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AVERAGE CASE ANALYSIS OF ON-LINE BIN PACKING ALGORITHMS

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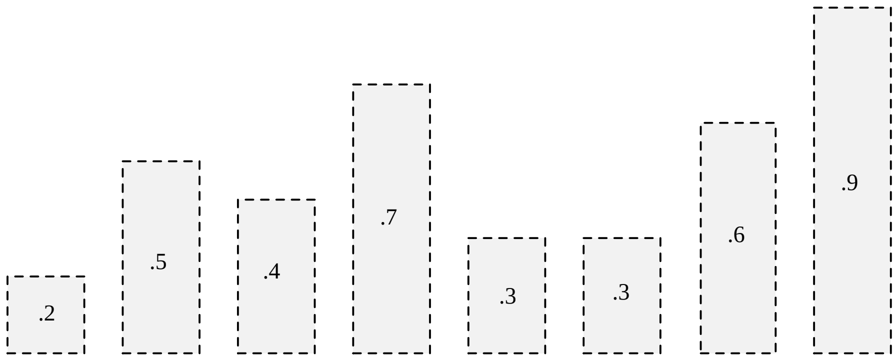
OUTLINE

1. Bin packing, an introduction
2. Discrete uniform distributions and the $\{8, 11\}$ conjecture for Best Fit
3. A Markov chain model and how to deal with it
4. Potential functions and the conjectures they resolve
5. Average case analysis of First Fit Decreasing

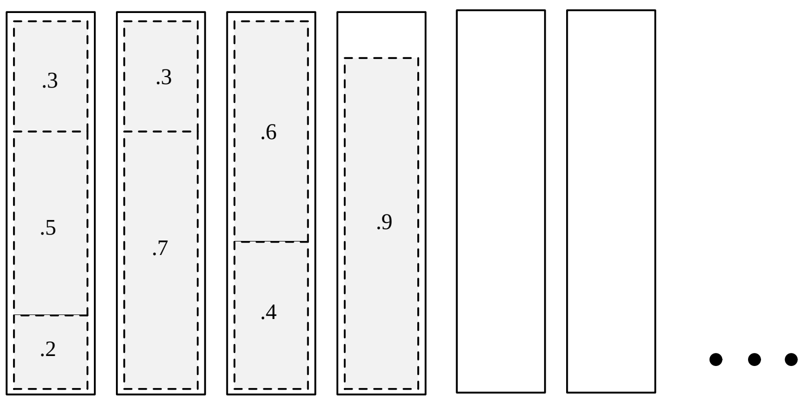
BIN PACKING

GIVEN: a list of n items L, with sizes

$a_1, \ldots, a_n \in (0, 1]$,



FIND: A packing of the items into a minimum number of unit capacity bins.



$$\mathbf{OPT}(L) = 4$$

DEFINITIONS

Given a list of items L with item sizes in $(0, 1]$ and an algorithm A,

$A(L)$ = number of unit capacity bins used when A packs L.

$\mathrm{OPT}(L)$ = minimum possible number of unit capacity bins into which L can be packed.

$s(L)$ = sum of sizes of items in L.
(note $\mathrm{OPT}(L) \geq s(L)$)

$w_A(L) = A(L) - s(L)$ is the 'waste' of the packing of L by A.

HOW DIFFICULT IS BIN-PACKING?

THEOREM (Karp 1972)

To find the optimal packing (off-line) is NP-hard.

THEOREM (Karmarkar and Karp 1982)

There is a $O(n^{10})$ polynomial-time off-line algorithm A such that for all L,

$$A(L) \leq \mathrm{OPT}(L) + \log^2(\mathrm{OPT}(L))$$

THEOREM (Brown 1980, Liang 1980)

Given any on-line algorithm A and $n > 0$ there is a list L with $|L| \geq n$ such that

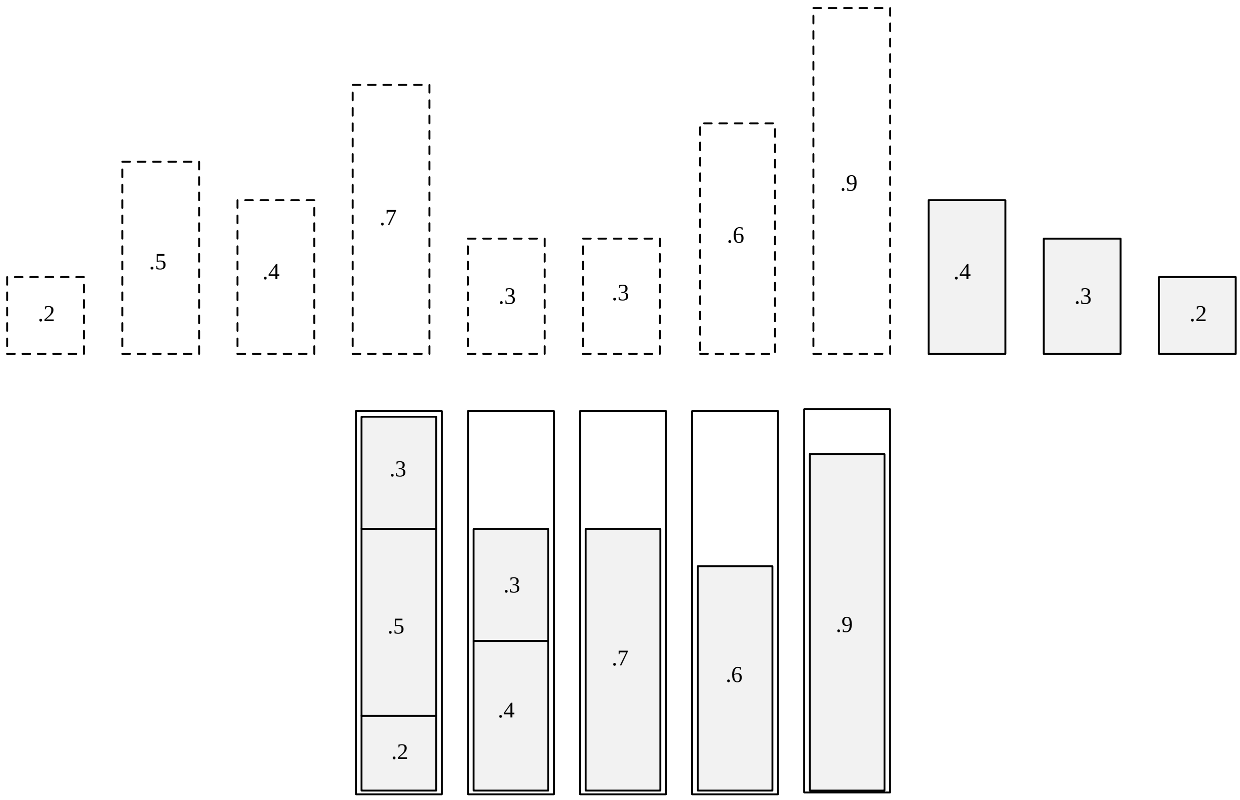
$$A(L) \geq 1.536\,\mathrm{OPT}(L)$$

(Van Vliet, 1991, 1.536→1.540.)

ALGORITHMS

FIRST FIT: place next item in lowest indexed bin into which it fits (FF).

This is an 'on-line' algorithm.



BEST FIT: place next item in smallest gap into which it fits (BF).

FIRST (BEST) FIT DECREASING: first sort the list into nonincreasing order (FFD/BFD).

This is an 'off-line' algorithm.

WORST CASE RESULTS

THEOREM (Garey, Graham, Ullman, 1972)

For all lists L,

$$FF(L), BF(L) \leq \frac{17}{10}\mathrm{OPT}(L) + 1$$

For all lists L with sizes in $(0, 1/2]$,

$$FF(L), BF(L) \leq \frac{3}{2}\mathrm{OPT}(L) + 1$$

For all lists L with sizes in $(0, 1/4]$,

$$FF(L), BF(L) \leq \frac{5}{4}\mathrm{OPT}(L) + 1$$

THEOREM (Johnson, 1973)

For all lists L,

$$FFD(L), BFD(L) \leq \frac{11}{9}\mathrm{OPT}(L) + 4$$

For all lists L with sizes in $(0, 1/2]$,

$$FFD(L), BFD(L) \leq \frac{71}{60}\mathrm{OPT}(L) + C$$

For all lists L with sizes in $(0, 1/4]$,

$$FFD(L), BFD(L) \leq \frac{23}{20}\mathrm{OPT}(L) + C$$

CONTINUOUS AND DISCRETE UNIFORM DISTRIBUTIONS

Continuous uniform:

$$U(0, u], \quad u \leq 1$$

sizes uniformly drawn from

$$(0, u].$$

Discrete uniform:

$$U\{j, k\}, \quad j \leq k$$

sizes uniformly drawn from

$$\{1/k, 2/k, \ldots, j/k\}.$$

AVERAGE CASE PERFORMANCE OF NF UNDER $U(0, u]$

NEXT FIT: if the current item doesn't fit in the last bin start a new bin (NF)

THEOREM (Coffman, Hofri, So and Yao, 1980)

$$E\left(\frac{\mathrm{NF}(L_n, 1)}{\mathrm{OPT}(L_n, 1)}\right) \to \frac{4}{3} \quad (\sim 1.333)$$

THEOREM (Karmarkar, 1982)

$$E\left(\frac{\mathrm{NF}(L_n, u)}{\mathrm{OPT}(L_n, u)}\right) \to \begin{array}{l} 1.359,\ u = 0.9 \\ 1.361,\ u = 0.8 \\ 1.328,\ u = 0.7 \\ 1.259,\ u = 0.6 \\ 1.195,\ u = 0.5 \end{array}$$

Simulations indicate that $\mathrm{NF}(L_n, u)/\mathrm{OPT}(L_n, u)$ peaks around $u = 0.8$.

NOTATION FOR ASYMPTOTIC GROWTH RATES

$$f(n) = O(g(n))$$

means $\exists \beta > 0$ s.t. $f(n) < \beta g(n)$ for all n.

$$f(n) = \Omega(g(n))$$

means $\exists \alpha > 0$ s.t. $\alpha g(n) < f(n)$ for all n.

$$f(n) = \Theta(g(n))$$

means $\exists \alpha, \beta > 0$ s.t. $\alpha g(n) < f(n) < \beta g(n)$ for all n.

AVERAGE CASE PERFORMANCE UNDER $U(0, u]$

List of n items, sizes $\sim U(0, u]$, $u < 1$

$Ew_{\text{OPT}} = O(1)$

$Ew_{FFD} = O(1)$, $u \leq 1/2$

$Ew_{FFD} = \Theta(n^{1/3})$, $1/2 < u < 1^*$

Minimal waste for an on-line algorithm $= \Theta(n^{1/2})^\dagger$

$Ew_{BF} =?$ $Ew_{FF} =?$

(Simulations indicate linear waste for all $u < 1$.)

List of n items, sizes $\sim U(0, 1]$

Ew_{OPT}, $Ew_{FFD} = \Theta(n^{1/2})$ ‡

Minimal waste for an on-line algorithm $= \Theta(n^{1/2} \log^{1/2} n)$

$Ew_{BF} = \Theta(n^{1/2} \log^{3/4} n)^\S$

$Ew_{FF} = \Theta(n^{2/3})^\P$

*Bentley, et al 1984

†Shor, 1986, 1991

‡Knoder, 1981, Lueker, 1982

§Shor, 1986

¶Shor, 1986, Coffman, Johnson, Shor and Weber, 1991

AN ON-LINE PACKING THEOREM

Suppose items can are chosen amongst the real-valued sizes $\{a_1, \ldots, a_j\}$ with probabilities $p = (p_1, \ldots, p_j)$ respectively.

A 'perfect packing configuration', can be specified by the vector $c = (c_1, \ldots, c_j)$, such that c_1 items of size a_1, plus c_2 items of size a_2, ..., plus c_j items of size a_j, will perfectly pack into a unit size bin.

Let C be the set of all vectors specifying possible perfect packing configurations and let $\Lambda \subset \mathcal{R}^j$ be the convex cone spanned by the elements of C.

EXAMPLE

Suppose item sizes are $\{\frac{1}{7}, \frac{2}{7}, \frac{3}{7}\}$. So packing configuration vectors are $(1, 0, 2)$, $(0, 2, 1)$, $(7, 0, 0)$, $(1, 3, 0)$, $(5, 1, 0)$, $(2, 1, 1)$, $(3, 1, 1)$, $(4, 0, 1)$.

AVERAGE CASE FOR OPTIMAL ALGORITHM UNDER DISCRETE DISTRIBUTION

Let C be the set of all vectors specifying possible perfect packing configurations and let $\Lambda \subset \mathcal{R}^j$ be the convex cone spanned by the elements of C.

THEOREM (Courcoubetis and Weber, 1986)

For on-line packing,

$$Ew_{\mathrm{OPT}} = \begin{cases} O(1) \\ \Theta(n^{1/2}) \\ \Theta(n) \end{cases} \text{ as } p \in \begin{cases} \Lambda^o \\ \mathrm{bdy}(\Lambda) \\ \Lambda^c \end{cases}$$

- The proof that $Ew_{\mathrm{OPT}} = O(1)$ in the first of these cases involves a randomizing algorithm with this property, i.e., one that decides into which partially full bin to put an item according to a probabilistic rule.
- Clearly, randomization is not necessary. It is possible, but not easy, to construct a complicated deterministic algorithm that has provably $O(1)$ expected waste.
- Can any well-known heuristic algorithms, such as BF or FF be proved to achieve $O(1)$ waste?

THE $U\{j,k\}$ CASE

In the $U\{j,k\}$ case, sizes are $1/k, 2/k, \ldots, j/k$ and $\bar{p} = (1/j, \ldots, 1/j)$.

We can prove a combinatorial result about perfect packings to show $\bar{p} \in \Lambda^o$, and hence

THEOREM Under $U\{j,k\}$,

$$Ew_{\mathrm{OPT}}(L) = \begin{cases} O(1), & 1 \leq j < k-1 \\ \Theta(n^{1/2}), & j \in \{k, k-1\}. \end{cases}$$

The combinatorial result we need in order to show $\bar{p} \in \Lambda$ is that there exist integers r and m, with

$$rj(j+1)/2 = mk,$$

such that rj items, r of each of the sizes $1, \ldots, j$, can be packed perfectly into m bins of size k.

A slightly stronger result is needed to show $\bar{p}$ is strictly inside Λ.

AN IRRESISTIBLE DIGRESSION

THE PERFECT PACKING THEOREM

For all r, j, k, $j < k$, such that

$$k \mid r(1 + 2 + \cdots + j) \quad [= rj(j+1)/2]$$

the set of rj items, comprising r each of sizes $1, 2 \ldots, j$, can be packed perfectly into bins of size k.

The proof of this is rather difficult. Even the following simpler theorem requires some ingenuity to prove.

THE SIMPLE PACKING THEOREM

For all j, k, $j < k$, such that

$$k \mid (1 + 2 + \cdots + j) \quad [= j(j+1)/2]$$

the set of j items, comprising one of each of the sizes $1, 2 \ldots, j$, can be packed perfectly into bins of size k.

AVERAGE CASE UNDER DISCRETE DISTRIBUTION $U\{j,k\}$

THEOREM (Coffman, Courcoubetis, Garey, Johnson, McGeoch, Shor, Weber and Yannakakis, 1991)

If $j \in \{k, k-1\}$, then

$$Ew_{BF} = \Theta(n^{1/2} \log^{3/4} k).$$
$$Ew_{FF} = \Theta(n^{1/2} k^{1/2}).$$

and for any on-line algorithm

$$Ew_A = \Omega(n^{1/2} \log^{1/2} k).$$

If $j \leq \sqrt{2k+2.5} - 1.5$, then

$$Ew_{BF} = O(1).$$

OPEN PROBLEM If $\sqrt{2k+2.5} - 1.5 < j \leq k-2$, then

$$Ew_{BF} = ?$$

Simulations suggest waste can be both $O(1)$ and $\Theta(n)$. Some results are known for small values of j and k.

OPEN PROBLEM

Prove that BF or FF has $\Theta(n)$ expected waste under some distribution, either the continuous uniform

$$U(0, u], \quad u < 1,$$

or the discrete uniform

$$U\{j, k\}, \quad j < k - 1.$$

For $U(0, u]$, one suspects $Ew_{BF} = \Theta(n)$, for all $u < 1$, but this problem remains open.

For $U\{j, k\}$, the first likely candidate for linear expected waste is $U\{8, 11\}$.

MARKOV CHAIN MODEL
FOR BF AND $U\{8, 11\}$

Empirical observation of the $\{8, 11\}$ case indicates that bins with a gap of size 1 are created faster than items of size 1 arrive, hence items of size 1 always go in gaps of size 1 under the BF rule. With regard to bins with gaps $2, \ldots, 9$ the process seems to be ergodic.

BF induces a Markov chain.

STATE:

$$(x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9) \in \mathcal{Z}_+^8,$$

where $x_i = \#$ of bins in current packing having a gap of size i.

(Making the conjecture $x_1 \to \infty$)

Note: $x_6 + x_7 + x_8 + x_9 \leq 1$ since at most one bin can be less than half-full

TRANSITIONS:

There are 8 transitions in each state, each having probability 0.125.

BEHAVIOR OF BF UNDER $U\{8, 11\}$

TRANSITIONS

There are 8 transitions in each state, each having probability 0.125.

Let $T(x)$ be the set of eight possible transitions from state x.

For $t \in T(x)$, let $w(t) = 1$ if t creates a new bin with a gap of 1.

CONJECTURE

If $P(x)$ is the equilibrium distribution of x, then

$$\sum_x P(x) \sum_{t \in T(x)} (0.125) w(t) > 0.125$$

EMPIRICAL OBSERVATION

Average rate at which excess 1–gaps are produced over a 10,000,000 item packing ~ 0.0012, which suggests the right hand side above ~ 0.1262, (> 0.125).

APPROXIMATING THE INFINITE MC

In observations of 1000 runs of BF for each n, the maximum component encountered in the state vector, $(x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9)$,

n	ave max	max max
10^6	13.8	20
10^7	18.4	24
10^8	23.1	27

TRUNCATING THE INFINITE CHAIN

$$t(x)_i = \min\{M, t(x)\}_i$$

NUMBER OF STATES

$$(M+1)^4 + (M+1)^3 + (M+1)^2 + (M+1) + 1$$

M	# states
10	16,105
20	204,205
25	475,225
30	954,305

Our simulation studies used $M = 30$.

MULTI-DIMENSIONAL MARKOV CHAINS
– WHAT'S KNOWN –

THEOREMS (Malyshev, Menshikov, 1979, Fayolle, 1989)

For Markov chains in $\mathcal{Z}_+^d$,

- Simple conditions for ergodicity, null-recurrence and transience for $d = 2$ when the Markov chain is jump-bounded, i.e., $||X_{n+1} - X_n|| < C$, and m-limited, i.e., $x \longrightarrow x + \Delta$ has the same probability as $\bar{x} \longrightarrow \bar{x} + \Delta$, where $\bar{x}_i = \min(x_i, m)$.

- Claims of results for $d = 3$, without proof.

- Suggested approach for larger d.

A LEMMA ABOUT DRIFTS

LEMMA (Hajek 1982)

Suppose $X_1, X_2, \ldots$ is a random path of an irreducible Markov chain; U is a finite subset of the state space S; γ, B are positive reals, and ϕ is a positive-valued function on S such that

(a)

$|\phi(X_{n+1}) - \phi(X_n)| \leq B$ almost surely,

and

(b)

$E[\phi(X_{n+1}) - \phi(X_n) \mid X_n = x] \leq -\gamma$, $x \in S \backslash U$.

Then

$$\limsup_{n \to \infty} E[\phi(X_n)] < \infty.$$

Note that if ϕ is linear with positive coefficients then this will imply $E(||X_n||) < \infty$, which will imply bounded waste.

CAN WE USE A LINEAR POTENTIAL FUNCTION ?

Let $t_i(x)$ is the transition effected by arrival of an item of size i.

$$t_i(x) = x + \delta_i(x), \quad i = 1, \ldots, 8,$$

Suppose we try a potential function $\phi(x) = y^\top x$. To prove ergodicity we want to show there are $y, \Delta > 0$ such that

$$0.0125 \sum_{i=1}^{8} y^\top t_i(x) - y^\top x < -\Delta < 0$$

for all x. Equivalently, we want

$$y^\top \delta(x) = y^\top \left(0.0125 \sum_{i=1}^{8} \delta_i(x)\right) < -\Delta$$

where we define expected drift out of state x,

$$\delta(x) = E(X_1 - X_0 \mid X_0 = x) = 0.0125 \sum_{i=1}^{8} \delta_i(x)$$

KEY OBSERVATION

If $x_i > 1$, then replacing x_i by 1 does not change $\delta(x)$, which depends only on the pattern of non-zeros. Thus there are actually only 2^8 constraints in the above.

LINEAR PROGRAM

Recall the definition of the expected drift out of state $\boldsymbol{x}$,

$$\boldsymbol{\delta}(\boldsymbol{x}) = \boldsymbol{E}(\boldsymbol{X}_1 - \boldsymbol{X}_0 \mid \boldsymbol{X}_0 = \boldsymbol{x}) = \mathbf{0.0125} \sum_{i=1}^{8} \boldsymbol{\delta}_i(\boldsymbol{x})$$

Then we want to find a solution to the LP:

Maximize $\boldsymbol{\Delta}$

subject to $\boldsymbol{y}^\top \boldsymbol{\delta}(\boldsymbol{x}) < -\boldsymbol{\Delta}$,

for all $\boldsymbol{x}$ with $\max(\boldsymbol{x}_i) = \mathbf{1}$, and $\boldsymbol{y} \geq \mathbf{0}$, $\boldsymbol{y}_2 = \mathbf{1}$.

SOLUTION

$$\boldsymbol{\Delta} = -\mathbf{0.09959} \not> \mathbf{0}$$

so there is no linear potential function that works.

m-STEP, m-LIMITED, MARKOV CHAIN

But now consider the expected drift over m steps:

$$\delta^m(x) = E(X_m - X_0 \mid X_0 = x)$$

This can be computed recursively.

NEW LINEAR PROGRAM

Maximize Δ

subject to $y^\top \delta^m(x) < -\Delta$,

for all x with $\max(x_i) = m$, and $y \geq 0$, $y_2 = 1$.

SOLUTION

Finally get $\Delta > 0$ for $m = 20$ (an LP with 35,784 constraints and solved by CPLEX in 37 seconds).

CONCLUSION

The 20–step Markov chain is ergodic and the equilibrium value of $E[\Sigma_{i=2}^{9} x_i] < \infty$.

COROLLARY

Same holds for the 1–step chain.

DEALING WITH THE 1-GAPS

STATES: $x_1, x_2, \ldots, x_9$,

$-\infty < x_1 < \infty$, and $x_2, \ldots, x_9 \geq 0$.

TRANSITIONS

As before except that creation of a 1-gap adds 1 to x_1 and arrival of a 1-item subtracts 1 from x_1, even if that makes x_1 go negative.

NEW LINEAR PROGRAM

Maximize Δ

subject to $y^\top \delta^m(x) < -\Delta$,

for all x, and $y_2, \ldots, y_9 \geq 0$, $y_1 = -1$.

If $\Delta > 0$ then this means we must have $x_1 \longrightarrow \infty$, i.e., explosion of 1-gaps.

PROBLEM

The difficult is showing that there is negative drift for all states, not just those with $\max(x_i) = m$. Restricted to that finite set of states, the LP solution works for $m = 15$.

HOW TO GET NEGATIVE DRIFT FROM ALL STATES

TRANSITIONS

We look at the expected drift over M steps, where this is now the minimum of 5,000 steps or 15 steps beyond the first exit from the set

$$U = \{x \,:\, x_i < 15, \text{ all } i \geq 2\}.$$

RECURSIVELY COMPUTE

the expected drift from every state, for the linear potential function $\phi(x) = y^\top x$, where

$$\begin{array}{rrrr} y_1 = & -1.000000 & & \\ y_2 = & 1.190854 & y_6 = & 0.003516 \\ y_3 = & 0.190854 & y_7 = & 0.091374 \\ y_4 = & 0.376722 & y_8 = & 0.020421 \\ y_5 = & 0.329896 & y_9 = & 0 \end{array}$$

(This is from the optimal solution to the LP restricted to U and $m = 15$.) Note that U is finite, and for all initial $x \notin U$ there are only finitely many values of $\delta^{15}(x)$ to compute.

We find that the drift is negative from all states.

CONCLUSION OF PROOF

CONCLUDE FROM THE ABOVE

For this chain the expected number of 1-gaps grows linearly to $+\infty$.

EVEN STRONGER

Almost surely there will only be a finite number of steps when $x_1 \leq 0$. (Proof uses supermartingale argument based on negative drift and fact that over 5014 steps the drift is bounded.)

COROLLARY

There exists a state, say $\bar{x}$, where there is positive probability that x_1 is never again ≤ 0.

OBSERVATION

Given any initial state with $x_1 > 0$ the probability that the true Markov chain reaches $\bar{x}$ is positive.

QED

ANOTHER DRIFT LEMMA

LEMMA (Coffman, Johnson, Shor, Weber, 1992)

Suppose $X_1, X_2, \ldots$ is a random path of an irreducible Markov chain; γ, B are positive reals, and ϕ is a positive-valued function on S such that

(a)

$|\phi(X_{n+1}) - \phi(X_n)| \leq B$ almost surely,

and

(b)

$E[\phi(X_{n+1}) - \phi(X_n) \mid X_n = x] \geq \gamma B$ for all x.

Then for all $\gamma' > \gamma$

$$\liminf_{n\to\infty} \left[\phi(X_n) - \gamma n + (1+\gamma')\sqrt{2n \log n}\right] > 0.$$

Note that this implies the Markov chain is transient.

USING LINEAR PROGRAMMING

TO DESIGN

QUADRATIC POTENTIAL FUNCTIONS

KEY OBSERVATION:

Suppose $f(x) = x^\top Ax$, where A is a non-negative and symmetric matrix. Then

$$\begin{aligned} f(x + \Delta) - f(x) &= 2x^\top A\Delta + \Delta^\top A\Delta \\ &\le 2\sum_i x_i \left[\sum_j a_{ij}\Delta_j\right] + C_1 \end{aligned}$$

So

$$\begin{aligned} &E[f(X_1) - f(X_0) \mid X_0 = x] \\ &\qquad \le 2\sum_i x_i \left[\sum_j a_{ij}\delta_j(x)\right] + C_2 \end{aligned}$$

Therefore we need to choose a_{ij} s.t

$$\sum_j a_{ij}\delta_j(x) < -z < 0$$

for all i such that $x_i \ge m$, and all x.

Actually, we take $\phi(x) = \sqrt{x^\top Ax}$.

KNOWN AND CONJECTURED RESULTS FOR AVERAGE WASTE UNDER BF

$j =$	3	4	5	6	7	8	9	10	11	12
$k = 5$	B-L2									
6	B-L2	B-L4								
7	B-L2	B-L3	B-L23							
8	B-Q1	B-Q1	B-Q1	B-Q2						
9	B-Th	B-Q1	B-Q1	B-Q5	B-Q7					
10	B-Th	B-Q1	B-Q1	B-Q1	B-Q15	B-Q13				
11	B-Th	B-Q1	B-Q1	B-Q1	B-Q2	Ln-P	B-x			
12	B-Th	B-Q1	B-Q1	B-Q1	B-Q2	B-x	Ln-P	B-x		
13	B-Th	B-Q1	B-Q1	B-Q1	B-Q1	B-x	Ln-x	Ln-x	B-x	
14	B-Th	B-Th	B-Q1	B-Q1	B-Q1	B-Q7	B-x	Ln-x	Ln-x	B-x

B= Bounded waste $O(1)$

Ln= Linear waste $\Theta(n)$

x= based on experimental evidence only

Th= proof based on theorem for $j \leq \sqrt{2k + 2.5} - 1.5$

P= proof based on linear potential function and drift calculation

Lm= proof based on linear potential function, m steps

Qm= proof based on quadratic potential function, m steps

SURPRISE OF $U\{6, 13\}$

THEOREM

Expected waste for BF under $U\{6, 13\}$ is $O(1)$.

THEOREM

Expected waste for BFD under $U\{6, 13\}$ is $\Theta(n)$.

This goes against one's intuition that the off-line algorithm BFD does better than the on-line algorithm BF.

CONCLUSIONS

- There's still a lot more to learn about expected behavior of bin packing algorithms.

  — For large k and $j < k - 3$ not only the 1-gaps go to infinity.

  — For $j = k - 2$, waste might be constant, but proportional to $k^2 \log k$, or something like that.

  — FF is more complicated.

  — Markov chain technique may have reached the limits of its usefulness. Can these results be proved some other way, not relying on long computation?

- Techniques here should be applicable to other problems (e.g., queues).

- This research has demonstrated the usefulness of the computer in the mathematics of operations research, both as an experimental tool to suggest conjectures, and as an aid in constructing the proof of theorems.

OFF-LINE BEHAVIOR

THEOREM (Coffman, Johnson, Shor and Weber, 1994) Suppose D is a discrete probability distribution over a finite set of item sizes S. Then the expected waste of FFD and BFD under D is one of

$$O(1) \qquad \Theta(n^{1/2}) \qquad \Theta(n)$$

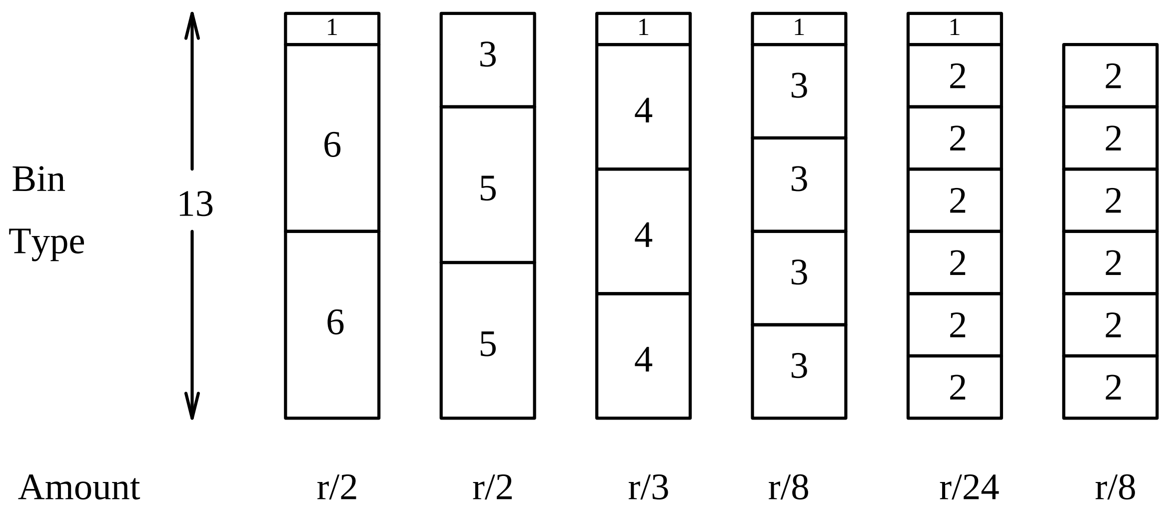
There is a polynomial-time algorithm that, given D, determines which case holds.

FFD FLUID ALGORITHM ANALYSIS

EXAMPLE OF $j = 6$, $k = 13$

$$\text{Size} = \quad 6 \;\; 5 \;\; 4 \;\; 3 \;\; 2 \;\; 1$$

$$\text{Amount} = \; r \;\; r \;\; r \;\; r \;\; r \;\; r$$

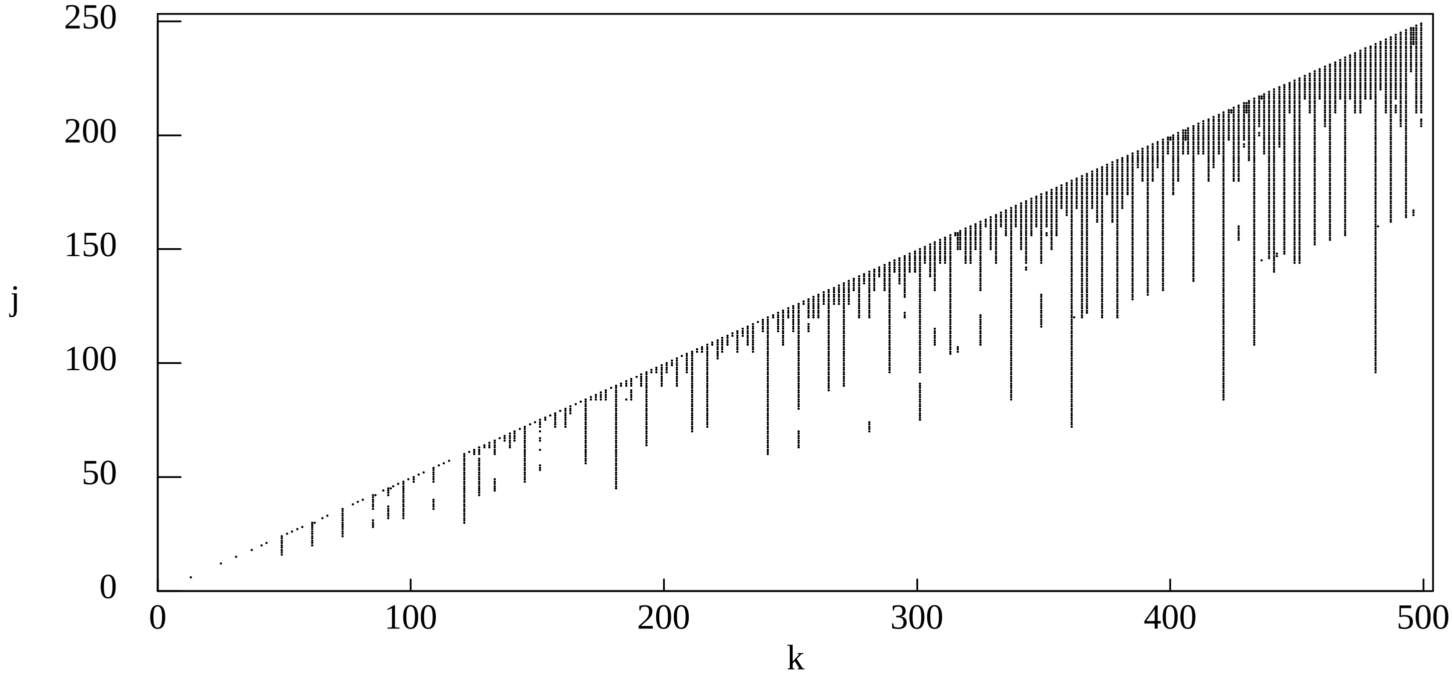


Thus the expected waste

$$= \frac{n}{6} \times \frac{1}{13} \times \frac{1}{8} = \frac{n}{624} = \Theta(n)$$

The idea is to show, by arguments using the Central Limit Theorem, that the average case behavior of FFD must be close to that predicted by running the fluid algorithm.

PAIRS j, k FOR WHICH FFD HAS LINEAR EXPECTED WASTE UNDER $U\{j, k\}$



FFD RESULTS

THEOREM

If $j < k^{1/2}$ then the expected waste for FFD is $O(1)$.

THEOREM

There exist pairs j, k with j arbitrarily large $j < k/2$, such that the expected waste is $\Theta(n)$.

THEOREM

For all pairs j, k with $j < k - 1$ the expected waste is no more than

$$\min\left\{\frac{n}{624}, \frac{0.00614n}{\sqrt{k}}\right\}$$

OPEN PROBLEM

Expected waste for FFD is provably $\Theta(n^{1/2})$ if the fluid algorithm terminates with just enough items of size 1 to avoid waste. Do there exist j, k for which this happens? (No. $j, k < 3000$.)

SUMMARY OF AVERAGE CASE RESULTS FOR UNIFORM DISTRIBUTIONS

	$U(0,u]$	$U\{j,k\}$
	$u = 1$	$j \in \{k-1, k\}$
Optimal packing	$\Theta(n^{1/2})$	$\Theta(n^{1/2})$
Best poly. time alg	$\Theta(n^{1/2})$	$\Theta(n^{1/2})$
Best poss on-line	$\Omega((n \log n)^{1/2})$	$\Omega(n^{1/2})$
Best known on-line	$\Theta((n \log n)^{1/2})$	$\Theta(n^{1/2})$
FFD	$\Theta(n^{1/2})$	$\Theta(n^{1/2})$
BF	$\Theta(n^{1/2} \log^{3/4} n)$	$\Theta(n^{1/2})$
FF	$\Theta(n^{2/3})$	$\Theta(n^{1/2})$
	$u < 1$	$j < k-1$
Optimal packing	$O(1)$	$O(1)$
Best poly. time alg	$O(1)$	$O(1)$
Best poss on-line	$\Omega(n^{1/2})$	$O(1)$
Best known on-line	$\Theta((n \log n)^{1/2})$	$O(1)$
FFD	$O(1), \Theta(n^{1/3})$	$O(1), \Theta(n^{1/2})?, \Theta(n)$
BF	$\Theta(n)?$	$O(1), \Theta(n)$
FF	$\Theta(n)?$	$O(1), \Theta(n)?$

COMPUTER PROOF

COMPUTING RESOURCES

- 128 Megabytes memory
- 33 MHz MIPS processor
- CPLEX™ linear programming software
- $\simeq$ 24 hours for largest problems
- Limiting factor is memory to store the LP constraints (e.g., proof of $O(1)$ waste for $(j, k) = (8, 14)$ requires an LP with 415,953 constraints, (=57 Mb))

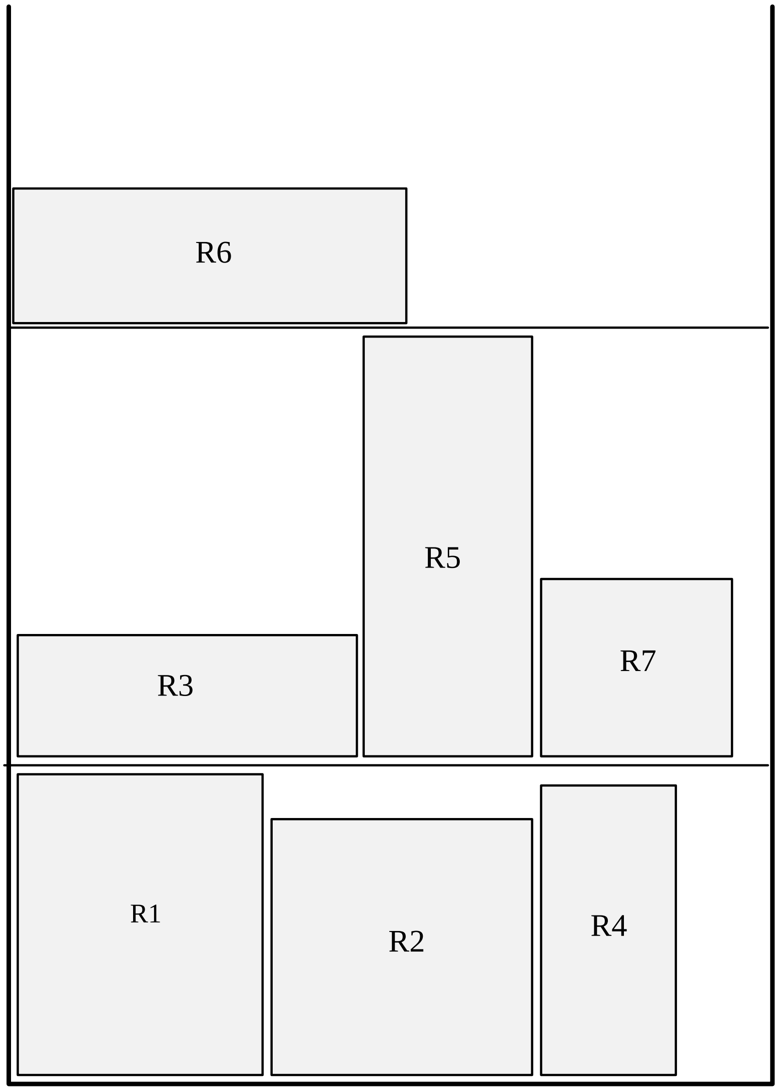
VALIDITY OF THE PROOFS DOES NOT DEPEND

- on correctness of the LP code

VALIDITY OF THE PROOFS DOES DEPEND

- on the check that the solution to the LP satisfies all the constraints that it should
- on recursive code that generates the constraints
- on maufacturer's certification that computer meets IEEE floating point standard

STRIP PACKING



First Fit Level Packing

There are many open problems.