Asymptotics for Provisioning Problems of Peering Wireless LANs with a Large Number of Participants

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Abstract. We consider a model in which wireless LANs are to be provided in a number of locations. The owners of these WLANs have decided to peer with one another so that they can roam in locations other than their own. We consider the question of designing a mechanism for determining the quantities of resources that agents should provide so that the qualities of service are achieved in the locations and a measure of expected welfare is maximized, subject to the constraints that this mechanism is incentive compatible, rational and feasible (in senses to be described). We show that as the number of participant becomes large the solution to a limiting problem takes a simple form. Namely, it is near optimal to demand that each participant in the same location makes a predetermined and equal contribution to the system, this being comprised of a contribution of resources (such as coverage area or the number of roaming peers he accepts), plus, possibly, a monetary transfer. A participant is permitted to use the WLANs in other locations when roaming if and only if he is willing to make this contribution. The advantage of this is that the provisioning policy and contribution requirements can be easily communicated to the participants.

1 Peer to peer networks of wireless LANs

Access to the Internet is still not as ubiquitous as access to the telephone network. This greatly reduces the economic value of many new portable devices, such as PDAs, tablet computers and smart-phones running the IP protocol. The users of these devices would benefit greatly from cost-effective Internet access that is wireless, always-on, ubiquitous and high-speed. However, deploying infrastructure with wide enough coverage to support this is a non-trivial task, especially from the business perspective.

Wireless Local Area Networks (WLANs) are an important developing infrastructure. Specifically, the IEEE 802.11 WLAN standard has grown steadily in popularity since its inception and, at least in metropolitan areas, is now well positioned to complement much more complex and costly technologies such as 3G. This is already happening. WLAN signals of networks set up by individuals for their own use already pervade many cities and such WLAN 'cells' frequently cover greater areas than were originally intended at their installation. Given how easy it is to gain access to a WLAN once a potential user is within its coverage area, and leaving out the obvious security issues involved, one wonders if individuals could share such infrastructure amongst themselves to achieve ubiquitous Internet access. Sharing comes as a natural idea since WLANs provide large amounts of bandwidth that is mostly underutilized by its local users. Also the pipe that connects the local WLAN users to the Internet is usually of a broadband nature (DSL) and may also be under-used over large time periods. Existing technology allows (or will soon allow) WLAN administrators to control access to their networks and to limit the consumption of network resources by remote (roaming) users.

In this paper we develop an economic model for sharing resources among WLANs. As in existing peer-to-peer (p2p) file sharing systems, such as Gnutella and Kazaa, each individual WLAN owner may choose whether or not he wished to join the peering group. If he decides to join, then he is must contribute a quantity of resources that are specified by some system participation rules. No central entity controls the interaction between the peers, each of whom has full control of his own participation level in the community. Our aim is to optimize participation rules to maximize economic efficiency and reduce free-riding. Without such rules, the free-riding problem could be debilitating, as each WLAN would be inclined to offer no resources to others (in order to minimize its own cost), while simultaneously trying to consume as much as it can of the resources that are provided by peering WLANs when it roams in remote areas. Altruism can go some way towards ameliorating this problem, and may partly explain why existing p2p systems Kazaa) operate with some degree of success, even though studies indicate that the majority of the users are free-riders (see [4] and [10]). However, we do not expect that altruism alone can completely correct such inefficiencies.

We suppose that the value that each peer places on being part of the peering group is known only to himself. The problem is one of 'mechanism design': to determine rules to specify the peers' contributions and usages that maximize social welfare. We note that existing p2p file sharing applications have recently started to incorporate simple rules. In Kazaa, for example, each peer has an associated 'participation level'. When two peers try contend for downloading a file then the peer with the higher participation level has priority. A peer can increase his participation level by increasing the amount of megabytes that other peers download from him, or by 'integrity-rating' the files he shares. However, this rule is only a heuristic and does not maximize social welfare. We seek rules that are social welfare maximizing, subject to appropriate informational participation constraints (the 'second-best' in mechanism design terminology). Unfortunately, such rules are likely to be very complicated. The contribution required of a peer may depend on the preference declarations of all the other peers and a central planner would be required to implement them. The remarkable result in this paper, is that there do exist simple rules that are near-optimal; namely, as the number of participants in the system becomes large, the social welfare maximizing contribution policy may be approximated by a simple fixed-fee rule. This rule says that a peer who wishes to roam must contribute a fixed amount of resources (such as coverage area, the number of roaming peers he accepts in his own WLAN, or a monetary payment). The required contribution can be computed off-line before the system is instantiated.

An economic model for peering WLANs was proposed in [5] and [6]. The model in these papers differs from ours in assuming that values which participates place on peering (their preference parameters) are known to a global planner. They also discuss security and architectural issues (which may be fundamental for the implementors). We focus on a more demanding, incomplete information model, seeking incentive policies so that participants gain by participating and by being truthful. We also discuss conditions under which no actual money may be part of a peer's contribution, but he may only contribute resources, i.e., make contributions 'in kind'. These issues can be crucial when implementing the system.

We must stress that the business aspect of ubiquitous wireless access is currently receiving lots of attention from communication providers. WISP (Wireless ISP) associations, like *Pass-One*, and large companies, such as *Cometa Networks* (with founders including AT&T, IBM and Intel), are attempting to standardize technologies, protocols and behaviors among existing WISPs in order to make WLAN roaming as seamless as possible. Cometa and other large WISPs attempt to set up new WLAN APs in hot spots and create their own standards, usually by investing a substantial amount of capital in the process. Due to its p2p character, our approach is fundamentally different. The network does not belong to a small number of telecom operators, but to the users themselves.

The paper is organized as follows. In Section 2 we formulate a WLAN peering model and discuss issues of cost. Section 3 formulates the optimization problem. In Section 4 we derive the limiting form and solution for a large system.

2 A WLAN peering model

Suppose that a number of WLANs are available in L locations. Each location is a large geographical area like a neighbourhood or a part of a city centre. Potentially n_ℓ WLANs are available in location ℓ. The owners of the WLANs may arrange to peer with one another, and thus agent a_{ij}, who is the owner of the jth WLAN in location i, benefits when he roams in areas covered by other WLANs. When agreeing to become a peer, a WLAN owner benefits, but he also incurs some cost in providing resources to the community. We seek a mechanism, defined in terms of certain rules, to specify what quantities of resources peers must contribute and what subsidies or payments they might have to make in addition to the benefits they obtain from peering. Our aim is that the incentives given by these rules should be such that when agents act to maximize their own benefits, social welfare is also maximized. To begin, we assume that there is some central authority, a 'global planner', who serves as an intermediary for implementing these rules. Later, we will show that these rule become very simple as the system becomes large.

There are many possible models of benefit and cost. Our basic model supposes coverage in all locations. Once a roaming peer is within coverage, he is accepted (perhaps with some fixed probability < 1). The coverage (quality of service) in location ℓ is defined as the probability Q_ℓ that a roaming peer obtains service in location ℓ, and hence is equal to the proportion of area ℓ that is covered by the footprint of the WLANs. Any congestion created by roaming peers is regarded as a negligible second order effect. We view coverage as a public good. That is, each roaming peer benefits by the amount of coverage available, and does not reduce the probability with which another roaming peer to obtain access. He benefits from, but does not consume, Q_ℓ. The important issue is to provide incentives for this area to grow, while balancing the resulting costs, assuming that existing WLAN owners can, at some cost, increase their area of coverage (say, by upgrading or increasing the number of base stations).

A different model supposes fixed coverage, i.e., footprints of stations are fixed. However, each individual WLAN owner can restrict the number of roaming customers who may simultaneously access the Internet through his infrastructure and so consume some of his bandwidth. The quality of service Q_ℓ models the geographically averaged probability that a roaming peer is granted service in location ℓ. Now incentives must be given to peers to accept more simultaneous roaming customers, while balancing the resulting opportunity cost of the bandwidth they consume.

We shall focus on the first model. Suppose that agent a_{ij} receives total benefit

$$\theta_{ij} \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell)\,,$$

where the preference parameters $\{\theta_{ij}\}_{j=1}^{n_i}$ are independent, identically distributed realizations of random variables with distribution function F_i on $[0,1]$. All the F_is are known to all agents, and to a global planner, but θ_{ij} is known to agent a_{ij} alone. We allow the possibility that agent a_{ij} may or may not be included in the set of agents who peer with one another, i.e., who share their WLANs, and we denote these possibilities by $\pi_{ij} = 1$ and $\pi_{ij} = 0$ respectively.

The cost (incurred by the agents) of ensuring quality Q_ℓ in location ℓ is $c_\ell(Q_\ell, n_\ell)$. Since this quality is realized through averaging the total sum of n_ℓ footprint contributions (each of which is random in space, equally provided by peers and required to cover a fraction Q_ℓ of the area), the cost increases with n_ℓ because of expected overlaps in the footprints. To simplify notation, in the rest of the paper we omit the dependence of c_ℓ on n_ℓ. The social welfare function is

$$\sum_{i=1}^{L}\sum_{j=1}^{n_i} \pi_{ij}\theta_{ij} \sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) - \sum_{i=1}^{L} c_i\left(Q_i\right)\,.$$

The decision variables π_{ij} and Q_i are to be chosen by the global planner as functions of $\boldsymbol{\theta}$, where this denotes a vector of all the θ_{ij}. Let Θ denote the domain of $\boldsymbol{\theta}$ and let $F(\boldsymbol{\theta})$ denote its distribution function, i.e., $F(\boldsymbol{\theta}) = \prod_{i,j} F_i(\theta_{ij})$. Thus the expected social welfare to be maximized is

$$\int_{\Theta} \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})\theta_{ij} \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta})\,. \tag{1}$$

Let us make some remarks on the cost. In the traditional model of public good provisioning, $c_i(\cdot)$ is the cost to a central authority of providing the public good. In our model, no central authority exists to provide and manage the WLAN access points. These belong to the peers themselves. If a global planner is to ensure a quality of service Q_i then he must make available a total amount of resource (area of coverage) by extracting it from the existing WLANs in location i. He acts as a middle man who both pays peers to provide resources and also collects fees from peers that benefit from roaming. Here, $c_i(\cdot)$ is the total cost of the resources that must be purchased by the global planner in location i.

There are two cases to consider: in the first monetary transfers (payments or subsidies) are possible, and agent a_{ij} pays a fee $p_{ij}(\boldsymbol{\theta})$ (positive or negative), which is his contribution towards the total cost of the services subcontracted by the planner. We require $\sum_{ij} p_{ij}(\boldsymbol{\theta}) \geq \sum_i c_i(\cdot)$. In the second case, monetary payments are not possible, but only payments in kind. This means that the cost must be measured in the units of the resource that is to be provided. For this to be possible, we need the cost to be linear in the percentage of area covered by WLAN services. Assuming that this is the case, we must then redefine our monetary unit to be a resource unit, and re-scale other functions appropriately. Now $c_i(\cdot)$ is the amount of resource required in location i and $p_{ij}(\boldsymbol{\theta})$ is the amount of resource that agent a_{ij} contributes. For example, we might take $c_i(Q_i) = Q_i$. When we maximize (1) with respect to the $\pi_{ij}(\boldsymbol{\theta})$ and $Q_i(\boldsymbol{\theta})$, it must be subject to L constraints $\sum_j p_{ij}(\boldsymbol{\theta}) \geq c_i(\cdot)$, $i = 1, \ldots, L$.

We must take account of two further constraints. Agent a_{ij} should expect to benefit by participating (individual rationality). He should also have the incentive to report to the global planner his true value θ_{ij} (incentive compatibility). Let f_i denote the density of F_i, and define

$$g_i(\theta_{ij}) = \theta_{ij} - \frac{1 - F_i(\theta_{ij})}{f_i(\theta_{ij})} \,. \tag{2}$$

We assume $g_i(\cdot)$ is nondecreasing. In Section 3 we consider the case in which monetary payments are possible and show that we can account for all the above constraints and compute the optimal policy of the global planner by maximizing (1) subject to

$$\int_{\Theta} \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) g_i(\theta_{ij}) \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \geq 0 \,. \tag{3}$$

Let $\mathcal{P}(n)$ denote the problem of maximizing (1) subject to (3)[3]

In Section 4 we show that as n becomes large, with $(n_1, \ldots, n_L) = (n\rho_1, \ldots, n\rho_L)$ for some given $\rho_1, \ldots, \rho_L)$, the limiting form of $\mathcal{P}(n)$ is $\hat{\mathcal{P}}(n)$, defined as:

$$\text{maximize} \sum_{i=1}^{L} \left[n_i \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i)\theta_i \, dF_i(\theta_i) - c_i\left(Q_i\right) \right] \tag{4}$$

with respect to the scalars $Q_1, \ldots, Q_L$ and functions $\pi_1(\cdot), \ldots, \pi_L(\cdot)$, subject to the constraint

$$\sum_{i=1}^{L} \left[n_i \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell) \displaystyle\int_0^1 \pi_i(\theta_i) g_i(\theta_i) \, dF_i(\theta_i) - c_i\left(Q_i\right) \right] \geq 0 \,. \tag{5}$$

Suppose $Q_1^*, \ldots, Q_L^*$ and $\pi_1^*(\cdot), \ldots, \pi_L^*(\cdot)$ solve $\hat{\mathcal{P}}(n)$. Then $Q_i(\boldsymbol{\theta}) = Q_i^*$ and $\pi_{ij}(\boldsymbol{\theta}) = \pi_i^*(\theta_{ij})$ are feasible for $\mathcal{P}(n)$ and maximize (1) to within $o(n)$ of its optimum value. In this case, it follows (by some algebra) that the optimal policy requires agents based in location i to contribute a fixed amount of coverage and to pay a fixed monetary fee (positive or negative), both of which depend only on i and not on the agents' declared θ_{ij}s. The global planner posts the expected values of coverage in the various locations and the required footprint contributions and monetary fees. Agents that find it profitable to join under these rules becomes members of the peer group. Agents that do not join are excluded from peering. It turns out that the sum of the monetary fees is zero, and these are essentially used to adjust the total cost of the agents. In this case, the sole role of the global planner is of a clearing agent who in addition checks that the right contributions are made in each location.

In the case where only payments in kind are possible, the optimal policy requires agents to contribute a fixed amount of coverage that depends only on the location. As above, the global planner posts the expected coverage in the various locations and the required footprint contributions, and agents decide whether to join. No money transfers are needed, and the role of the global planner in now simply to check that participating peers do indeed contribute the amounts of resource required.

3 Mechanism design

The basic problem in maximizing social welfare subject to covering the cost by payments is the fact that the global planner does not have complete information regarding the θ_{ij}s of the agents. The situation corresponds to a two stage game where the global planner first posts his policy defined by the functions $Q_i(\boldsymbol{\theta})$, $\pi_{ij}(\boldsymbol{\theta})$, $p_{ij}(\boldsymbol{\theta})$, $i = 1, \ldots, L$, $j = 1, \ldots, n_i$, and then the agents make their declarations of the θ_{ij}s based on the policy of the global planner, and the system is dimensioned accordingly. The global

[3] If only payments in kind are possible, then, as in (21), we would need L constraints, equivalent to requiring that each term in the sum on i on the left hand side of (3) is individually nonnegative.

planner seeks to design his policy so that the end result of the game maximizes social welfare subject to the constraint that the payments cover cost. This falls with the traditional 'mechanism design' paradigm studied by economists. In this section we apply this approach to our problem, exploiting the fact that we have modelled coverage in location i as a public good.

As we have said, there are two possibilities. Either there are monetary transfers, so resources can be purchased from those agents based in location i, or there are no monetary transfers, but agents in location i provide resources in exchange for being allowed to roam in other regions. Throughout what follows we take the first viewpoints: monetary transfers are allowed. We also suppose that there is a mechanism for excluding agents from the peering set. If exclusion is not an option, then we simply make the restriction $\pi_{ij}(\boldsymbol{\theta}) = 1$ for all i, j and $\boldsymbol{\theta}$. We adapt as the ideas in [9] as follows.

An allocation is said to be *feasible* if the sum of the payments made by agents in location i covers the cost of providing quality of service Q_i in that location, i.e.,

$$\sum_{i=1}^{L} \left(\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) p_{ij}(\boldsymbol{\theta}) - c_i\left(Q_i(\boldsymbol{\theta})\right) \right) \geq 0 \tag{6}$$

for all $\boldsymbol{\theta} \in \Theta$. An allocation is *weakly feasible* if the expected sum of the payments covers the expected cost, i.e.,

$$E_{\boldsymbol{\theta}} \left[\sum_{i=1}^{L} \left(\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta}) p_{ij}(\boldsymbol{\theta}) \ - c_i\left(Q_i(\boldsymbol{\theta})\right) \right) \right] \geq 0\,. \tag{7}$$

Note that agents who are excluded do not pay.

Suppose agent a_{ij} pays $p_{ij}(\boldsymbol{\theta})$. Let us define

$$V_{ij}(\theta_{ij}) = \int_{\Theta_{-ij}} \pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \textstyle\sum_{\ell} u_i(Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij}))\, dF(\boldsymbol{\theta}_{-ij}) \tag{8}$$

$$P_{ij}(\theta_{ij}) = \int_{\Theta_{-ij}} \pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) p_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij})\, dF(\boldsymbol{\theta}_{-ij})\,. \tag{9}$$

Here $\boldsymbol{\theta}_{-ij}$ denotes the vector of all preference parameters other than that of agent a_{ij}. Its distribution function is $F(\boldsymbol{\theta}_{-ij})$ and its domain is Θ_{-ij}. Agent a_{ij} must expect to have a positive net benefit and an incentive to report truthfully his value of θ_{ij}. These are the condition of *individual rationality*[4]:

$$\theta_{ij} V_{ij}(\theta_{ij}) - P_{ij}(\theta_{ij}) \geq 0 \tag{10}$$

and *incentive compatibility*:

$$\theta_{ij} V_{ij}(\theta_{ij}) - P_{ij}(\theta_{ij}) \geq \theta_{ij} V_{ij}(\hat{\theta}_{ij}) - P_{ij}(\hat{\theta}_{ij})\,, \quad \text{for all } \hat{\theta}_{ij} \in [0, 1]. \tag{11}$$

We have the following.

Lemma 1. *(a) It is necessary and sufficient for incentive compatibility that (i) $V_{ij}(\theta_{ij})$ is nondecreasing in θ_{ij}, and (ii)*

$$P_{ij}(\theta_{ij}) = P_{ij}(0) + \theta_{ij} V_{ij}(\theta_{ij}) - \int_0^{\theta_{ij}} V_{ij}(\eta)\, d\eta\,. \tag{12}$$

(b) Given incentive compatibility, a necessary and sufficient condition for individual rationality is $P_i(0) \leq 0$.

Proof. Firstly, we must have

$$[\theta^*_{ij} V_{ij}(\theta^*_{ij}) - P_{ij}(\theta^*_{ij})] + [\bar{\theta}_{ij} V_{ij}(\bar{\theta}_{ij}) - P_{ij}(\bar{\theta}_{ij})] \geq [\theta^*_{ij} V_{ij}(\bar{\theta}_{ij}) - P_{ij}(\bar{\theta}_{ij})] + [\bar{\theta}_{ij} V_{ij}(\theta^*_{ij}) - P_{ij}(\theta^*_{ij})]$$

If this were not true then it would be better to declare θ^*_{ij} when $\theta_{ij} = \bar{\theta}_{ij}$, and/or to declare $\bar{\theta}_{ij}$ when $\theta_{ij} = \theta^*_{ij}$. The above gives $(\theta^*_{ij} - \bar{\theta}_{ij})[V_{ij}(\theta^*_{ij}) - V_{ij}(\bar{\theta}_{ij})] \geq 0$ and hence we find the condition that (i) $V_{ij}(\theta_{ij})$ is nondecreasing in θ_{ij}.

[4] Since this is a function of an agent's expected benefit, there can be times when he is be required to pay more than he benefits, in which cases he might decide not to participate. Our model make most sense if users make binding agreements to participate, or if there are repeated rounds, so a user who reneges on participating in one round can be punished in subsequent rounds.

Secondly, since θ_{ij} maximizes $\theta_{ij}V_{ij}(\bar{\theta}_{ij})-P_{ij}(\bar{\theta}_{ij})$ with respect to $\bar{\theta}_{ij}$, we must have, taking derivatives with respect to $\bar{\theta}_{ij}$,

$$\theta_{ij}V'_{ij}(\theta_{ij}) - P'_{ij}(\theta_{ij}) = 0\,.$$

Integrating the above, we find (12). Thus (i) and (ii) are necessary for incentive compatibility. It is easy to check that they are also sufficient.

Individual rationality is the condition that $\theta_{ij}V_{ij}(\theta_{ij}) - P_{ij}(\theta_{ij}) \geq 0$ for all θ_{ij}. Considering this as $\theta_{ij} \to 0$, we see that individual rationality requires $P_{ij}(0) \leq 0$. Conversely, $P_{ij}(0) \leq 0$ and incentive compatibility, implies individual rationality via (8) and (12). ■

Now consider the problem of maximizing expected social welfare (1), subject to the constraint that our mechanism is weakly feasible[5], individually rational and incentive compatible. Since the scheme is to be incentive compatible, we can deduce from (12) that the expected sum of the payments in location i is given by

$$\begin{aligned}
&\sum_{j=1}^{n_i}\int \pi_{ij}(\theta_{ij},\boldsymbol{\theta}_{-ij})p_{ij}(\theta)\,dF(\boldsymbol{\theta}) \qquad (13)\\
&\quad= \sum_{j=1}^{n_i}\int P_{ij}(\theta_{ij})\,dF(\boldsymbol{\theta})\\
&\quad= \sum_{j=1}^{n_i}P_{ij}(0) + \sum_{j=1}^{n_i}\int\left[\theta_{ij}V_{ij}(\theta_{ij}) - \int_0^{\theta_{ij}} V_{ij}(\eta)d\eta\right]\,dF(\boldsymbol{\theta})\\
&\quad= \sum_{j=1}^{n_i}P_{ij}(0) + \sum_{j=1}^{n_i}\int \pi_{ij}(\boldsymbol{\theta})g(\theta_{ij})\textstyle\sum_\ell u_{i\ell}(Q_\ell(\boldsymbol{\theta}))\,dF(\boldsymbol{\theta})\,. \qquad (14)
\end{aligned}$$

Since the scheme is to be weakly feasible, we can use (14) to deduce that our problem is one of maximizing (1) subject to

$$-\sum_{ij}P_{ij}(0) \leq \sum_{ij}\int \left[\pi_{ij}(\boldsymbol{\theta})g(\theta_{ij})\textstyle\sum_\ell u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i\left(Q_i(\boldsymbol{\theta})\right)\right]\,dF(\boldsymbol{\theta})\,,$$

The maximization is with respect to a choice of the functions $Q_i(\boldsymbol{\theta})$ and the constants $P_{ij}(0)$. Restricting ourself to individually rational payments means we must take $P_{ij}(0) \leq 0$ for all i. These enter only through their sum, and we may take every $P_{ij}(0) = 0$. This gives $P_{ij}(\theta_{ij}) \geq 0$ for all θ_{ij}, which is as we wish if the payments are to be made in kind. Hence the problem reduces to one of maximizing (1) subject to

$$\int_\Theta \sum_{i=1}^{L}\left[\sum_{j=1}^{n_i}\pi_{ij}(\boldsymbol{\theta})g_i(\theta_{ij})\textstyle\sum_\ell u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - c_i\left(Q_i(\boldsymbol{\theta})\right)\right]\,dF(\boldsymbol{\theta}) \geq 0\,. \qquad (15)$$

The maximum is to be found by pointwise choice of $\pi_{ij}(\boldsymbol{\theta})$ and $Q_i(\boldsymbol{\theta})$. Having found it, we can calculate $V_{ij}(\theta_{ij})$, and then the $P_{ij}(\theta_{ij})$ from (12). Finally, we set $p_{ij}(\boldsymbol{\theta}) = P_{ij}(\theta_{ij})/E_{\boldsymbol{\theta}_{-ij}}\pi_{ij}(\theta_{ij},\boldsymbol{\theta}_{-ij})$ if $\pi_{ij}(\boldsymbol{\theta}) > 0$ and $p_{ij}(\boldsymbol{\theta}) = 0$ if $\pi_{ij}(\boldsymbol{\theta}) = 0$, so that agent a_{ij} pays only if he is included and (9) holds.

We can establish several more important lemmas. Lemma 3 is used to prove Theorem 1 and establish the limiting problem $\hat{\mathcal{P}}(n)$. Lemma 4 guarantees one of the conditions that we require for incentive compatibility, i.e., (i) on page 5. The following lemma is easy to prove.

Lemma 2. *If for two agents in location i we have $\theta_{ij} > \theta_{ih}$ then an optimal solution must have $\pi_{ij}(\boldsymbol{\theta}) \geq \pi_{ih}(\boldsymbol{\theta})$.*

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[5] In practice, we would like to have the stronger condition of feasibility, so that the required resources to be provided with probability 1. If we are allowed to charge excluded agents, then an argument of [7] shows that a scheme which is weakly feasible, incentive compatible and individual rational can always be turned into one that is feasible, incentive compatible and individual rational. However, this requires some monetary transfer payments between the agents, so we are no longer in a market where the only currency is payment in kind. If excluded agents cannot be charged, then it is not yet clear to us whether a similar result can be proved. Perhaps one can only hope for weak feasibility. But the fact that we are providing the required resources on average may be enough. It is possible to modify the optimal weakly feasible scheme so that as $n \to \infty$, with $n_i = n\rho_i$, the probability the scheme is feasible tends to 1 and the percentage reduction in expected social welfare tends to 0.

Lemma 3. *There exists a Lagrange multiplier λ such that an optimal solution to $\mathcal{P}(n)$ can be found by maximizing the Lagrangian*

$$\int_{\Theta} \sum_{i=1}^{L} \left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})(\theta_{ij} + \lambda g_i(\theta_{ij})) \textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - (1+\lambda) c_i(Q_i(\boldsymbol{\theta})) \right] dF(\boldsymbol{\theta}) \,. \tag{16}$$

The proof is in the Appendix.

Lemma 4. *$V_{ij}(\theta_{ij})$ is nondecreasing in θ_{ij}.*

Proof. Assume first that F_i is the uniform distribution on $[0,1]$. By details we omit the argument generalizes to arbitrary F_i. It is sufficient to show that $\pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \sum_\ell u_{i\ell}(Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij}))$ is nondecreasing in θ_{ij}. For then integrating with respect to $\boldsymbol{\theta}_{-ij}$ would complete the proof. So suppose this were not true and that for a fixed $\boldsymbol{\theta}_{-ij}$, and $\theta'_{ij} > \theta_{ij}$ we have

$$\pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \textstyle\sum_\ell u_{i\ell}(Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij})) > \pi_{ij}(\theta'_{ij}, \boldsymbol{\theta}_{-ij}) \sum_\ell u_{i\ell}(Q_\ell(\theta'_{ij}, \boldsymbol{\theta}_{-ij}))$$

Denote the integrand in (16) by $s(\boldsymbol{\theta})$ and consider $s(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ and $s(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$. Suppose we make a change in which the values of $Q_\ell(\theta_{ij}, \boldsymbol{\theta}_{-ij})$, $\pi_{ij}(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ are interchanged with $Q_\ell(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$, $\pi_{ij}(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$ for all ℓ. With these changes, denote the integrand by $\bar{s}(\boldsymbol{\theta})$. Then as $\theta_{ij} + \lambda_i g_i(\theta_{ij}) \leq \theta'_{ij} + \lambda_i g_i(\theta'_{ij})$ we find

$$s(\theta_{ij}, \boldsymbol{\theta}_{-ij}) + s(\theta'_{ij}, \boldsymbol{\theta}_{-ij}) < \bar{s}(\theta_{ij}, \boldsymbol{\theta}_{-ij}) + \bar{s}(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$$

The integral over all other values of $\boldsymbol{\theta}$ other than $(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ and $(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$ is unchanged. This contradicts the fact that the original choices of the $Q_\ell(\boldsymbol{\theta})$, $\pi_{ij}(\boldsymbol{\theta})$ were optimal, since that would require would require $s(\theta_{ij}, \boldsymbol{\theta}_{-ij}) \geq \bar{s}(\theta_{ij}, \boldsymbol{\theta}_{-ij})$ and $s(\theta'_{ij}, \boldsymbol{\theta}_{-ij}) \geq \bar{s}(\theta'_{ij}, \boldsymbol{\theta}_{-ij})$. ■

4 The provisioning problem for a large system

It is difficult to compute and communicate the $\pi_{ij}(\boldsymbol{\theta})$, $Q_i(\boldsymbol{\theta})$ (which maximize (16)) and the payments that the participants are to make. A central authority would have to learn the preference parameters of all the agents and then communicate the required payments to the agents. Fortunately, when n is large the problem becomes easier. Recall that $\mathcal{P}(n)$ is to maximize (1) subject to (3). We define problem $\hat{\mathcal{P}}(n)$ as one of maximizing (4) subject to (5) when $n_i = n\rho_i$, for some given $\rho_1, \ldots, \rho_L$, where $\sum_i \rho_i = 1$.

Theorem 1. *Let Φ_n and $\hat{\Phi}$ denote the optimal values of $\mathcal{P}(n)$ and $\hat{\mathcal{P}}(n)$ respectively. Then*

$$\hat{\Phi}_n \leq \Phi_n \leq \hat{\Phi}_n + o(n) \,.$$

Moreover, if the decision variables $Q_1^*, \ldots, Q_L^*$ and $\pi_1^*(\cdot), \ldots, \pi_L^*(\cdot)$ solve $\hat{\mathcal{P}}(n)$, then by taking $\pi_{ij}(\boldsymbol{\theta}) = \pi_i^*(\theta_{ij})$ and $Q_i(\boldsymbol{\theta}) = Q_i^*$, for all $\boldsymbol{\theta}$ and i, j, we have a feasible solution for $\mathcal{P}(n)$ for which the value of the expected social welfare is $\hat{\Phi}_n$, i.e., suboptimal by only $o(n)$.

Form of the limiting solution Note that the optimal solution to $\hat{\mathcal{P}}(n)$ can be computed off-line. In particular, $\pi_i^*(\theta_i)$ is 1 or 0 as θ_i does or does not exceed a threshold, say θ_i^*. We find that (8) becomes

$$V_i(\theta_{ij}) = \pi_i(\theta_{ij}) \textstyle\sum_\ell u_i(Q_\ell) \tag{17}$$

and from (12) we have

$$P_i(\theta_{ij}) = \theta_i^* V_i(\theta_{ij}) \,. \tag{18}$$

Thus every WLAN in location i that is included for roaming is required to do two things: (i) pay the same fixed fee $\sum_\ell \theta_i^* u_i(Q_\ell)$. This fee can be calculated (knowing the n_i, u_i and F_i) and communicated before hearing the values of the preference parameters; (ii) contribute its share of the cost in terms of coverage, this being Q_i/n_i. Assuming that all agents have identical cost function for coverage, each should provide an equal share. $\hat{\mathcal{P}}(n)$ is now just

$$\underset{Q_1,\ldots,Q_L,\theta_1^*,\ldots,\theta_L^*}{\text{maximize}} \sum_{i=1}^{L} \Big[n_i(1 - F_i(\theta_i^*)) \textstyle\sum_\ell u_i(Q_\ell) - c_i(Q_i) \Big] \tag{19}$$

subject to

$$\sum_{i=1}^{L}\Big[n_i(1-F_i(\theta_i^*))\,\theta_i^*\sum_\ell u_i(Q_\ell)-c_i(Q_i)\Big]\geq 0\,. \tag{20}$$

The proof is in the Appendix and is given for the more general case in which only payments in kind are possible. Now we require L constraints and so replace (3) by

$$\int_\Theta\left[\sum_{j=1}^{n_i}\pi_{ij}(\boldsymbol{\theta})g_i(\theta_{ij})\sum_{\ell=1}^{L}u_{i\ell}(Q_\ell(\boldsymbol{\theta}))-c_i(Q_i(\boldsymbol{\theta}))\right]dF(\boldsymbol{\theta})\geq 0\,. \tag{21}$$

$i=1,\dots,L$. Lemma 3 must be generalized to Lemma 5. Its proof is in the appendix.

Lemma 5. *There exist $\lambda_1,\dots,\lambda_L>0$ such that $\mathcal{P}(n)$ can be solved by maximizing the Lagrangian*

$$\int_\Theta\sum_{i=1}^{L}\left[\sum_{j=1}^{n_i}\pi_{ij}(\boldsymbol{\theta})(\theta_{ij}+\lambda_i g_i(\theta_{ij}))\sum_{\ell=1}^{L}u_{i\ell}(Q_\ell(\boldsymbol{\theta}))-(1+\lambda_i)c_i\left(Q_i(\boldsymbol{\theta})\right)\right]dF(\boldsymbol{\theta})\,. \tag{22}$$

The maximization is carried out pointwise. That is, for each $\boldsymbol{\theta}$, the values of $\pi_{ij}(\boldsymbol{\theta})$ and $Q_i(\boldsymbol{\theta})$ are chosen to maximize the integrand in (22).

Acknowledgement We thank Panos Antoniadis for helping to formulate the WLAN peering model.

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Appendix

A Proofs of Lemma 3 and 5

We prove Lemma 5, which we need for $\mathcal{P}(n)$ when only payments in kind are possible. The proof of Lemma 3 is similar. This is the problem of maximizing (1) subject to (21). Let us rewrite this as the problem of maximizing

$$\int_\Theta\sum_{i=1}^{L}\left[\sum_{j=1}^{n_i}x_{ij}(\boldsymbol{\theta})-c_i(Q_i(\boldsymbol{\theta}))\right]dF(\boldsymbol{\theta})\,, \tag{23}$$

with respect to $x_{ij}(\boldsymbol{\theta})$, $Q_i(\boldsymbol{\theta})$, subject to

$$Q_i(\boldsymbol{\theta})\geq 0\,,\quad x_{ij}(\boldsymbol{\theta})\geq 0\,, \tag{24}$$

$$x_{ij}(\boldsymbol{\theta})-\theta_{ij}\sum_{\ell=1}^{L}u_{i\ell}(Q_\ell(\boldsymbol{\theta})\leq 0\,,\quad \text{for all } i,\ j,\ \boldsymbol{\theta} \tag{25}$$

and

$$-\int_{\Theta}\left[\sum_{j=1}^{n_i} x_{ij}(\boldsymbol{\theta})\frac{g_i(\theta_{ij})}{\theta_{ij}} - c_i(Q_i(\boldsymbol{\theta}))\right] dF(\boldsymbol{\theta}) \leq 0\,, \quad i=1,\dots,L\,. \tag{26}$$

Assuming that $u_i(\cdot)$ is concave and $c_i(\cdot)$ is convex, the objective function (23) is a concave function of the decision variables, and (24)–(26) define a region that is convex in the decision variables. These are sufficient conditions for the problem to be solvable by maximizing a Lagrangian. That is, there exist $\lambda_1,\dots,\lambda_L$ such that we can solve the problem by maximizing

$$\int_{\Theta}\sum_{i=1}^{L}\left[\sum_{j=1}^{n_i} x_{ij}(\boldsymbol{\theta})\left(1+\lambda_i\frac{g_i(\theta_{ij})}{\theta_{ij}}\right) - (1+\lambda_i)c_i(Q_i(\boldsymbol{\theta}))\right] dF(\boldsymbol{\theta})\,, \tag{27}$$

with respect to $Q_i(\boldsymbol{\theta})$, $x_{ij}(\boldsymbol{\theta})$, subject to (25). This is equivalent to maximizing

$$\int_{\Theta}\sum_{i=1}^{L}\left[\sum_{j=1}^{n_i} \pi_{ij}(\boldsymbol{\theta})(\theta_{ij}+\lambda_i g_i(\theta_{ij}))\textstyle\sum_{\ell=1}^{L} u_{i\ell}(Q_\ell(\boldsymbol{\theta}) - (1+\lambda_i)c_i(Q_i(\boldsymbol{\theta}))\right] dF(\boldsymbol{\theta})\,, \tag{28}$$

with respect to $Q_i(\boldsymbol{\theta})$, $\pi_{ij}(\boldsymbol{\theta})$, subject to $0 \leq \pi_{ij}(\boldsymbol{\theta}) \leq 1$. ■

B Proof of Theorem 1

Note that solving $\hat{\mathcal{P}}(n)$ is equivalent to solving $\mathcal{P}(n)$ under the additional constraints that $\pi_{ij}(\boldsymbol{\theta})$ depends only on θ_{ij} and $Q_i(\cdot)$ depends only on i. This fact immediately gives $\hat{\Phi}_n \leq \Phi_n$. Moreover, if we take as a solution to $\mathcal{P}(n)$, $\pi_{ij}(\boldsymbol{\theta}) = \pi_i^*(\theta_{ij})$ and $Q_i(\boldsymbol{\theta}) = Q_i^*$ for all $\boldsymbol{\theta}$ and i,j, then these define a weakly feasible, incentive compatible and individually rational scheme that has expected social welfare equal to $\hat{\Phi}_n$. We can set $p_{ij}(\theta_{ij})$ equal to $P_{ij}(\theta_{ij})$, where $P_{ij}(\theta_{ij})$ is calculated via (8) and (12).

It remains to show that $\Phi_n \leq \hat{\Phi}_n + o(n)$. By Lemma 5, the problem can be solved by maximizing a Lagrangian with Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1,\dots,\bar{\lambda}_L)$. Then for $\bar{\lambda}$ and all other λ we have

$$\begin{aligned}\Phi_n &= \max_{Q_\ell(\cdot),\pi_{\ell j}(\cdot)}\int_{\Theta}\sum_{\ell=1}^{L}\left[\sum_{j=1}^{n_\ell}\pi_{\ell j}(\boldsymbol{\theta})(\theta_{\ell j}+\bar{\lambda}_\ell g_\ell(\theta_{\ell j}))\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h(\boldsymbol{\theta})) - (1+\bar{\lambda}_\ell)c_\ell(Q_\ell(\boldsymbol{\theta}))\right] dF(\boldsymbol{\theta})\\ &\leq \max_{Q_\ell(\cdot),\pi_{\ell j}(\cdot)}\int_{\Theta}\sum_{\ell=1}^{L}\left[\sum_{j=1}^{n_\ell}\pi_{\ell j}(\boldsymbol{\theta})(\theta_{\ell j}+\lambda_\ell g_\ell(\theta_{\ell j}))\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h(\boldsymbol{\theta})) - (1+\lambda_\ell)c_\ell(Q_\ell(\boldsymbol{\theta}))\right] dF(\boldsymbol{\theta}) \end{aligned} \tag{29}$$

We will show that the integral in (29) is bounded above by $\hat{\Phi}_n + o(n)$, where

$$\hat{\Phi}_n = \inf_{\lambda}\max_{Q_\ell,\pi_\ell(\cdot)}\sum_{\ell=1}^{L} n_\ell\left[\textstyle\sum_{h=1}^{L} u_{\ell h}(Q_h)\displaystyle\int_0^1 \pi_\ell(\theta_\ell)(\theta_\ell+\lambda_\ell g_\ell(\theta_\ell))\,dF_\ell(\theta_\ell) - (1+\lambda_\ell)c_\ell(Q_\ell)/n_\ell\right] \tag{30}$$

Let us suppose that each F_ℓ is the uniform distribution. It is notationally more elaborate, but routine, to prove the theorem for general F_ℓ. Imagine dividing the interval $[0,1]$ into k equal parts, defining

$$I_i = \left[\frac{i-1}{k},\frac{i}{k}\right)\,, \quad i=1,\dots,k.$$

Let the random variable $X_{\ell i}$ be the number of the $\theta_{\ell 1},\dots,\theta_{\ell n_\ell}$ that are in I_i, Note that $X_{\ell i}$ has a binomial distribution with mean n_ℓ/k, and that by Chebyshev's inequality we have

$$P(|X_{\ell i} - n_\ell/k| > \epsilon) \leq \frac{n_\ell(1-1/k)(1/k)}{\epsilon^2} \leq \frac{n}{\epsilon^2}\,.$$

We shall use this with $\epsilon = n^{2/3}$. Let us define the set $S = \{\boldsymbol{\theta} : |X_{\ell i} - n_\ell/k| \leq n^{2/3},\ \text{for all } \ell, i\}$. Then

$$P(S^c) = P\left(\bigcup_{\ell=1}^{L}\bigcup_{i=1}^{k}\left\{|X_{\ell i}-n_\ell/k| > n^{2/3}\right\}\right) \leq \sum_{\ell=1}^{L}\sum_{i=1}^{k} P\left(\left\{|X_{\ell i}-n_\ell/k| > n^{2/3}\right\}\right) \leq \frac{1}{n^{1/3}}\,.$$

Let us assume that the minimizing λ in (30) tends to a finite limit as $n \to \infty$. Let $\varLambda$ be a some compact subset of $\mathbb{R}^L$ which contains both $\bar{\lambda}$ and also the minimizing λ in (30) for all n. Let $s(\boldsymbol{\theta})$ denote the integrand in (29) for a given $\lambda \in \varLambda$. Then we have for (29)

$$\max_{Q_\ell(\cdot),\,\pi_{\ell j}(\cdot)} \int_{\Theta} s(\boldsymbol{\theta})\,dF(\boldsymbol{\theta}) \leq \max_{Q_\ell(\cdot),\,\pi_{\ell j}(\cdot)} \int_{S} s(\boldsymbol{\theta})\,dF(\boldsymbol{\theta}) + \max_{Q_\ell(\cdot),\,\pi_{\ell j}(\cdot)} \int_{S^c} s(\boldsymbol{\theta})\,dF(\boldsymbol{\theta}) \tag{31}$$

Since $P(S^c) \leq 1/n^{1/3}$ we can bound the final term in (31) by $(1/n^{1/3})(nB_\varLambda)$, where $B_\varLambda$ is a bound such that for all i, j, $\boldsymbol{\theta}$, and $\lambda \in \varLambda$.

$$\pi_{ij}(\boldsymbol{\theta})(\theta_{ij} + \lambda_i g_i(\theta_{ij})) \textstyle\sum_{\ell=1}^L u_{i\ell}(Q_\ell(\boldsymbol{\theta})) - (1+\lambda_i)c_i(Q_i(\boldsymbol{\theta})) \leq B_\varLambda\,.$$

We bound the first term in on the right hand side of (31) by

$$\max_{Q_\ell(\cdot),\pi_{j\ell}(\cdot)} \int_S s(\boldsymbol{\theta})\,dF(\boldsymbol{\theta}) \tag{32}$$

$$\leq \max_{\substack{Q_1,\dots,Q_L,\theta\in S,\\ \theta_1\in I_1,\dots,\theta_k\in I_k\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^L \left[\sum_{i=1}^k X_{\ell i}\pi_\ell(\theta_i)(\theta_i + \lambda_\ell g_\ell(\theta_i)) \textstyle\sum_{h=1}^L u_{\ell h}(Q_h) - (1+\lambda_\ell)c_\ell(Q_\ell)\right] \tag{33}$$

$$\leq \max_{\substack{Q_1,\dots,Q_L,\theta\in S,\\ \theta_1\in I_1,\dots,\theta_k\in I_k\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^L \left[\sum_{i=1}^k (n_\ell/k)\pi_\ell(\theta_i)(\theta_i + \lambda_\ell g_\ell(\theta_i)) \textstyle\sum_{h=1}^L u_{\ell h}(Q_h) - (1+\lambda_\ell)c_\ell(Q_\ell)\right] \tag{34}$$

$$+ B_\varLambda \sum_{\ell=1}^L \sum_{i=1}^k |X_{\ell i} - n_\ell/k| \tag{35}$$

The second term in (35) is bounded by $n^{2/3}LkB_\varLambda$.

Given any $\epsilon > 0$ we can choose k sufficiently large so that the intervals I_i have very small widths, of $1/k$, and so we can have (using continuity and approximation of an integral by a Riemann sum)

$$\max_{\substack{Q_1,\dots,Q_L,\\ \theta_1\in I_1,\dots,\theta_k\in I_k\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^L \left[\sum_{i=1}^k (n_\ell/k)\pi_\ell(\theta_i)(\theta_i + \lambda_\ell g_\ell(\theta_i)) \textstyle\sum_{h=1}^L u_{\ell h}(Q_h) - (1+\lambda_\ell)c_\ell(Q_\ell)\right]$$

$$\leq \max_{\substack{Q_1,\dots,Q_L\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^L n_\ell \left[\textstyle\sum_{h=1}^L u_{\ell h}(Q_h) \displaystyle\int_0^1 \pi_\ell(\theta_i)(\theta_\ell + \lambda_\ell g_\ell(\theta_\ell))\,dF_\ell(\theta_i) - (1+\lambda_\ell)c_\ell(Q_\ell)/n_\ell\right] + n\epsilon/2$$

Given this k we can then choose n sufficiently large that $n^{2/3}LkB_\varLambda$ is less than $n\epsilon/2$. It follows, that given any $\epsilon > 0$ it is possible to choose k sufficiently large and then n sufficiently large to deduce that for n sufficiently large (but depending on $\varLambda$),

$$\varPhi_n \leq \max_{\substack{Q_1,\dots,Q_\ell\\ \pi_1(\cdot),\dots,\pi_L(\cdot)}} \sum_{\ell=1}^L n_\ell \left[\textstyle\sum_{h=1}^L u_{\ell h}(Q_h) \displaystyle\int_0^1 \pi_\ell(\theta_i)(\theta_\ell + \lambda_\ell g_\ell(\theta_\ell))\,dF_\ell(\theta_i) - (1+\lambda_\ell)c_\ell(Q_\ell)/n_\ell\right] + n\epsilon$$

By taking an infimum over $\lambda \in \varLambda$ on the right hand side we deduce $\varPhi_n \leq \hat{\varPhi}_n + o(n)$. The fact that the limiting optimization problem in (4), (5) is solved by the Lagrangia corresponding to $\hat{\varPhi}_n$ follows by a proof similar to that of Lemma 5.

■