Observations on the Bomber Problem

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Sequential allocation problems

Groundwater Management Burt (1965)

$$F(x,t) = \max_{y \in [0,x]} \{a(y) - c(x,y) + \delta E F(x - y + R_t, t-1)\}$$

$F(x,0) = 0.$ $\quad x$ is level of water in an aquifer.

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Investment Derman, Lieberman and Ross (1975)

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Fighter

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k \in \{1,\ldots,n\}} \{a(k) + F(n-k, t-1)\}$$

$F(n,0) = 0.$ $\quad$ k is remaining stock of missiles.

Fighter Problems

Invincible Fighter Bartroff et al (2010)

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k \in \{1,\ldots,n\}} \{a(k) + F(n-k,t-1)\}$$

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Frail Fighter Weber (1985)

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k\in\{1,\dots,n\}} \{a(k) + a(k)F(n-k,t-1)]\}$$

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General Fighter

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k \in \{1,\ldots,n\}} \{a(k) + c(k)F(n-k,t-1)\}$$

Might take $c(k) = a(k) + u(1 - a(k))$.

Monotonicity properties **(A)**, **(B)** and **(C)**

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k\in\{1,\dots,n\}} \{a(k) + c(k)F(n-k,t-1)\}$$

Let $k(n,t)$ be the maximizing k in the above.

Intuitively obvious properties of an optimal policy are:

(A)	$k(n,t)$	$\searrow$	as $t \nearrow$
(B)	$k(n,t)$	$\nearrow$	as $n \nearrow$
(C)	$n - k(n,t)$	$\nearrow$	as $n \nearrow$

(A) (B) (C) s of fighter problems

If

(i) $\{p_t\}_{t=1,\dots}$ is any sequence of probabilities;

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(ii) $a(k)$ is nondecreasing and concave in k, then

(A) holds for the invincible fighter, in the strong sense that

$$\textbf{(A)}^* : \; k(n, p_{t-1}, \dots, p_1) \text{ is nonincreasing in each } p_i.$$

and for the frail fighter, nonincreasing in p_1.

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Does **(A)** *hold for the general fighter?*

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(B) holds for the invincible fighter, but not frail fighter.

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Does **(A)** *hold for the general fighter?*

(B) holds for the invincible fighter, but not frail fighter.

If also,

(iii) $c(k)$ is nondecreasing and log-concave in k, then

(C) holds for the general fighter.

Bomber Problem

Klinger and Brown (1968)

With discrete ammunition, and attacks occurring as a Poisson process of rate 1, the continuous-time bomber problem (CBP) has defining equations:

$$
\begin{aligned}
P(n,t) &= P(\text{survive to until time } t) \\
&= e^{-t} + \int_0^t \max_{k\in\{1,\dots,n\}} c(k)P(n-k,s)e^{-(t-s)}\,ds. \\
P(n,0) &= 1.
\end{aligned}
$$

Bernoulli model: $a(k) = 1 - \theta^k$, a concave function of k.

Doubly-discrete Bomber Problem (DBP)

Aim is to survive t periods. With s periods to go, an attack occurs with probability p_s $(= 1 - q_s)$.

$$P(n,t) = q_t P(n,t-1) + p_t \max_{k \in \{1,\dots,n\}} c(k) P(n-k, t-1)$$

$$P(n,0) = 1.$$

Again we are interested in whether the following are true of false.

(A)	$k(n,t)$	$\searrow$	as $t \nearrow$	proved
(C)	$n - k(n,t)$	$\nearrow$	as $n \nearrow$	proved
(B)	$k(n,t)$	$\nearrow$	as $n \nearrow$	**?**

(A) (B) (C)s and open problems

$$F(n,t) = q_t F(n,t-1) + p_t \max_{k\in\{1,\dots,n\}} \{a(k) + c(k)F(n-k,t-1)\}$$

$F(n,0)$	$a(k)$	$c(k)$	p_t	**(A)**	**(B)**	**(C)**	
0	Bernoulli	$= u + (1-u)a(k)$	$= p$	**?**	no	yes	general fighter
0	concave	$= u + (1-u)a(k)$	$= 1$	yes	no	yes	general fighter
0	concave	$= \delta$		(A)*	yes	yes	invincible fighter
0	concave	$= a(k)$		yes	no	yes	frail fighter

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0	concave	$= a(k)$		yes	no	yes	frail fighter
1	0	Bernoulli	$= p$	yes	?	yes	bomber
1	0	log-concave	$= 1$	yes	yes	yes	bomber
1	0	log-concave		yes	no	yes	bomber
1	0	log-concave	$= p$	yes	no	yes	bomber
1	0	concave	$= p$	yes	no	yes	bomber

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1	0	log-concave		yes	no	yes	bomber
1	0	log-concave	$= p$	yes	no	yes	bomber
1	0	concave	$= p$	yes	no	yes	bomber
≥ 0	concave	log-concave				yes	bomber/fighter

(A) (B) (C) s of the bomber problem

Theorem 1 **(A)** *and* **(C)** *hold for the DBP (and CBP, CDBP and CCBP) under generous assumptions that*

(i) $\{p_t\}_{t=1,\dots}$ *is any sequence of probabilities (i.e. nonstationary).*

(iii) $c(k)$ *is any nondecreasing and log-concave function of* k*.*

Is **(B)** true under these same generous assumptions?

Suppose $c(k)$ is merely log-concave in k (rather than concave in k)

(A) and **(C)** are true.

(B) is not true.

$p_t = \frac{10}{11}$ for all t.

$\{c(0), c(1), c(2), c(3), c(4), \dots\} = \{0, \frac{3}{16}, \frac{1}{2}, 1, 1, \dots\}$.

Note that $c(i)^2 \geq c(i+1)c(i-1)$ for all $i \geq 1$.

$\{k(n,4)\}_{n=1,2,\dots} = \{1, 1, 1, 2, 2, 3, 2, \dots\}$.

i.e. $3 = k(6,4) > k(7,4) = 2$.

Suppose $c(k)$ is merely concave in k (rather than of Bernoulli form $c(k) = 1 - v\theta^k$)

(A) and **(C)** are true. **(B)** is not true.

Let ammunition be continuous (CDBP).

$$P(x,1) = q + pc(x)$$

$$P(x,2) = q^2 + qpc(x) + \max_y \Big\{ pqc(y) + p^2 c(y)c(x-y) \Big\}$$

We design $c(\cdot)$ so that it is not log-concave in the neighbourhood of $x = 3$, and so that in this neighbourhood,

$$y(x,2) = \arg\max_y \{c(y)P(x-y,1)\} = \frac{1+x}{2}.$$

$c(x)$ for which $P(x,2)$ is not log-concave in the neighbourhood of $x=3$

$$c(x) = \min\left\{\tfrac{1}{96} + \tfrac{31}{96}x, \tfrac{17}{96} + \tfrac{31}{192}x, \tfrac{5}{12} + \tfrac{31}{384}x, 1\right\}$$

$$= \begin{cases} \frac{1}{96} + \frac{31}{96}x, & x \in \left[0, \frac{32}{31}\right] \\ \frac{17}{96} + \frac{31}{192}x, & x \in \left[\frac{32}{31}, \frac{92}{31}\right] \\ \frac{5}{12} + \frac{31}{384}x, & x \in \left[\frac{92}{31}, \frac{224}{31}\right] \\ 1, & x \geq \frac{224}{31} \end{cases}$$

(B) is not true under generous assumptions

$$y(32,3) = \arg \max_{y \in [0,5.24]} \Big[c(y) F(31.4 - y, 2) \Big] = 14.0079$$

$$y(33,3) = \arg \max_{y \in [0,5.25]} \Big[c(y) F(31.5 - y, 2) \Big] = 13.9174.$$

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(B) fails because

$$3.4027 = \arg \max_{y \in [0,6.39]} \Big[c(y) P(6.39 - y, 2) \Big]$$

$$3.3965 = \arg \max_{y \in [0,6.40]} \Big[c(y) P(6.40 - y, 2) \Big].$$

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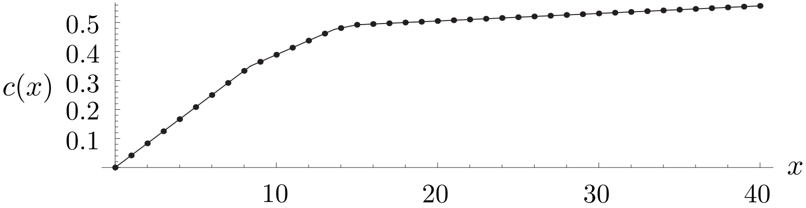
$6.39 - 3.4027 = 2.9873$
$6.40 - 3.3965 = 3.0035$
lie just either side of $x = 3$, where $P(x,2)$ is not log-concave.

Discrete ammunition counterexample

$$c(x) = \min\left\{\tfrac{1}{24}x, \tfrac{7}{48} + \tfrac{7}{288}x, \tfrac{371}{1152} + \tfrac{49}{4320}x, \tfrac{29}{64} + \tfrac{1}{384}x, 1\right\}.$$



$$\{k(n,2)\}_{n=1}^{40} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 9, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 13, 14, 14, 14, 14, 14, 14, 15, 15, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25\}$$

$$\{k(n,3)\}_{n=1}^{40} = \{1, 2, 3, 4, 5, 6, 6, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 10, 11, 12, 13, 13, 13, 13, 14, 14, 14, 14, 14, 14, 15, 14, 15, 14, 14, 15, 15, 15, 15, 15\}.$$

So $k(31,3) = 15 > 14 = k(32,3)$, in contradiction to **(B)**.

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   $P(n,1) = q + pc(n)$ is concave in n.
   $P(n,2)$ is log-concave in n (for $c(k) = 1 - \theta^k$ model).
   $P(n,t)$ is not necessarily log-concave when $t \geq 3$.

Log-concavity of $P(n,t)$ can fail in DBP

$P(n,t)$ fails to be log-concave when

$$\Delta(n,t) = \frac{P(n+1,t)}{P(n,t)} - \frac{P(n,t)}{P(n-1,t)} > 0$$

for some n, t and some p, θ.

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$p = 0.58$, $\theta = 0.6$, $\Delta(8,3) = \frac{93682400617500}{668426731570135139} = 0.0001402$.
Simons and Yao (1990)

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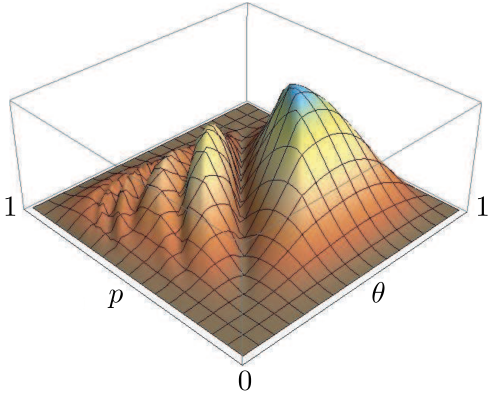
$p = 0.7207,\ \theta = 0.7254,\ \Delta(8,3) = 0.0004779.$

(most positive Δ found)

Regions of Log-concavity in DBP



$P(8,3)^2 - P(7,3)P(9,3)$ as a function of p and θ.
The region where this quantity is negative lies in the central trench, where p is a bit less than θ.

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   Take p a bit less than θ and both approaching 1.
4. Continuous time is the limit as $p \to 0$.
   So what about small p?

   For $p = 0.01$, $\theta = 0.01000048$, $\Delta(8,3) = 4.58768 \times 10^{-15}$.
   (This really is positive; checked in exact arithmetic).

Log-concavity in CBP

No examples have (yet!) been found in CBP for which $P(n, t)$ is not log-concave (continuous time, discrete ammunition).

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However, for a slightly different model $P(n, t)$ fails to be log-concave (and nonetheless **(B)** appears to hold).

We take $c(k) = 1 - (7/8)^k$ and make the restriction that only 1, 2 or 3 missiles may be fired.

Log-concavity in CBP

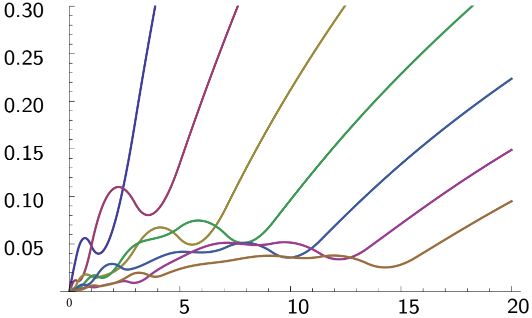

Figure: $-\Delta(n,t) = P(n,t)/P(n-1,t) - P(n+1,t)/P(n,t)$, for the continuous-time bomber problem with $\theta = 1/2$, for $0 \leq t \leq 20$ and $n = 2, \ldots, 8$ (reading left to right across the asymptotes). Although we see that $P(n,t)$ is log-concave in n, the fact that these functions are not monotone increasing, in either n or t, means that it is probably difficult to prove that $P(n,t)$ is log-concave in n by some sort of induction on n, or using differential equations in t.

Continuous ammunition

CDBP and CCBP (continuous ammunition)

$$P(x,t) = qP(x,t-1) + p \max_{0<y\leq x} c(y)P(x-y,t-1)$$

or

$$\frac{d}{dt}P(x,t) = \max_{0<y\leq x} c(y)P(x-y,t)$$

1. $P(x,t)$ is log-concave in $x \iff$ **(B)** is true.

$$P(x,t)P''(x,t) - P'(x,t)^2 < 0.$$

Towards an iterative approach to proof of **(B)**

Consider iterating, from a start of $P_0(n,t)=1$, with

$$P_i(n,t) = e^{-t} + \int_0^t \max_{k\in\{1,\dots,n\}} c(k)P_{i-1}(n-k,s)e^{-(t-s)}\,ds$$

Might we inductively show that $P_i(n,t)$ is log-concave?
This poses a problem of maximizing the probability of surviving until time t, or until the first i attacks have been repelled.
Discrete time equivalent problem is

$$P_i(n,t) = q^t + \sum_{s=1}^{t-1} \max_{k\in\{1,\dots,n\}} c(k)P_{i-1}(n-k,s)\,q^{t-1-s}p$$

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Denote the maximizer of $c(k)P_i(n-k, s-1)$ as $k_i(n, s)$.

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Denote the maximizer of $c(k)P_i(n-k,s-1)$ as $k_i(n,s)$.

With $p = 1/2$, $c(k) = 1-(3/5)^k$, we find
$k_8(7,18) = 2 > k_8(8,18) = 1$.

So **(B)** does not hold for $k_8(n,18)$.

Also, rather surprisingly, $k_7(7,17) = 1$ and $k_8(7,17) = 2$.

Varying the final missile's miss probability **(B)**

Suppose that if the last missile is fired in a volley of k then

$$a(k) = 1 - \psi\theta^{k-1}, \quad v \in [\theta, 1].$$

Might we find $k(n, t, \psi)$ nonincreasing in ψ so that
$k(n, t) = k(n, t, \theta) \geq k(n, t, 1) = k(n - 1, t)$?

No counterexample to this has (yet) been found.

Another variation in which **(B)** fails

Suppose the boundary condition $P(n,0)=1$ is changed to

$$\begin{aligned} P(0,0) &= 1 \\ P(n,0) &= 1+\lambda, \quad n=1,2,\ldots \end{aligned}$$

Then $k(n,t) \to k(n-1,t)$ as $\lambda \to \infty$.

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$$\begin{aligned} P(0,0) &= 1 \\ P(n,0) &= 1 + \lambda, \quad n = 1, 2, \ldots \end{aligned}$$

Then $k(n,t) \to k(n-1,t)$ as $\lambda \to \infty$.

But with $p = \theta = 3/5$, we find $p(8,3,\lambda)$ is not nonincreasing in λ, and indeed

$k(8,3,0.6) = 4$ and $k(9,3,0.6) = 3$.

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So **(B)** fails, with this slight change of boundary condition.

Interestingly, for a boundary condition of $P(n,0)=n$, we find no counterexample to $P(n,t)$ being log-concave.

Special cases when **(B)** is true

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Conclusions

1. Proofs of **(A)** and **(C)** make no special use of $c(k) = 1 - \theta^k$.
   In discrete-time models they do not need $p_t = p$.
   They need only that $c(k)$ be log-concave.
   Yet **(B)** does not hold under such generous assumptions.

Conclusions

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   In discrete-time models they do not need $p_t = p$.
   They need only that $c(k)$ be log-concave.
   Yet **(B)** does not hold under such generous assumptions.

2. Experimental evidence still suggests the following are true (in the doubly discrete versions of the problems):

   (A) in the general fighter problem, when p_t is nonstationary and $a(k) = 1 - \theta^k$, $c(k) = 1 - v\theta^k$.

   (B) in the bomber problem, when p_t is nonstationary and $c(k) = 1 - \theta^k$.

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