, and μ_{ad} be the composition of μ with $G \to G_{\mathrm{ad}}$. Let μ_i be the projection of μ_{ad} to the factor G_i, and $\{\mu_i\}$ be the geometric conjugacy class of μ_i in G_i. Then it is easy to see that

$$X(\{\mu_{\mathrm{ad}}\}, \dot{\tau}_{\{\mu_{\mathrm{ad}}\}}) \cong X(\{\mu_1\}, \dot{\tau}_{\{\mu_1\}}) \times \cdots \times X(\{\mu_n\}, \dot{\tau}_{\{\mu_n\}}).$$

It follows from Corollary 4.4 that we have a cartesian diagram

$$\begin{array}{ccc} X(\{\mu\}, \dot{\tau}_{\{\mu\}}) & \longrightarrow & X(\{\mu_{\mathrm{ad}}\}, \dot{\tau}_{\{\mu_{\mathrm{ad}}\}}) \\ \big\downarrow & & \big\downarrow \\ \pi_1(G)^{\sigma}_{\Gamma_0} & \longrightarrow & \pi_1(G_{\mathrm{ad}})^{\sigma}_{\Gamma_0}, \end{array}$$

where we consider $\pi_1(G)^{\sigma}_{\Gamma_0}$ and $\pi_1(G_{\mathrm{ad}})^{\sigma}_{\Gamma_0}$ as discrete schemes. This induces a Cartesian diagram of connected components

$$\begin{array}{ccc} \pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})) & \longrightarrow & \pi_0(X(\{\mu_{\mathrm{ad}}\}, \dot{\tau}_{\{\mu_{\mathrm{ad}}\}})) \\ \big\downarrow & & \big\downarrow \\ \pi_1(G)^{\sigma}_{\Gamma_0} & \longrightarrow & \pi_1(G_{\mathrm{ad}})^{\sigma}_{\Gamma_0}. \end{array}$$

See [5, Lemma 2.4.3] for a special case. Therefore we can recover $\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}}))$ from $\pi_0(X(\{\mu_i\}, \dot{\tau}_{\mu_i})$.

6.2. The rest of the section will be devoted to the proof of Theorem 6.3. We begin with several results concerning the structure of the admissible set $\mathrm{Adm}(\{\mu\})$.

Lemma 6.4. *Let $\mathcal{O}$ be a(n ordinary) straight conjugacy class of $\breve{W}$. If $\mathcal{O} \cap \mathrm{Adm}(\{\mu\}) \neq \emptyset$, then all the straight elements in $\mathcal{O}$ are contained in $\mathrm{Adm}(\{\mu\})$.*

Proof. Let $w \in \mathcal{O} \cap \mathrm{Adm}(\{\mu\})$. By [16, Corollary 2.6], there exists a straight element w' in $\mathcal{O}$ with $w' \leqslant w$. Since $\mathrm{Adm}(\{\mu\})$ is closed under Bruhat order, we have $w' \in \mathrm{Adm}(\{\mu\})$. Let w'' be another straight element in $\mathcal{O}$. By Theorem 1.2 (2), $w'' \approx w'$. Let $\tau \in \breve{\Omega}$ such that $\tau w' \tau^{-1} \approx w''$. Since $w' \in \mathrm{Adm}(\{\mu\})$, there exists $\lambda \in \breve{W}_0(\underline{\mu})$ such that $w' \le t^{\lambda}$. It follows that $\tau w' \tau^{-1} \le t^{\tau(\lambda)}$ and therefore $\tau w' \tau^{-1} \in \mathrm{Adm}(\{\mu\})$. The statement then follows from [13, Lemma 4.5]. □

Lemma 6.5. *Suppose that $G_{\breve{F},ad}$ is simple and μ is not central. Then $s_j \tau_{\{\mu\}} \in \mathrm{Adm}(\{\mu\})$ for all $j \in \breve{\mathbb{S}}$.*

Remark 6.6. Note that $G_{\breve{F},\mathrm{ad}}$ is simple if and only if the corresponding affine Dynkin diagram is connected. The group G_{ad} is simple if and only if the action of σ on the set of connected components of the affine Dynkin diagram is transitive.

Proof. Set $w = t^{\underline{\mu}} \tau_{\{\mu\}}^{-1} \in \breve{W}_a$. Let K be the minimal $\mathrm{Ad}(\tau_{\{\mu\}})$-stable subset of $\breve{\mathbb{S}}$ that contains the set $\{\iota; s_\iota \in \mathrm{supp}(w)\}$. If $K \neq \breve{\mathbb{S}}$, then W_K is a finite Coxeter group since $G_{\breve{F},\mathrm{ad}}$ is simple. In this case, for sufficiently divisible n, we have $t^{n\underline{\mu}} = (w\tau_{\{\mu\}})^n = \tau_{\{\mu\}}^n$. This is a central element in $\breve{W}$, which contradicts the assumption that μ is non-central.

Hence $K = \breve{\mathbb{S}}$. In particular, there exists $s_{j'} \in \mathrm{supp}(w)$ and $i \in \mathbb{N}$ such that $\mathrm{Ad}(\tau_{\{\mu\}})^i s_{j'} = s_j$. We have

$$s_{j'} \tau_{\{\mu\}} \approx \tau_{\{\mu\}} s_{j'} = \mathrm{Ad}(\tau_{\{\mu\}})(s_{j'}) \tau_{\{\mu\}} \approx \cdots \approx \mathrm{Ad}(\tau_{\{\mu\}})^i (s_{j'}) \tau_{\{\mu\}} = s_j \tau_{\{\mu\}}.$$

Since $s_{j'} \tau_{\{\mu\}} \leqslant w \tau_{\{\mu\}}$, we have $s_{j'} \tau_{\{\mu\}} \in \mathrm{Adm}(\{\mu\})$. By [13, Lemma 4.5], $s_j \tau_{\{\mu\}} \in \mathrm{Adm}(\{\mu\})$. □

Note that $\nu_{\dot{\tau}_{\{\mu\}}}$ is central. Hence $\breve{\mathbb{S}}_{\nu_{\dot{\tau}_{\{\mu\}}}} = \breve{\mathbb{S}}$. We then use this lemma to construct certain curves in $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$.

Proposition 6.7. *Suppose that G_{ad} is simple and μ is not central. Let $\tau = \mathrm{Ad}(\dot{\tau}_{\{\mu\}}) \circ \sigma$. Let $J \in \breve{\mathbb{S}}(\tau)$ and u_{-J} be the corresponding element of $J_{\dot{\tau}_{\{\mu\}}}$ constructed in 5.4. Then $u_{-J}\breve{\mathcal{I}}$ and $\breve{\mathcal{I}}$ lie in the same connected component of $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$.*

Proof. By Lemma 6.5 and Remark 6.6, τ acts transitively on the set of connected components of the affine Dynkin diagram of $\breve{\mathbb{S}}$ and there exists $j \in J$ such that $s_j \tau_{\{\mu\}} \in \mathrm{Adm}(\{\mu\})$. Let $\breve{\mathcal{K}}$ be the standard parahoric subgroup of $G(\breve{F})$ associated to J. Then we identify $\breve{\mathcal{K}}/\breve{\mathcal{I}}$ with the finite type flag variety for the reductive quotient $\overline{\breve{\mathcal{K}}}^{red}$ of the special fiber of $\breve{\mathcal{K}}$. Set

$$Y = \{g \in \breve{\mathcal{K}}/\breve{\mathcal{I}}; g^{-1} \dot{\tau}_{\{\mu\}} \sigma(g) \in \breve{\mathcal{I}} \dot{\tau}_{\{\mu\}} \cup \breve{\mathcal{I}} \dot{s}_j \dot{\tau}_{\{\mu\}} \breve{\mathcal{I}}\} = \{g \in \breve{\mathcal{K}}/\breve{\mathcal{I}}; g^{-1} \tau(g) \in \breve{\mathcal{I}} \cup \breve{\mathcal{I}} \dot{s}_j \breve{\mathcal{I}}\}$$

Since $s_j \tau_{\{\mu\}} \in \mathrm{Adm}(\{\mu\})$, $Y \subset X(\{\mu\}, \dot{\tau}_{\{\mu\}})$. By definition, $u_{-J}\breve{\mathcal{I}}$ and $\breve{\mathcal{I}}$ are contained in Y. By [9, Theorem 1.1], Y is connected. It is a curve in $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$ that connects $u_{-J}\breve{\mathcal{I}}$ with $\breve{\mathcal{I}}$. □

6.3. **Proof of Theorem 6.3.** First assume μ is central. In this case, $\mathrm{Adm}(\{\mu\}) = \{\tau_{\{\mu\}}\}$ and we may choose $\dot{\tau}_{\{\mu\}}$ to be central. Then

$$\begin{aligned} X(\{\mu\}, \dot{\tau}_{\{\mu\}}) &= \{g \in G(\breve{F})/\breve{\mathcal{I}}; g^{-1} \dot{\tau}_{\{\mu\}} \sigma(g) \in \breve{\mathcal{I}} \dot{\tau}_{\{\mu\}} \breve{\mathcal{I}}\} \\ &= \{g \in G(\breve{F})/\breve{\mathcal{I}}; g^{-1} \sigma(g) \in \breve{\mathcal{I}}\} \\ &= G(F)/\mathcal{I}, \end{aligned}$$

where the last equality follows from Lang's theorem on $\breve{\mathcal{I}}$. This proves (1).

For part (2), we may assume G is adjoint and simple; see the discussion in §6.1. Let $\mathcal{O}$ be the unique straight σ-conjugacy class of $\breve{W}$ associated to $[\dot{\tau}_{\{\mu\}}]$ in the sense of Theorem 1.1. It is easy to see that the only σ-straight elements in $\mathcal{O}$ are $\sigma^i(\tau_{\{\mu\}})$ for some $i \in \mathbb{N}$. Hence the intersection $\mathcal{O} \cap \mathrm{Adm}(\{\mu\})$ contains a unique σ-straight element, which is $\tau_{\{\mu\}}$. By Theorem 4.1, every point of $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$ is connected to a point in $X_{\tau_{\{\mu\}}}(\dot{\tau}_{\{\mu\}})$.

By Theorem 5.1, $J_{\dot{\tau}_{\{\mu\}}}$ acts transitively on $X_{\tau_{\{\mu\}}}(\dot{\tau}_{\{\mu\}})$. For any $J \in \breve{\mathbb{S}}(\tau)$, u_{-J} is contained in the parahoric subgroup of $G(\breve{F})$ corresponding to J and thus is contained in the kernel of the Kottwitz map $\tilde{\kappa}$. By Theorem 5.5, $\tilde{\kappa} : J_{\dot{\tau}_{\{\mu\}}} \to \pi_1(G)^{\sigma}_{\Gamma_0}$ is surjective and the kernel is generated by u_{-J} for $J \in \breve{\mathbb{S}}(\tau)$ and $\breve{\mathcal{I}}^{\tau}$. It remains to show that for any $j \in \ker(\tilde{\kappa}) \cap J_{\dot{\tau}_{\{\mu\}}}$, $j\breve{\mathcal{I}}$ and $\breve{\mathcal{I}}$ are in the same connected component of $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$.

We have $j = j_1 \cdots j_n$, where $j_i \in \breve{\mathcal{I}}^{\tau}$ or $j_i = u_{-J}$ for some $J \in \breve{\mathbb{S}}(\tau)$. Set $j' = j_1 \cdots j_{n-1}$. If $j_n \in \breve{\mathcal{I}}^{\tau}$, then $j\breve{\mathcal{I}} = j'\breve{\mathcal{I}}$. If $j_n = u_{-J}$, then by Proposition 6.7, $j_n\breve{\mathcal{I}}$ and $\breve{\mathcal{I}}$ lie in the same connected component of $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$, and hence $j\breve{\mathcal{I}}$ and $j'\breve{\mathcal{I}}$ are in the same connected component of $X(\{\mu\}, \dot{\tau}_{\{\mu\}})$. The statement follows from induction on n.

7. Second reduction theorem

7.1. In this section, we assume that G is residually split in the sense of [44, 1.10.2]. This means that G has the same split rank as $G_{\breve{F}}$. Since the split rank of a group is equal to the dimension of an apartment in its building, it follows from [44, 1.10] that the property of being residually split is equivalent to σ acting trivially on the apartment, and also equivalent to σ acting trivially on $\breve{W}$. Note that this condition is more general than G being split, however the assumption does imply that G is quasi-split.

As σ acts trivially on $\breve{W}$, the σ-conjugacy classes in $\breve{W}$ are just the ordinary conjugacy classes.

Let $[b] \in B(G, \{\mu\})$ and $\mathcal{O}_{[b]} \subset \breve{W}$ be the unique straight conjugacy class associated to it in the sense of Theorem 1.1. By [16, Proposition 4.1], $\mathrm{Adm}(\{\mu\})$ contains an element of $\mathcal{O}_{[b]}$. By Lemma 6.4,

(a) Any straight element in $\mathcal{O}_{[b]}$ is contained in $\mathrm{Adm}(\{\mu\})$.

Let $\breve{\mathcal{K}}_0$ be the special parahoric subgroup corresponding to the special vertex $\mathfrak{s}$. Then the Weyl group of $\breve{\mathcal{K}}_0$ is $\breve{W}_0$. Let $w \in \breve{W}$ be a straight element with $\dot{w} \in [b]$. Then by (a), $w \in \mathrm{Adm}(\{\mu\})$. By definition, $w = t^{\lambda_w} x$ for some $\lambda_w \in X_*(\breve{T})_{\Gamma_0}$ and $x \in \breve{W}_0$. Let $\bar{\lambda}_w$ be the dominant representative of λ_w. Then $t^{\bar{\lambda}_w} \in \breve{W}_0 \,\mathrm{Adm}(\{\mu\}) \breve{W}_0 = \mathrm{Adm}(\{\mu\})^{K_0}$ (here $K_0 \subset \breve{\mathbb{S}}$ is the subset corresponding to $\breve{\mathcal{K}}_0$). Moreover, $t^{\bar{\lambda}_w}$ is the minimal length element in the coset $\breve{W}_0 t^{\bar{\lambda}_w}$. By [16, Theorem 6.1], $t^{\bar{\lambda}_w} \in \mathrm{Adm}(\{\mu\})$. Hence

(b) For any straight element $w \in \breve{W}$ with $\dot{w} \in [b]$, we have $t^{\lambda_w} \in \mathrm{Adm}(\{\mu\})$.

Let $\breve{W}_{\nu_w,0}$ be the relative Weyl group of M_{ν_w}. Let $\lambda_{w,dom} \in X_*(T)_{\Gamma_0}$ be the unique M_{ν_w}-dominant element in the $\breve{W}_{\nu_w,0}$-orbit of λ_w. Then by Lemma 6.4, $t^{\lambda_{w,dom}} \in \mathrm{Adm}(\{\mu\})$.

Let $\tilde{\lambda}_{w,dom} \in X_*(T)$ be an M_{ν_w}-dominant lift of $\lambda_{w,dom}$. Let $\{\tilde{\lambda}_w\}_{M_{\nu_w}}$ be the M_{ν_w}-conjugacy class of characters containing $\tilde{\lambda}_{w,dom}$.

We say that $[b]$ is *strongly noncentral* in $M_{\bar{\nu}_b}$ if for some (or equivalently, any) straight element $w \in \breve{W}$ with $\dot{w} \in [b]$, $\tilde{\lambda}_w$ is noncentral in any simple factor of $M_{\nu_w,\mathrm{ad}}$.

7.2. Recall that we denote the Bruhat order on $\breve{W}$ by $\leqslant$. Let $\breve{W}_{\nu_w}$ be the Iwahori-Weyl group of M_{ν_w} and $\leqslant_{\nu_w}$ be the Bruhat order on M_{ν_w}. By definition, for any $x \in \breve{W}_{\nu_w}$ and an affine reflection r of $\breve{W}_{\nu_w}$, we have $x \leqslant_{\nu_w} rx$ if and only if $x \leqslant rx$. Therefore, we have

(a) Let $x, y \in \breve{W}_{\nu_w}$. If $x \leqslant_{\nu_w} y$, then $x \leqslant y$.

In particular,

(b) $\mathrm{Adm}^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}) \subset \mathrm{Adm}(\{\mu\})$.

We denote by $\breve{W}_{\mathrm{str}} \subset \breve{W}$ the subset consisting of straight elements.

The main result of this section is the following

Theorem 7.1. *Assume that G is residually split. Then for any $[b] \in B(G, \{\mu\})$, we have a natural morphism*

$$\coprod_{w \in \breve{W}_{str} \text{ with } \dot{w} \in [b]} X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w}) \longrightarrow X(\{\mu\}, b),$$

which induces a surjection

$$\coprod_{w \in \breve{W}_{str} \text{ with } \dot{w} \in [b]} \pi_0(X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w})) \twoheadrightarrow \pi_0(X(\{\mu\}, b)). \qquad \text{(a)}$$

If moreover, $[b]$ is strongly noncentral in the Levi subgroup $M_{\bar{\nu}_b}$, then the natural morphism induces a surjection

$$\coprod_{w \in \breve{W}_{str} \text{ with } \dot{w} \in [b]} \pi_1(M_{\nu_w})_{\Gamma_0} \twoheadrightarrow \pi_0(X(\{\mu\}, b)).$$

Remark 7.2. If G is not residually split, the situation can be much worse. For example, if $\mathcal{O}_{[b]}$ was the conjugacy class of $t^{\underline{\mu}}$, i.e. $\mathcal{O}_{[b]}$ consists of maximal translation elements in $\mathrm{Adm}(\{\mu\})$, then $X(\{\mu\}, b)$ is discrete and we have a bijection

$$\coprod_{w \in \breve{W}_{str}, \text{ with } \dot{w} \in [b]} M_{\nu_w}(F)/\breve{\mathcal{I}}^{\sigma}_{M_{\nu_w}} \cong X(\{\mu\}, b),$$

where $\breve{\mathcal{I}}_{M_{\nu_w}} = M_{\nu_w}(\breve{F}) \cap \breve{\mathcal{I}}$ is an Iwahori subgroup of M_{ν_w}.

Proof. Let $[b] \in B(G, \{\mu\})$ and $w \in \breve{W}_{\mathrm{str}}$ with $\dot{w} \in [b]$. Then by §7.2 (b), we have

$$X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w}) \subset X(\{\mu\}, \dot{w}) \cong X(\{\mu\}, b).$$

Here the second map is given by $g\breve{\mathcal{I}} \mapsto h_w^{-1} g \breve{\mathcal{I}}$, where h_w is an element in $G(\breve{F})$ with $h_w^{-1} \dot{w} \sigma(h_w) = b$.

This defines a morphism

$$\coprod_{w \in \breve{W}_{\mathrm{str}} \text{ with } \dot{w} \in [b]} X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w}) \longrightarrow X(\{\mu\}, b)$$

and a map

$$\coprod_{w \in \breve{W}_{\mathrm{str}} \text{ with } \dot{w} \in [b]} \pi_0(X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w})) \longrightarrow \pi_0(X(\{\mu\}, b)).$$

Both maps depend on the choice of h_w. However, since $J_{\dot{w}} \subset M_{\nu_w}$, the image of the second map in $\pi_0(X(\{\mu\}, b))$ does not depend on the choice of h_w.

By Corollary 4.2, every element in $X(\{\mu\}, b)$ is connected to an element in $X_w(b)$ for a straight element w with $\dot{w} \in [b]$. By [10, Theorem 2.1.4], $X_w(\dot{w}) \cong X_w^{M_{\nu_w}}(\dot{w}) \subset X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w})$. This proves the surjectivity of (a).

By definition, ν_w is central in M_{ν_w}. Therefore the σ-conjugacy class of M_{ν_w} that contains $\dot{w}$ is basic. The "moreover" part follows from the surjectivity of (a) and Theorem 6.3. □

8. Passing from Iwahori level to parahoric level

In this section, we study the connected components of $X(\{\mu\}, b)_K$, for $\breve{\mathcal{K}}$ a σ-invariant standard parahoric subgroup of $G(\breve{F})$. We first describe the case where $[b]$ is basic. We also remind the reader that unless specified, the group G is no longer residually split.

Theorem 8.1. *Assume that G_{ad} is simple.*

(1) If μ is central, then

$$X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K = G(F)/\mathcal{K},$$

where $\mathcal{K}$ is the group of $\mathcal{O}_F$-points of the parahoric group scheme corresponding to K.

(2) If μ is noncentral, then

$$\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K) \cong \pi_1(G)^{\sigma}_{\Gamma_0}.$$

Proof. Part (1) can be proved in the same in the same way as Theorem 6.3 (1).

For part (2), we have a commutative diagram

$$\begin{array}{ccc} X(\{\mu\}, \dot{\tau}_{\{\mu\}}) & \longrightarrow & X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K \\ \big\downarrow & & \big\downarrow \\ \mathcal{FL} & \longrightarrow & Gr_{\breve{\mathcal{K}}}. \end{array}$$

By Theorem 2.3 (2), the map $X(\{\mu\}, \dot{\tau}_{\{\mu\}}) \to X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K$ is surjective. Hence the induced map $\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})) \to \pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K)$ is also surjective.

We have $\pi_0(\mathcal{FL}) \cong \pi_0(Gr_{\breve{\mathcal{K}}}) \cong \pi_1(G)_{\Gamma_0}$. By Theorem 6.3, the map

$$\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})) \longrightarrow \pi_0(\mathcal{FL}) \cong \pi_0(Gr_{\breve{\mathcal{K}}})$$

is injective. By the commutativity of the above diagram, the map

$$\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})) \longrightarrow \pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K)$$

is also injective. Hence $\pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K) \cong \pi_1(G)^\sigma_{\Gamma_0}$.

□

Remark 8.2. As in the discussion in §6.1, the case of general G (i.e. G_{ad} is no longer assumed to be simple) reduces to the case in the Theorem. In particular, it follows from Remark 4.5 that we have the following cartesian diagram

$$\begin{array}{ccc} \pi_0(X(\{\mu\}, \dot{\tau}_{\{\mu\}})_K) & \longrightarrow & \pi_0(X(\{\mu_{\mathrm{ad}}\}, \dot{\tau}_{\{\mu_{\mathrm{ad}}\}})_{K_{\mathrm{ad}}}) \\ \downarrow & & \downarrow \\ \pi_1(G)^\sigma_{\Gamma_0} & \longrightarrow & \pi_1(G_{\mathrm{ad}})^\sigma_{\Gamma_0} \end{array} .$$

We have the following result in the general case.

Proposition 8.3. *Every point in $X(\{\mu\}, b)_K$ is connected to a point in $X_{K,x}(b)$ for some σ-straight element $x \in \mathrm{Adm}(\{\mu\}) \cap {}^K\breve{W}$.*

Proof. Let

$$\begin{aligned} X(\{\mu\}, b)^K &= \{g \in G(\breve{F})/\breve{\mathcal{I}}; g^{-1}b\sigma(g) \in \cup_{w \in \mathrm{Adm}(\{\mu\})_K} \breve{\mathcal{K}}\dot{w}\breve{\mathcal{K}}\} \\ &= \sqcup_{w \in \mathrm{Adm}(\{\mu\})^K} X_w(b) \subset \mathcal{FL}. \end{aligned}$$

Then the projection map $\mathcal{FL} \to Gr_{\breve{\mathcal{K}}}$ induces a surjective map $X(\{\mu\}, b)^K \to X(\{\mu\}, b)_K$ and each fiber is isomorphic to $\breve{\mathcal{K}}/\breve{\mathcal{I}}$. Therefore

(a) $\pi_0(X(\{\mu\}, b)^K) \cong \pi_0(X(\{\mu\}, b)_K)$.

Note that $\mathrm{Adm}(\{\mu\})^K$ is closed under the Bruhat order. By Theorem 4.1, every point in $X(\{\mu\}, b)^K$ is connected to a point in $X_w(b)$ for a σ-straight element w in $\mathrm{Adm}(\{\mu\})^K$. By [20, Theorem 6.17], there exists a σ-straight element $x \in \mathrm{Adm}(\{\mu\}) \cap {}^K\breve{W}$ that is σ-conjugate to w by an element in $\breve{W}_K$.

Let $g\breve{\mathcal{I}} \in X_w(b)$. Then by Theorem 5.2, we may assume $g^{-1}b\sigma(g) = \dot{w}$. Consider the subscheme $g\breve{\mathcal{K}}/\breve{\mathcal{I}} \subset \mathcal{FL}$; it is a translate of $\breve{\mathcal{K}}/\breve{\mathcal{I}}$ hence is connected. Let $z \in \breve{W}_K$ such that $z^{-1}w\sigma(z) = x$. Let $\dot{z}$ be a lift of z to $\breve{\mathcal{K}}$. Then $g\breve{\mathcal{I}}$ and $g\dot{z}\breve{\mathcal{I}}$ both lie in $g\breve{\mathcal{K}}/\breve{\mathcal{I}}$ and $g\dot{z}\breve{\mathcal{I}} \in X_x(b)$. Therefore $g\breve{\mathcal{I}}$ is connected to point in $X_x(b)$. The result then follows from (a). □

Similar to the proof in Theorem 7.1, we have

Theorem 8.4. *Assume that G is residually split. Then for any $[b] \in B(G, \{\mu\})$, we have a natural morphism*

$$\coprod_{w \in {}^K\breve{W} \cap \breve{W}_{str} \text{ with } \dot{w} \in [b]} X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w}) \longrightarrow X(\{\mu\}, b)_K,$$

which induces a surjection

$$\coprod_{w \in {}^K\breve{W} \cap \breve{W}_{str} \text{ with } \dot{w} \in [b]} \pi_0(X^{M_{\nu_w}}(\{\tilde{\lambda}_w\}_{M_{\nu_w}}, \dot{w})) \twoheadrightarrow \pi_0(X(\{\mu\}, b)_K).$$

If moreover, $[b]$ is strongly noncentral in the Levi subgroup $M_{\bar{\nu}_b}$, then the natural morphism induces a surjection

$$\coprod_{w \in {}^K\breve{W} \cap \breve{W}_{str} \text{ with } \dot{w} \in [b]} \pi_1(M_{v_w})^\sigma_{\Gamma_0} \twoheadrightarrow \pi_0(X(\{\mu\}, b)_K).$$ □

Compared to the Iwahori case, the number of straight elements one needs to consider here could be smaller. For example, if G is residually split, $\breve{\mathcal{K}}$ is a special maximal parahoric subgroup and b is a translation element, then there is only one straight element in ${}^KW$ with $\dot{w} \in [b]$.

9. Verification of the Axioms in [20] for PEL-type Shimura varieties

In this section we apply the previous construction to study the connected components in the basic locus of some PEL-type Shimura varieties and to verify the axioms of [20] for integral models of these Shimura varieties. We first recall the general set-up of PEL-type Shimura varieties following [42, Chapter 6].

9.1. We let $(B, *, V, (-,-))$ denote a PEL-datum as in [42, Chapter 6], i.e. B is a finite dimensional semisimple algebra over $\mathbb{Q}$, $*$ is a positive involution of B and V is a $\mathbb{Q}$-vector space equipped with an alternating bilinear form $(-,-)$. Moreover V is equipped with a B-module structure such that:

$$(bv, w) = (v, b^* w), \quad v, w \in V, \ b \in B.$$

We let $\mathbf{G}$ be the similitude group

$$\mathbf{G}(\mathbb{Q}) = \{g \in \mathbf{GL}_B(V); (gv, gw) = c(g)(v, w) \text{ for some } c(g) \in \mathbb{Q}^\times\}$$

which arises as a reductive group over $\mathbb{Q}$.

Let $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to \mathbf{G}_\mathbb{R}$ be a homomorphism which defines a Hodge structure of type $(1,0),(0,1)$ on $V_\mathbb{R}$ and such that the pairing $(v, h_0(\sqrt{-1})w)$ is symmetric positive definite; we let X denote the $\mathbf{G}(\mathbb{R})$ conjugacy class of h_0. For a sufficiently small compact open subgroup $\mathcal{K} \subset \mathbf{G}(\mathbb{A}_f)$, the formalism in [6] associates to the above data the Shimura variety $Sh_\mathcal{K}(\mathbf{G}, X)$ defined over a canonical number field $\mathbf{E}$ called the reflex field. Let p be a prime and we let G denote the base change of $\mathbf{G}$ to $\mathbb{Q}_p$.

Theorem 9.1. *The five Axioms in* [20] *hold for PEL-type Shimura varieties in the following two cases:*

i) $\mathbf{G}$ *satisfies the Hasse principle, only involves groups of type A and C, and G is unramified; i.e. G is quasi-split and splits over an unramified extension of $\mathbb{Q}_p$.*

ii) $\mathbf{G}$ *is an odd unitary similitude group as in* [39]. *In particular G splits over a ramified extension of $\mathbb{Q}_p$.*

The axioms stated in [20] postulate the existence of integral models of Shimura varieties with parahoric level structure which satisfy certain nice properties. We refer to §9.5 for their statements, and to [20] for more details and explanations. The Siegel case has already been verified in [20]. In case i), the integral models are constructed in [42], and in case ii) they are studied in [39]. In both these cases, for $v|p$ a prime of $\mathbf{E}$, there is the construction of a moduli space of abelian varieties with extra structure defined over $\mathcal{O}_{\mathbf{E}_{(v)}}$ whose generic fiber agrees with $Sh_\mathcal{K}(\mathbf{G}, X)$ by the Hasse principle (note that this is automatically satisfied in case ii), see [39, §1.4]).

In the rest of this section, we focus on case ii), i.e. on odd ramified unitary groups. The same proofs go through for case i). In fact, in these unramified cases, there are even some simplifications. For example, the naive local model is flat, see [9], [8], and hence there is no need to take closures in the construction of the integral models. Therefore one does not need the coherence conjecture in the verification of Axiom (2).

As to other PEL-types, we expect that the axioms in [20] still hold. However, there are several technical difficulties to overcome. First, the argument we have uses Dieudonné theory, which requires a moduli description of integral models. Second, for even ramified unitary groups and even orthogonal groups, the special fiber of the stabilizer of a lattice chain might not be connected and thus is not a parahoric subgroup. This gives extra difficulties in understanding the integral models. We do not investigate these cases here.

9.2. We now specialize to the case of odd unitary groups; we begin by recalling the constructions in [39, §1]. Let E be a quadratic imaginary field and $\epsilon : E \to \mathbb{C}$ be a fixed embedding. Let W be an n-dimensional E-vector space equipped with a Hermitian form $\phi : W \times W \to E$. We assume $n = 2m + 1$ is odd. Let $\mathbf{G}$ be the unitary similitude group

$$\mathbf{G}(\mathbb{Q}) = \{g \in GL_E(W); \phi(gv, gw) = c(g)\phi(v, w) \text{ for some } c(g) \in \mathbb{Q}^\times\}.$$

Then $\mathbf{G}$ arises as a reductive group over $\mathbb{Q}$.

We let (r,s) be the signature of the Hermitian space W. Since n is odd, we may assume upon replacing ϕ by $-\phi$ that $s<r$. Then as in [39, §1.1] there is a homomorphism

$$h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\,\mathbb{G}_m \to \mathbf{G}_{\mathbb{R}},$$

which induces a complex structure on $W\otimes_{\mathbb{Q}}\mathbb{R}$ such that

$$\mathrm{Tr}(a, W\otimes_{\mathbb{Q}}\mathbb{R}) = s\epsilon(a) + r\overline{\epsilon}(a) \text{ for } a\in E$$

with respect to this complex structure. Let X denote the $\mathbf{G}(\mathbb{R})$ conjugacy class of h_0. We obtain a Shimura datum $(\mathbf{G}, X)$ with reflex field E.

The homomorphism h_0 induces a decomposition

$$W_{\mathbb{C}} = W_0 \oplus W_1, \tag{9.1}$$

where $z\in\mathbb{C}^\times = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{R})$ acts on the first factor by z and the second factor by $\overline{z}$. We let $\mu:\mathbb{C}^\times\to\mathbf{G}(\mathbb{C})\cong\mathbf{GL}_n(\mathbb{C})\times\mathbb{C}^\times$ denote the cocharacter of $\mathbf{G}$ over $\mathbb{C}$ given by $z\mapsto h_0(z,1)$; explicitly, up to conjugacy it is given by

$$z\longmapsto(\mathrm{diag}(z^{(s)},1^{(r)}),z).$$

The geometric conjugacy class of μ is defined over E.

Let $\alpha\in E$ be an element such that $\overline{\alpha}=-\alpha$. Then we have an alternating $\mathbb{Q}$-bilinear form $\psi: W\otimes_{\mathbb{Q}} W\to\mathbb{Q}$ given by

$$\psi(v,w)=\mathrm{Tr}_{E/\mathbb{Q}}(\alpha^{-1}\phi(v,w)).$$

Upon replacing α by $-\alpha$, we may assume the $\mathbb{R}$-bilinear form on $W\otimes_{\mathbb{Q}}\mathbb{R}$ given by $\psi(v,h_0(i)w)$ is positive definite.

We also obtain an embedding of Shimura data

$$\iota:(\mathbf{G},X)\hookrightarrow(\mathbf{GS}p(W,\psi),S^{\pm})$$

into the standard Shimura datum for the symplectic group by forgetting the E structure on W.

9.3. Now let $p>2$ be a prime which ramifies in E. We fix an embedding $\overline{\mathbb{Q}}\to\overline{\mathbb{Q}}_p$ and we let v denote the place of E induced by this embedding; it is the unique place of E above p. We let $F=\mathbb{Q}_p$ and F' the completion of E at the place v. Assume that ϕ is split over F. This means that there exists an F' basis $e_1,\dots,e_n$ of $W\otimes_{\mathbb{Q}}\mathbb{Q}_p$ such that

$$\phi(e_i,e_j)=\delta_{ij},\forall i,j.$$

Here we also use ϕ to denote the scalar extension of ϕ to $W\otimes_{\mathbb{Q}}\mathbb{Q}_p$. We denote by G the base change of $\mathbf{G}$ to $\mathbb{Q}_p$; then G is quasi-split. We recall that $\mathbf{k}$ is an algebraic closure of the residue field of $\mathcal{O}_F$.

We have that F' is a ramified quadratic extension of $\mathbb{Q}_p$ and we let $\varpi\in\mathcal{O}_{F'}$ be a uniformizer with $\overline{\varpi}=-\varpi$, which exists since p is odd. We let T be the maximal torus of G given by the elements

$$\mathrm{diag}(a_1,\dots,a_m,a,a\overline{a}\overline{a}_m^{-1},\dots,a\overline{a}\overline{a}_1^{-1}),\quad a\in F'.$$

We have the associated Iwahori-Weyl group $\breve{W}$ and the affine Weyl group

$$\breve{W}_a\cong\mathbb{Z}^m\rtimes\breve{W}_0,$$

where $\breve{W}_0=S_m\times\{\pm1\}^m$, see [39, §2.4.2] for these calculations.

For any $\mathcal{O}_{F'}$-lattice $\Lambda\subset W\otimes_{\mathbb{Q}}\mathbb{Q}_p$, we define its dual lattice

$$\hat{\Lambda}=\{v\in W\otimes_{\mathbb{Q}}\mathbb{Q}_p;\phi(v,\Lambda)\subset\mathcal{O}_{F'}\}.$$

For $j\in\{0,\dots,n-1\}$, we define an $\mathcal{O}_{F'}$-lattice Λ_j by

$$\Lambda_j=\mathrm{span}_{\mathcal{O}_{F'}}\{\varpi^{-1}e_1,\dots,\varpi^{-1}e_j,e_{j+1},\dots,e_n\}.$$

In what follows, we will use slightly different notations for parahorics subgroups of G than in the rest of the paper. For $J\subset\{0,\dots,m\}$ a non-empty subset, we associate the common stabilizer $\mathcal{K}_J$ of the lattices $\{\Lambda_j;j\in J\}$ in $G(\mathbb{Q}_p)$. Then $\mathcal{K}_J$ is a parahoric subgroup by [38, 1.2.3] and we write $\breve{\mathcal{K}}_J$ for the $\mathcal{O}_{\breve{F}}$-points of the corresponding group scheme. The parahoric subgroup $\mathcal{K}_J$ is special maximal when $J=\{0\},\{m\}$ and is Iwahori when $J=\{0,\dots,m\}$. We write $\mathcal{I}$ for this Iwahori subgroup and $\breve{\mathcal{I}}$ for its group of $\mathcal{O}_{\breve{F}}$-points. We may identify J with the set $\breve{\mathbb{S}}$ for this choice of Iwahori and so for each non-empty $J \subset \{0, \dots, m\}$, we have the corresponding Weyl group $\breve{W}_J$ and $\mathrm{Adm}(\{\mu\})_J$ as in §2.1 and §2.2. Here in the definition of $\mathrm{Adm}(\{\mu\})$ we use the embedding $\overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$ to view $\{\mu\}$ as a geometric conjugacy of cocharacters of G. Then we may represent $t^{\underline{\mu}}$ by the element $\mathrm{diag}(a_1, \dots, a_m, a, a\overline{a}\overline{a}_m^{-1}, \dots, a\overline{a}\overline{a}_1^{-1}) \in T(\breve{F})$, where

$$a_1 = \dots = a_s = \varpi^2, \quad a_{s+1} = \dots = a_m = a = \varpi.$$

See [38, §2.4.2] for these calculations.

We may extend $\{\Lambda_j; j \in J\}$ to a periodic self-dual lattice chain $\mathcal{L}$ by first including all the duals $\hat{\Lambda}_j$, and then all ϖ-multiples of the lattices. The lattice chain $\mathcal{L}$ is naturally indexed by the set $\{kn \pm j; k \in \mathbb{Z}, j \in J\}$; see [39, §1.4].

Upon choosing a suitable F'-basis $e_1, \dots, e_n$ of $W \otimes_{\mathbb{Q}} \mathbb{Q}_p$, every parahoric arises in the above way. Thus in order to verify the axioms, we may fix such a choice of basis and verify the axioms for the standard parahorics $\mathcal{K}_J$.

9.4. We fix $\mathcal{K}^p \subset \mathbf{G}(\mathbb{A}_f^p)$ a sufficiently small compact open subgroup. Then we have the associated Shimura variety $Sh_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X)$ over E. Let v be the unique place of E above p. The completion E_v of E at v is identified with F'.

In order to define integral models, we follow the construction in [39, §1.4], see also [42, Chapter 6]. We consider the moduli functor $\mathscr{S}^{naive}_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X)$, whose set of points valued in an $\mathcal{O}_{F'}$-scheme S is given by the set of isomorphism classes of triples (A, λ, η), where:

- $A = \{A_j\}_{j \in \{kn \pm j; k \in \mathbb{Z}, j \in J\}}$, is an $\mathcal{L}$-set of abelian varieties over S in the sense of [42, Definition 6.5]. In particular each A_j is equipped with a map $i : \mathcal{O}_E \otimes \mathbb{Z}_{(p)} \to \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$. Here $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p.
- λ is a $\mathbb{Q}$-homogeneous polarization of the $\mathcal{L}$-set A, [42, Definition 6.6].
- η is a $\mathcal{K}^p$-level structure

$$\eta : H_1(A, \mathbb{A}_f^p) \cong W \otimes_{\mathbb{Q}} \mathbb{A}_f^p \mod \mathcal{K}^p$$

which respects the bilinear forms on either side up to a constant in $\mathbb{A}_f^{p\times}$.

Moreover A is required to satisfy the determinant condition of [42, Definition 6.9]; in other words we have an equality of polynomial functions

$$\det{}_{\mathcal{O}_S}(b; \mathrm{Lie}(A_j)) = \det{}_L(b; W_0).$$

Here L is a finite extension of $\mathbb{Q}_p$ over which the decomposition 9.1 is defined; see loc. cit. for details. Then $\mathscr{S}^{naive}_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X)$ is representable by a quasi-projective scheme over $\mathcal{O}_{F'}$. Moreover, since the Hasse principle holds for $\mathbf{G}$, the generic fiber of $\mathscr{S}^{naive}_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X)$ is naturally identified with $Sh_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X) \otimes_E F'$. The integral model $\mathscr{S}_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X)$ is defined to be the scheme-theoretic closure of the generic fiber in $\mathscr{S}^{naive}_{\mathcal{K}_J\mathcal{K}^p}(\mathbf{G}, X)$.

For notational convenience, and because the compact open $\mathcal{K}^p \subset G(\mathbb{A}_f^p)$ plays a minor role in what follows, we will usually write $\mathscr{S}_{\mathcal{K}_J}$ (resp. $\mathscr{S}^{naive}_{\mathcal{K}_J}$) for the integral model (resp. naive integral model) model defined above. We write $\overline{\mathscr{S}}_{\mathcal{K}_J}$ for the geometric special fiber of $\mathscr{S}_{\mathcal{K}_J}$, i.e. $\mathscr{S}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{F'}} \mathbf{k}$.

9.5. We now recall and verify the axioms of [20] for these models.

Axiom (1) *Compatibility with change in parahoric: For an inclusion $\mathcal{K}_J \subset \mathcal{K}_{J'}$ of parahorics, there exists a map*

$$\pi_{\mathcal{K}_J, \mathcal{K}_{J'}} : \mathscr{S}_{\mathcal{K}_J} \to \mathscr{S}_{\mathcal{K}_{J'}}$$

which is proper and surjective.

As in [20, §7], it suffices to prove the desired properties when J arises from J' by adding a single element j. Let $j' \in J'$ be maximal with $j' < j$ (or $j' < j + m$ if $j = 0$).

Using the description of the naive integral models as a moduli space, one sees that there exists a natural forgetful map

$$\pi^{naive}_{\mathcal{K}_J, \mathcal{K}_{J'}} : \mathscr{S}^{naive}_{\mathcal{K}_J} \to \mathscr{S}^{naive}_{\mathcal{K}_{J'}}.$$

This induces a map of integral models

$$\pi_{\mathcal{K}_J, \mathcal{K}_{J'}} : \mathscr{S}_{\mathcal{K}_J} \to \mathscr{S}_{\mathcal{K}_{J'}}.$$

For any S-point (A, λ, η) of $\mathscr{S}^{naive}_{\mathcal{K}_{J'}}$, to give a point in the preimage of $\pi_{\mathcal{K}_J,\mathcal{K}_{J'}}$ is to give a subgroup of $A_{j'}[p]$ satisfying certain properties, which define closed conditions. Indeed, it is representable by a closed subscheme of a Hilbert scheme. Thus $\pi^{naive}_{\mathcal{K}_J,\mathcal{K}_{J'}}$ is proper and hence so is $\pi_{\mathcal{K}_J,\mathcal{K}_{J'}}$. Since $\pi_{\mathcal{K}_J,\mathcal{K}_{J'}}$ is surjective on the generic fiber, it has dense image and thus $\pi_{\mathcal{K}_J,\mathcal{K}_{J'}}$ is surjective.

Axiom (2) *Local model diagram: There exists a local model $M^{loc}_{\mathcal{K}_J}$ attached to the triple $(G, \{\mu\}, \mathcal{K}_J)$. This is a projective flat $\operatorname{Spec}\mathcal{O}_{F'}$-scheme equipped with an action of $\mathcal{G}_{J,\mathcal{O}_{F'}} = \mathcal{G}_J \otimes_{\mathbb{Z}_p} \mathcal{O}_{F'}$. Here $\mathcal{G}_J$ is the parahoric group scheme over $\mathbb{Z}_p$ associated to $\mathcal{K}_J$. This local model is required to satisfy the following properties:*

There is a smooth morphism of algebraic stacks

$$\tilde{\lambda}_{\mathcal{K}_J} : \mathscr{S}_{\mathcal{K}_J} \to [M^{loc}_{\mathcal{K}_J}/\mathcal{G}_{J,\mathcal{O}_{F'}}]$$

compatible with change of parahorics (i.e. the maps $\pi_{\mathcal{K}_J,\mathcal{K}_{J'}}$). Moreover the action of $\mathcal{G}_J \otimes_{\mathbb{Z}_p} \mathbf{k}$ on $M^{loc}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{F'}} \mathbf{k}$ has finitely many orbits O_w indexed by $w \in \operatorname{Adm}(\{\mu\})_J$, and O_w lies in the closure of $O_{w'}$ if and only $w \leq w'$ in the partially ordered set $\breve{W}_J \backslash \breve{W} / \breve{W}_J$.

As in [38, §1.4.4] there is a diagram

$$\mathscr{S}^{naive}_{\mathcal{K}_J} \xleftarrow{\pi} \tilde{\mathscr{S}}^{naive}_{\mathcal{K}_J} \xrightarrow{q} M^{naive}_{\mathcal{K}_J}$$

of $\mathcal{O}_{F'}$-schemes, where π is a $\mathcal{G}_{J,\mathcal{O}_{F'}}$-torsor and q is smooth of relative dimension $\dim G$. Here $M^{naive}_{\mathcal{K}_J}$ is the naive local model defined in [42]. It is equipped with an action of $\mathcal{G}_{J,\mathcal{O}_{F'}}$ and its geometric special fiber is a union of $\mathcal{G}_J \otimes_{\mathbb{Z}_p} \mathbf{k}$-orbits index by a subset of $\breve{W}_J \backslash \breve{W} / \breve{W}_J$ containing $\operatorname{Adm}(\{\mu\})_J$. We thus obtain a diagram

$$\mathscr{S}_{\mathcal{K}_J} \xleftarrow{\pi} \tilde{\mathscr{S}}_{\mathcal{K}_J} \xrightarrow{q} M^{loc}_{\mathcal{K}_J},$$

where $M^{loc}_{\mathcal{K}_J}$ is the scheme-theoretic closure of the generic fiber of $M^{naive}_{\mathcal{K}_J}$. Since the coherence conjecture has been proved in [48], the geometric special fiber of $M^{loc}_{\mathcal{K}_J}$ is the union of orbits corresponding to $\operatorname{Adm}(\{\mu\})_J$, see [39, §4.1]. We have a Cartesian diagram of morphisms of stacks:

$$\begin{array}{ccc} \mathscr{S}_{\mathcal{K}_J} & \longrightarrow & [M^{loc}_{\mathcal{K}_J}/\mathcal{G}_{J,\mathcal{O}_{F'}}] \\ \downarrow & & \downarrow \\ \mathscr{S}^{naive}_{\mathcal{K}_J} & \longrightarrow & [M^{naive}_{\mathcal{K}_J}/\mathcal{G}_{J,\mathcal{O}_{F'}}]. \end{array}$$

We obtain a map

$$\lambda^{naive}_{\mathcal{K}_J} : \mathscr{S}^{naive}_{\mathcal{K}_J}(\mathbf{k}) \to \breve{W}_J \backslash \breve{W} / \breve{W}_J,$$

which associates to a point of $\mathscr{S}^{naive}_{\mathcal{K}_J}$ the corresponding orbit of $M^{naive}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{F'}} \mathbf{k}$. It follows from the Cartesian diagram above that $x \in \mathscr{S}^{naive}_{\mathcal{K}_J}(\mathbf{k})$ lies in $\mathscr{S}_{\mathcal{K}_J}(\mathbf{k})$ if and only if $\lambda^{naive}_{\mathcal{K}_J}(x) \in \operatorname{Adm}(\{\mu\})_J$. By restriction, we obtain the map

$$\lambda_{\mathcal{K}_J} : \mathscr{S}_{\mathcal{K}_J}(\mathbf{k}) \to \breve{W}_J \backslash \breve{W} / \breve{W}_J$$

This gives the Kottwitz-Rapoport stratification of $\overline{\mathscr{S}}_{\mathcal{K}_J}$ and we write $KR_{\mathcal{K}_J,w}$ for $\lambda^{-1}_{\mathcal{K}_J} O_w$. The closure relations on $M^{loc}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{F'}} \mathbf{k}$ follow from the embedding of $M^{naive}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{F'}} \mathbf{k}$, and hence $M^{loc}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{F'}} \mathbf{k}$, into an appropriate (equal characteristic) affine flag variety; see [39, 3.3].

Axiom (3) *Newton stratification: There is a map*

$$\delta_{\mathcal{K}_J} : \mathscr{S}_{\mathcal{K}_J}(\mathbf{k}) \to B(G)$$

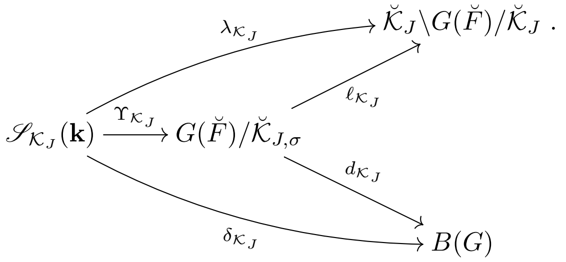
compatible with change in parahorics, and such that for each $[b] \in B(G)$, the fiber of $\delta_{\mathcal{K}_J}$ over $[b]$ is the set of $\mathbf{k}$-rational points of a locally closed subvariety $\overline{\mathscr{S}}_{\mathcal{K}_J,[b]}$ of $\overline{\mathscr{S}}_{\mathcal{K}_J}$. Furthermore, if $\overline{\mathscr{S}}_{\mathcal{K}_J,[b]}$ has non-empty intersection with the closure of $\overline{\mathscr{S}}_{\mathcal{K}_J,[b']} \neq \emptyset$, then $[b] \leq [b']$ in the the partial order on $B(G)$, cf. (2.3).

The verification of this Axiom is the same as in [20, §7] in the Siegel case.

Axiom (4) *Joint stratification: a) There exists a natural map*

$$\Upsilon_{\mathcal{K}_J} : \mathscr{S}_{\mathcal{K}_J}(\mathbf{k}) \to G(\breve{F})/\breve{\mathcal{K}}_{J,\sigma}$$

compatible with changes in the parahoric, such that the following diagram commutes



Here $G(\breve{F})/\breve{\mathcal{K}}_{J,\sigma}$ denotes the quotient of $G(\breve{F})$ by the σ-conjugation action of $\breve{\mathcal{K}}_J$ and $l_{\mathcal{K}_J}$, $d_{\mathcal{K}_J}$ are the natural projection maps.

b) Furthermore, $\operatorname{Im}\Upsilon_{\mathcal{K}_J} = l_{\mathcal{K}_J}^{-1}(\operatorname{Im}(\lambda_{\mathcal{K}_J}))$.

c) For $\mathcal{K}_J \subset \mathcal{K}_{J'}$ and any element $y \in \operatorname{Im}(\Upsilon_{\mathcal{K}_J})$ with image $y' \in G(\breve{F})/\breve{\mathcal{K}}_{J,\sigma}$, the natural map

$$\pi_{\mathcal{K}_J,\mathcal{K}_{J'}} : \Upsilon_{\mathcal{K}_J}^{-1}(y) \to \Upsilon_{\mathcal{K}_{J'}}^{-1}(y')$$

is surjective with finite fibers.

a) & c) The verifications are again the same as in [20, §7] in the Siegel case. We recall how the map $\Upsilon_{\mathcal{K}_J} : \mathscr{S}_{\mathcal{K}_J}(\mathbf{k}) \to G(\breve{F})/\breve{\mathcal{K}}_{J,\sigma}$ is defined. It is induced from a map

$$\Upsilon_{\mathcal{K}_J} : \mathscr{S}_{\mathcal{K}_J}^{\mathrm{naive}}(\mathbf{k}) \to G(\breve{F})/\breve{\mathcal{K}}_{J,\sigma}.$$

For $x \in \mathscr{S}_{\mathcal{K}_J}^{naive}(\mathbf{k})$ corresponding to (A, λ, η), we let $\mathbb{D}_i$ be the Dieudonné module of $A_i[p^\infty]$ and let N be the common rational Dieudonné module. The $\mathbb{D}_i$ form an $\mathcal{O}_{\breve{F}'}$-lattice chain inside N. Then by [42, App. to Chapter 3] there is a $\breve{F}'$-linear isomorphism

$$N \cong W \otimes_{\mathbb{Q}} \breve{F}$$

taking $\mathbb{D}_i$ to $\Lambda_i \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}$ for all $i \in \{kn \pm j; k \in \mathbb{Z}, j \in J\}$ and respecting the forms up to a scalar. Under this isomorphism, the Frobenius is given by $\delta(\mathrm{id}_W \otimes \sigma)$ for some

$$\delta \in G(\breve{F}) \tag{9.2}$$

which is well-defined up to σ-conjugation by $\breve{\mathcal{K}}_J$. It can be checked from the definition of the embedding of $M_{\mathcal{K}_J}^{naive} \otimes_{\mathcal{O}_{\breve{F}'}} \mathbf{k}$ into the affine flag variety that $\delta \in \breve{\mathcal{K}}_J \dot{w} \breve{\mathcal{K}}_J$ for $w \in \breve{W}_J \backslash \breve{W}/\breve{W}_J$ if and only if $\lambda_{\mathcal{K}_J}^{naive}(x) = w$.

b) By [20, Lemma 3.11], we may assume $J = \{0, \ldots, m\}$. By Axiom (5) below and [20, Theorem 4.1], the image of the map $\lambda_{\mathcal{I}} : \mathscr{S}_{\mathcal{I}}(\mathbf{k}) \to \breve{W}$ is $\mathrm{Adm}(\{\mu\})$. Let $\overline{b} \in l_{\mathcal{I}}^{-1}(w)$ for some $w \in \mathrm{Adm}(\{\mu\})$, and fix a lift $b \in \breve{\mathcal{I}} \dot{w} \breve{\mathcal{I}}$. It follows that $1 \in X(\{\mu\}, b)$ and so $[b] \in B(G, \{\mu\})$ by Theorem 2.3. By [20, Theorem 5.4], $\delta_{\mathcal{I}}$ is surjective, hence there exists $x \in \mathscr{S}_{\mathcal{I}}(\mathbf{k})$ such that after fixing an identification $N \cong W \otimes_{\mathbb{Q}} \breve{F}$ as above, the element $\delta \in G(\breve{F})$ associated to x as in (9.2) satisfies $[\delta] = [b]$ in $B(G)$.

We let $\mathbb{D}_j \subset N, j \in \mathbb{Z}$ be the Dieudonné modules as in Axiom (4) above. For any $g \in G(\breve{F})$, the relative position of the lattices $g\mathbb{D}_j$ and $\delta\sigma(g)\mathbb{D}_j$ for $j \in \mathbb{Z}$ only depend on the element $x \in \breve{W}$ such that $g^{-1}\delta\sigma(g) \in \breve{\mathcal{I}} \dot{x} \breve{\mathcal{I}}$ (here we use the identification $N \cong W \otimes_{\mathbb{Q}} \breve{F}$ to view $G_{\breve{F}} \subset \mathrm{End}_{F'}(N)$). Now let $g \in G(\breve{F})$ such that $g^{-1}\delta\sigma(g) = b \in \breve{\mathcal{I}} \dot{w} \breve{\mathcal{I}}$. Let $x' \in \lambda_{\mathcal{I}}^{-1}(w)$ and N', $\mathbb{D}'_j$, δ' the corresponding objects associated to x' as in Axiom (4); then $\delta' \in \breve{\mathcal{I}} \dot{w} \breve{\mathcal{I}}$. By construction, for $j \in \mathbb{Z}$ the lattices $\mathbb{D}'_j$ satisfy

$$p\mathbb{D}'_j \subset \delta'\mathbb{D}'_j \subset \mathbb{D}'_j.$$

It follows that $g\mathbb{D}_j$ also satisfy

$$pg\mathbb{D}_j \subset \delta\sigma(g)\mathbb{D}_j \subset g\mathbb{D}_j$$

for $j \in \mathbb{Z}$. As in the proof of Proposition 9.2 below, this produces a triple $(gA, g\lambda, \eta)$ which corresponds to a point $gx \in \mathscr{S}_{\mathcal{I}}(\mathbf{k})$, and we have $\Upsilon_{\mathcal{I}}(gx) = g^{-1}\delta\sigma(g) = b$ in $G(\breve{F})/\breve{\mathcal{I}}_\sigma$.

Axiom (5) *Basic non-emptiness: Recall $\tau_{\{\mu\}}$ is the minimal element in* $\mathrm{Adm}(\{\mu\})$. *Then $KR_{\mathcal{I},\tau_{\{\mu\}}}$ intersects with every geometric connected component of $\overline{\mathscr{S}}_{\mathcal{I}}$.*

By [24], the basic locus is nonempty. Let $x \in \mathscr{S}_{\mathcal{K}_J}(\mathbf{k})$ be a point which lies in the basic locus. This corresponds to a triple (A, λ, η) as above. As in (9.2) we obtain a $\delta \in G(\breve{F})$ where $[\delta] = [b]_{basic} \in B(G, \{\mu\})$ the unique basic σ-conjugacy class. A simple calculation as in [38, §2.4.2] shows that $\tau_{\{\mu\}}$ is represented by the element $\dot{\tau}_{\{\mu\}} = \mathrm{diag}(\varpi, \dots, \varpi) \in T(\breve{F})$.

Let $g \in X_{\tau_{\{\mu\}}}(\delta)$; such an element exists since δ is basic. Then $g^{-1}\delta\sigma(g) \in \breve{\mathcal{I}}\dot{\tau}_{\{\mu\}}\breve{\mathcal{I}}$. Since $\dot{\tau}_{\{\mu\}}$ is central, we have $\delta\sigma(g) \in g\dot{\tau}_{\{\mu\}}\breve{\mathcal{I}}$. Let $\mathbb{D}_j$ denote the Dieudonné module of $\mathscr{G}_j = A_j[p^\infty]$ for $j \in \mathbb{Z}$. Then we have

$$pg\mathbb{D}_j \subset \delta\sigma(g)\mathbb{D}_j = \dot{\tau}_{\{\mu\}}g\mathbb{D}_j \subset g\mathbb{D}_j.$$

Thus $g\mathbb{D}_j$ corresponds to a p-divisible $g\mathscr{G}_j$ which is isogenous to $A_j[p^\infty]$. Since g is $\breve{F}'$-linear, the action of $\mathcal{O}_{F'}$ on $\mathscr{G}_j$ extends to an action on $g\mathscr{G}_j$. Therefore $g\mathscr{G}_j$ corresponds to an abelian variety, denoted gA_j, equipped with an action of $\mathcal{O}_E$. We thus obtain an $\mathcal{L}$-set of abelian varieties $gA_j, j \in \mathbb{Z}$. Since $\mathbf{G} \subset \mathbf{GS}p(W, \psi)$, the polarization λ extends to a polarization $g\lambda$ of the $\mathcal{L}$-set gA_j. The prime to p level structure η extends to gA_j; we thus obtain a triple $(gA_j, g\lambda, \eta)$ as in §9.4. The determinant condition holds since $\mathrm{Lie}A_j \cong \dot{\tau}_{\{\mu\}}g\mathbb{D}_j/pg\mathbb{D}_j \cong \dot{\tau}\mathbb{D}_j/p\mathbb{D}_j$. Hence we obtain a $\mathbf{k}$-point of $gx \in \mathscr{S}^{naive}_{\mathcal{I}}(\mathbf{k})$. It follows from the pullback diagram in the verification of Axiom (2) that $gx \in \mathscr{S}_{\mathcal{I}}(\mathbf{k})$, and by construction we have $\lambda_{\mathcal{I}}(gx) = \tau_{\{\mu\}}$.

To show

$$\lambda_{\mathcal{I}}^{-1}(\tau_{\{\mu\}}) \to \pi_0(\mathscr{S}_{\mathcal{I}} \otimes_{\mathcal{O}_{F'}} \mathbf{k})$$

is surjective, we follow the same strategy as in [20]. Indeed in this case a good theory of compactifications exists by [33]. See also [30]. We thus have an isomorphism

$$\pi_0(\mathscr{S}_{\mathcal{I}\mathcal{K}^p}(\mathbf{G}, X) \otimes_{\mathcal{O}_{F'}} \mathbf{k}) \cong \pi_0(Sh_{\mathcal{I}\mathcal{K}^p}(\mathbf{G}, X) \otimes_E \mathbb{C}).$$

See [33, 4.11].

By [22, Lemma 2.2.5], $\mathbf{G}(\mathbb{A}_f^p)$ acts transitively on $\varprojlim_{\mathcal{K}^p} \pi_0(Sh_{\mathcal{I}\mathcal{K}^p}(\mathbf{G}, X) \otimes_{\mathcal{O}_{E_{(v)}}} \mathbb{C})$ and hence on $\varprojlim_{\mathcal{K}^p} \pi_0(\mathscr{S}_{\mathcal{I}\mathcal{K}^p}(\mathbf{G}, X) \otimes_{\mathcal{O}_{E_{(v)}}} \mathbf{k})$. Thus for $x \in \lambda_{\mathcal{I}}^{-1}(\tau_{\{\mu\}})$ corresponding to (A, λ, η) we may modify the prime-to-p level structure to pass to any connected component. □

9.6. In fact in this case we have the following stronger result. We write $\mathscr{S}_{\mathcal{K}_J, [b]_{basic}} = \delta_{\mathcal{K}_J}^{-1}([b]_{basic})$, a closed subvariety of the special fiber $\mathscr{S}_{\mathcal{K}_J} \otimes_{\mathcal{O}_{\breve{F}'}} \mathbf{k}$.

Proposition 9.2. *Let $x \in \mathscr{S}_{\mathcal{I}, [b]_{basic}}(\mathbf{k})$. Then there exists a point x' such that x and x' are connected in $\mathscr{S}_{\mathcal{I}}^{[b]_{basic}}$ and $\lambda_{\mathcal{I}}(x') = \tau_{\{\mu\}}$.*

Proof. Let (A, λ, η) denote the triple corresponding to x and $\delta \in G(\breve{F})$ be the element associated to x in (9.2). By Theorem 4.1, if $Y \subset X(\{\mu\}, \delta)$ denotes the connected component containing 1, we have $Y \cap X_{\tau_\mu}(\delta) \neq \emptyset$. By Theorem A.4, there exists $g_0 \in X_{\tau_\mu}(\delta)$ such that $1 \sim g_0$ in the sense of Definition A.3.

Let $g \in X(\{\mu\}, \delta)(\mathscr{R})$ (cf. §A.1), where $\mathscr{R}$ is a frame for a smooth integral $\mathbf{k}$-algebra R (see Definition A.1). Let $\mathbb{D}_j$ denote the Dieudonné module associated to $\mathscr{G}_j = A_j[p^\infty]$. By [47, §6.9] (see also [23, Lemma 1.4.6]), upon replacing $\mathscr{R}$ by $\mathscr{R}'_n$, $g\mathbb{D}_j$ corresponds to a chain of p-divisible groups $g\mathscr{G}_j$ over R with an identification $\mathbb{D}(g\mathscr{G}_j)(\mathscr{R}) \cong g\mathbb{D}_j$. Here, $\mathscr{R}'$ denotes the canonical frame for an étale covering $R \to R' = \mathscr{R}'/p\mathscr{R}'$, and $\mathscr{R}'_n$ denotes the $\mathcal{O}_{\breve{F}}$-algebra with underlying ring $\mathscr{R}'$ and whose structure map is σ^n.

The $g\mathscr{G}_j$ correspond to abelian varieties gA_j over R together with a quasi-isogeny $gA_j \to A_j \otimes_{\mathbf{k}} R$. As before gA_j is equipped with an action of $\mathcal{O}_E$, a polarization $g\lambda$ and prime to p level structure η. We thus obtain a triple $(gA, g\lambda, \eta)$ over R. To check the determinant condition, we have to check the equality of two polynomials with coefficients over R. It thus suffices to check this for all $\mathbf{k}$-points $s : R \to \mathbf{k}$.

Let $g(s) \in G(\breve{F})$ denote the point induced by the unique σ-equivariant lift $\mathscr{R} \to \mathcal{O}_{\breve{F}}$ of s. The Dieudonné module of $gA_j[p^\infty]$ at s is given by $g(s)\mathbb{D}_j \subset \mathbb{D}_j \otimes \breve{F}$, and the Lie algebra $\mathrm{Lie}(g(s)A_j)$ of gA_j at s can be identified with $\delta\sigma(g(s))\mathbb{D}_j/pg(s)\mathbb{D}_j \cong g(s)^{-1}\delta\sigma(g(s))\mathbb{D}_j/p\mathbb{D}_j$.

By definition of g, we have $g(s)^{-1}\delta\sigma(g(s)) \in \breve{\mathcal{I}}\dot{w}\breve{\mathcal{I}}$ for some $\dot{w} \in \mathrm{Adm}(\{\mu\})$. The isomorphism class of the $\mathcal{O}_E$-module $\mathrm{Lie}(g(s)A_j)$ depends only on w. Write $M_{j,w}$ for the $\mathcal{O}_E$-module $\dot{w}(\Lambda_j \otimes_{\mathbb{Z}_p} \mathcal{O}_{\breve{F}})/p(\Lambda_j \otimes_{\mathbb{Z}_p} \mathcal{O}_{\breve{F}})$. We have an isomorphism $\mathrm{Lie}(g(s)A_j) \cong M_{j,w}$.

Since $\lambda_{\mathcal{I}}$ is surjective, we may take $x' \in \lambda_{\mathcal{I}}^{-1}(w)$ corresponding to a triple (A', λ', η'). Then by definition we have $\mathrm{Lie}(A'_j) \cong M_{j,w}$. In particular for any $a \in \mathcal{O}_E$, we have the following equalities of polynomial functions:

$$\det_{\mathbf{k}}(a, \mathrm{Lie}(g(s)A_j)) = \det_{\mathbf{k}}(a, M_{j,w}) = \det_{\mathbf{k}}(a, \mathrm{Lie}(A'_j)) = \det_E(a, V_0).$$

It follows that $\det_R(a, \mathrm{Lie}(gA)) = \det_L(a, W_0)$, and so the triple (gA, λ, η) corresponds to a morphism

$$i_R : \mathrm{Spec}\, R \to \mathscr{S}_{\mathcal{I}}^{naive}.$$

For each $s : R \to \mathbf{k}$ this map factors through $\mathscr{S}_{\mathcal{I}}$, hence i_R factors through $\mathscr{S}_{\mathcal{I}}$ and in fact through $\mathscr{S}_{\mathcal{I},[b]_{basic}}$. Since $1 \sim g_0$, we may find a sequence of maps i_R which connects x to a point x' in $\lambda_{\mathcal{I}}^{-1}(\tau_{\{\mu\}})$. □

Appendix A. Various definitions of connected components

In this section we relate the notions of connected components in this paper to that studied in [5]. Although the algebraic structure on the affine Deligne-Lusztig varieties was not known at the time, the authors were still able to define a notion of connected components. That paper deals with the case of unramified groups and hyperspecial level structure; however, it is relatively straightforward to generalize its notion of connected components to our context (i.e. parahoric level structure). Although it is much more natural to talk about connected components in the Zariski topology, nevertheless the notion in [5] is useful for applications to Shimura varieties and Rapoport-Zink spaces. Therefore it is useful to know that the two notions coincide.

We recall some of the definitions of [5]. For simplicity, we assume $F = \mathbb{Q}_p$ and we only consider Iwahori level; the proofs can be adapted for F any finite extension of $\mathbb{Q}_p$ and for any parahoric with minor modifications. Thus let $\mathcal{G}$ denote the group scheme over $\mathcal{O}_F$ associated to the σ-invariant Iwahori $\breve{\mathcal{I}}$. Recall that $\mathbf{k}$ is an algebraic closure of $\mathbb{F}_p$.

Definition A.1. Let R be $\mathbf{k}$-algebra. A *frame* for R is a p-torsion free, p-adically complete and separated $\mathcal{O}_{\breve{F}}$-algebra $\mathscr{R}$ equipped with an isomorphism $R \cong \mathscr{R}/p\mathscr{R}$ and a lift (again denoted σ) of the Frobenius σ on R.

Let R be as above and fix a frame $\mathscr{R}$ for R. We write $\mathscr{R}_{\breve{F}}$ for $\mathscr{R}[\frac{1}{p}]$. If κ is any perfect field of characteristic p and $s : R \to \kappa$ is a map, then there is a unique σ-equivariant map $\mathscr{R} \to W(\kappa)$ lifting s. By abuse of notation, this lifting will also be denoted by s.

Let $g \in G(\mathscr{R}_{\breve{F}})$. For $C \subset \breve{W}$, we write

$$S_C(g) = \bigcup_{w \in C} \{s \in \mathrm{Spec}\, R; s(g) \in \mathcal{G}(W(\mathbf{k}(s)))\dot{w}\mathcal{G}(W(\mathbf{k}(s)))\},$$

where $\mathbf{k}(s)$ is an algebraic closure of residue field $k(s)$ of s. Note that this only depends on the image of $g \in G(\mathscr{R}_{\breve{F}})/\mathcal{G}(\mathscr{R})$, hence we can define $S_C(g)$ for any element of $g \in G(\mathscr{R}_{\breve{F}})/\mathcal{G}(\mathscr{R})$. For $b \in G(\breve{F})$, we define the set

$$X_C(b)(\mathscr{R}) = \{g \in G(\mathscr{R}_{\breve{F}})/\mathcal{G}(\mathscr{R}); S_C(g^{-1}b\sigma(g)) = \mathrm{Spec}\, R\}. \tag{A.1}$$

When $C = \mathrm{Adm}(\{\mu\})$ we write $X(\{\mu\}, b)(\mathscr{R})$ for $X_C(b)(\mathscr{R})$.

Remark A.2. This is the definition given in [23, §1.2.9], which is slightly different from the definition in [5]. It is possible to define a version of the mixed characteristic affine flag variety as in [5] in our context using $\mathcal{G}$-torsors. However using that $\mathcal{G}$-torsors are étale locally trivial, it can be shown that "étale locally" on $\mathscr{R}$, these notions coincide. In particular, the connected components of $X_C(b)$ will coincide. Indeed any path connecting g_0 and g_1 using the definition in [5] gives a (sequence of) paths using our definition upon passing to an étale cover.

For $R = \mathbf{k}$, we write $X_C(b) = X_C(b)(W(\mathbf{k}))$, this is compatible with the definitions in Section 2.1.

Note that any $g \in X_C(b)(\mathscr{R})$ defines an R^{perf}-point of $X_C(b)$ (considered as a perfect scheme). Indeed, let $\mathscr{R}^\infty = \varinjlim \mathscr{R}$, where the transition maps are given by σ; thus, $\mathscr{R}^\infty$ is a flat $\mathcal{O}_{\breve{F}}$-algebra lifting R^{perf}. Then if we denote by $\widehat{\mathscr{R}^\infty}$ the p-adic completion of $\mathscr{R}^\infty$, we have an isomorphism $\widehat{\mathscr{R}^\infty} \cong W(R^{perf})$ since $W(R^{perf})$ is the unique p-adically complete flat $\mathbb{Z}_p$-algebra lifting R^{perf} and $\widehat{\mathscr{R}^\infty}$ gives such a lifting. The composition $\mathscr{R} \to \widehat{\mathscr{R}^\infty} \cong W(R^{perf})$ induces a point $g \in G(W(R^{perf})[\frac{1}{p}])/\mathcal{G}(W(R^{perf}))$ and hence an element, also denoted g, of $\mathcal{FL}(R^{perf})$. The conditions defining $X_C(b)(\mathscr{R})$ imply that the corresponding point in $\mathcal{FL}$ lies in $X_C(b)$, since $\operatorname{Spec} R$ and $\operatorname{Spec} R^{perf}$ have the same points.

Definition A.3. For $g_0, g_1 \in X(\{\mu\}, b)$ and R a smooth $\mathbf{k}$-algebra with connected spectrum and frame $\mathscr{R}$, we say g_0 *is connected to* g_1 *via* R if there exists $g \in X(\{\mu\}, b)(\mathscr{R})$ and two $\mathbf{k}$-points s_0, s_1 of $\operatorname{Spec} R$ such that $s_0(g) = g_0$ and $s_1(g) = g_1$.

We write $\sim$ for the equivalence relation on $X(\{\mu\}, b)$ generated by the relation $g_0 \sim g_1$ if g_0 is connected to g_1 via some R as above, and we write $\pi_0'(X(\{\mu\}, b))$ for the set of equivalence classes.

Theorem A.4. *Let $g_0, g_1 \in X(\{\mu\}, b)$. Then $g_0 \sim g_1$ if and only if g_0 is connected to g_1 in the Zariski topology. In particular*

$$\pi_0(X(\{\mu\}, b)) = \pi_0'(X(\{\mu\}, b)).$$

A.1. **Proof of Theorem A.4.** If $g_0 \sim g_1$, then g_0 and g_1 are connected in the Zariski topology. Indeed, without loss of generality we may assume g_0 is connected to g_1 via some R, and the above construction gives a $\operatorname{Spec} R^{perf}$-point in $X(\{\mu\}, b)$ which connects the two points. The rest of this section will be devoted to proving the converse.

Suppose g_0 and g_1 are connected in the Zariski topology. Since $\mathcal{FL}$ is an increasing union of perfections of projective schemes, we may assume $g_0, g_1 \in S^{perf} \subset \mathcal{FL}$, where S^{perf} is the perfection of a projective scheme S over $\mathbf{k}$. Considering g_0 and g_1 as points on S, we can find a morphism $C \to S$, where C is a smooth curve over $\mathbf{k}$, whoxse image contains g_0 and g_1, or at least we can connect them up with finitely many such curves. Taking affine open covers of C, we may reduce to the case $C = \operatorname{Spec} R$, for a smooth $\mathbf{k}$-algebra R, and where g_0 and g_1 are in the image of C^{perf} in $\mathcal{FL}$.

We need to find a frame $\mathscr{R}$ for R, and an element $g \in X(\{\mu\}, b)(\mathscr{R}) \subset G(\mathscr{R}_{\breve{F}})/\mathcal{G}(\mathscr{R})$. which connects up the two points g_0 and g_1. By assumption, the curve $C \in X(\{\mu\}, b)(R^{perf}) \subset \mathcal{FL}(R^{perf})$ connects g_0 and g_1. Upon passing to an fpqc cover, we may assume C^{perf} comes from a point $g \in G(W(R^{perf})[\frac{1}{p}])$.

Now let $\mathscr{R}$ be any frame for R. Indeed a frame exists since we may choose any p-adically complete $W(\mathbf{k})$-algebra $\mathscr{R}$ lifting R. Then all obstructions to lifting σ lie in positive degree coherent cohomology and hence vanish. As before we have an isomorphism $\widehat{\mathscr{R}^\infty} \cong W(R^{perf})$. We would like to show $g \in G(\widehat{\mathscr{R}^\infty_{\breve{F}}})/\mathcal{G}(\widehat{\mathscr{R}^\infty})$ arises from an element of $G(\mathscr{R}^\infty_{\breve{F}})/\mathcal{G}(\mathscr{R}^\infty)$. This follows from the more general Proposition A.5 below.

Proposition A.5. *Let $\mathcal{R}$ be a flat $\mathcal{O}_{\breve{F}}$-algebra such that $\mathcal{R}/p\mathcal{R}$ is an integral domain and let $\mathscr{R}$ be its p-adic completion. Suppose every element $r \in \mathcal{R}$ whose reduction mod p is a unit is itself a unit. Then for any flat affine algebraic group $\mathcal{G}/\mathcal{O}_{\breve{F}}$ with generic fiber G, the natural map*

$$G(\mathcal{R}_{\breve{F}})/\mathcal{G}(\mathcal{R}) \to G(\mathscr{R}_{\breve{F}})/\mathcal{G}(\mathscr{R})$$

is a bijection.

Granting this proposition we may prove the Theorem. Indeed $\mathscr{R}^\infty$ is flat over $\mathcal{O}_{\breve{F}}$ since it is a direct limit of flat $\mathcal{O}_{\breve{F}}$-algebras. Also, for any $r \in \mathscr{R}^\infty$ which reduces to a unit in $\mathscr{R}^\infty/p\mathscr{R}^\infty$, we have $r \in \mathscr{R}_n$, where $\mathscr{R}_n$ is the ring $\mathscr{R}$ regarded as an $\mathcal{O}_{\breve{F}}$ algebra via the map σ^n. Increasing n if necessary we may assume the image of r in $\mathscr{R}_n/p\mathscr{R}_n$ is a unit, hence since $\mathscr{R}_n$ is p-adically complete, $r \in \mathscr{R}_n^\times \subset \mathscr{R}^{\infty,\times}$. Thus we may take $\mathscr{R}^\infty$ as $\mathcal{R}$ in the above proposition and hence $g \in G(\widehat{\mathscr{R}^\infty_{\breve{F}}})/\mathcal{G}(\widehat{\mathscr{R}^\infty})$ arises from an element $h' \in G(\mathscr{R}^\infty_{\breve{F}})$. Since $\mathcal{G}$ is finite type over $\mathcal{O}_{\breve{F}}$, the element h' descends to an element $h \in G(\mathscr{R}_n)$ for some n. As $\mathscr{R}_n$ is a frame for the smooth $\mathbf{k}$-algebra R_n, the element h connects the points g_0 and g_1 in $X(\{\mu\}, b)$; hence $g_0 \sim g_1$. Theorem A.4 is now proved modulo Proposition A.5. ☐

Proof of Proposition A.5. We first consider the case of GL_n. Since $\mathcal{R}$ is p-torsion free, we have $\mathcal{R}_{\breve{F}}/\mathcal{R} \to \mathscr{R}_{\breve{F}}/\mathscr{R}$ is a bijection (cf. [1]). Hence the natural map

$$GL_n(\mathcal{R}_{\breve{F}})/GL_n(\mathcal{R}) \to GL_n(\mathscr{R}_{\breve{F}})/GL_n(\mathscr{R})$$

is an injection: If $A, B \in GL_n(\mathcal{R}_{\breve{F}})$ and $C \in GL_n(\mathscr{R})$ is such that $A = BC$, we have $B^{-1}A = C \in GL_n(\mathcal{R}_{\breve{F}}) \cap GL_n(\mathscr{R}) \subset \mathrm{Mat}_n(\mathcal{R})$. We have $\det(C) \in \mathcal{R}_{\breve{F}}^\times \cap \mathscr{R}^\times = \mathcal{R}^\times$ where the equality follows by our assumption on $\mathcal{R}$; hence $C \in GL_n(\mathcal{R})$.

To show surjectivity, let $g \in GL_n(\mathscr{R}_{\breve{F}})$. Let $s \in \mathbb{N}$ be such that $g, g^{-1} \in \frac{1}{p^s}\mathrm{Mat}_n(\mathscr{R})$. Then for any $m > 0$, there exists $h \in \frac{1}{p^s}\mathrm{Mat}_n(\mathcal{R})$ such that

$$g - h = \delta \in p^m\mathrm{Mat}_n(\mathscr{R}).$$

For m sufficiently large, we have

$$g^{-1}h = 1 - g^{-1}\delta \in 1 + p\mathrm{Mat}_n(\mathscr{R}) \subset GL_n(\mathscr{R})$$

and hence $h \in GL_n(\mathcal{R}_{\breve{F}})$ since $\det(h) \in \mathcal{R}_{\breve{F}} \cap \mathscr{R}_{\breve{F}}^\times = \mathcal{R}_{\breve{F}}^\times$. We have $h = g(1 - g^{-1}\delta)$, and hence for m sufficiently large, the image of h in $GL_n(\mathscr{R}_{\breve{F}})/GL_n(\mathscr{R})$ is equal to g.

For the case of general G, we will need the following result of [2, Theorem 2.1.5.5].

Lemma A.6. *Let Y be a flat $\mathcal{O}_{\breve{F}}$-scheme. Let $\mathcal{F}$ denote the category of exact, faithful tensor functors from representations of $\mathcal{G}$ on finite free $\mathcal{O}_{\breve{F}}$-modules to vector bundles on Y.*

If P is a $\mathcal{G}$-bundle on Y, and V is a representation of $\mathcal{G}$ on a finite free $\mathcal{O}_{\breve{F}}$ module, write $F_P(V) = \mathcal{G}\backslash(P \times V)$. Then $P \mapsto F_P$ is an equivalence between the category of $\mathcal{G}$-bundles on Y, and the category $\mathcal{F}$.

From an element $g \in G(\mathscr{R})$, we can construct the pair (P, τ) consisting of the trivial $\mathcal{G}$-torsor P and a trivialization τ of P over $\mathscr{R}_{\breve{F}}$. The set of isomorphism classes of such objects can be identified with $G(\mathscr{R}_{\breve{F}})/\mathcal{G}(\mathscr{R})$. We will show that (P, τ) descends to a $\mathcal{G}$-torsor X over $\mathcal{R}$ equipped with trivialization over $\mathcal{R}_{\breve{F}}$.

By Lemma A.6, the pair (P, τ) gives rise to an exact faithful tensor functor F_P which associates to a representation of $\mathcal{G}$ on a finite free $\mathcal{O}_{\breve{F}}$-module V, the vector bundle $F_P(V) = \mathcal{G}\backslash P \times V$ on $\mathscr{R}$, together with an isomorphism

$$\tau_V : V \otimes_{\mathcal{O}_{\breve{F}}} \mathscr{R}_{\breve{F}} \cong F_P(V) \otimes_{\mathscr{R}} \mathscr{R}_{\breve{F}}.$$

Since P is trivial, $F_P(V)$ is a trivial vector bundle, hence the pair $(F_P(V), \tau_V)$ corresponds to an element $g \in GL_n(\mathscr{R}_{\breve{F}})/GL_n(\mathscr{R})$. By the case of GL_n proved above, this corresponds to an element $h \in GL_n(\mathcal{R}_{\breve{F}})/GL_n(\mathcal{R})$ and hence a pair $(F'_P(V), \tau'_V)$ consisting of the trivial vector bundle over $\mathcal{R}$ together with a trivialization τ'_V over $\mathcal{R}_{\breve{F}}$. Then $F'_P(V)$ is an exact faithful tensor functor from the category of representations of $\mathcal{G}$ on finite free $\mathcal{O}_{\breve{F}}$-modules to vector bundles on $\mathcal{R}$, hence corresponds to a $\mathcal{G}$-torsor X over $\mathcal{R}$, and τ'_V gives a trivialization of X over $\mathcal{R}_{\breve{F}}$. Since $F'_P(V)$ is trivial for all V, we have X is trivial. Hence the pair (X, τ'_V) corresponds to an element $h \in G(\mathcal{R}_{\breve{F}})/G(\mathcal{R})$ which maps to g. □

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