The influence of conjugacy class sizes on the structure of finite groups: a survey

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1   Introduction

This survey considers the important problem of the influence of the size of conjugacy classes on the structure of finite groups. Over the last thirty years this has been a flourishing and active area of research. Now is an appropriate time to bring together many of the major results in one place. This is especially relevant as some authors seem unaware both of others writing in the field and of some of the older results. It is hoped that by writing this survey we will bring these ideas to an even wider audience.

The paper is divided into sections, both to make it easier to read but also to enable the reader to concentrate on the particular aspects which might take their interest. In addition, we have posed a number of questions throughout the survey, which, hopefully, will inspire the reader to become involved: solving questions and developing new techniques.

How much information can one expect to obtain from the size of conjugacy classes? In 1872, Sylow examined what happened if there was information about the size of all conjugacy classes, whereas in 1904 Burnside showed that strong results could be obtained if there was particular information about the size of just one conjugacy class. In 1903, Landau bounded the order of the group in terms of the number of conjugacy classes whilst in 1919, Miller gave

a detailed analysis of groups with very few conjugacy classes. Then very little seems to have been done, apart from two short papers by Tchounikhin in 1930 and 1934, until 1953. In that year both Baer and Itô published papers on this topic but with different conditions on the sizes.

By looking at these early results, it can been seen that much will depend on how much information is given and it is important to be explicit. For example if it is known that there is only one conjugacy class size, then the group is abelian but this can be any abelian group. If, however, the conjugacy class sizes and their multiplicities are given, then the order of the group can be determined. However, it would still not be possible to identify the group. Some authors have considered the situation where the multiplicities of the conjugacy class sizes are used only if the size is not 1. This is particularly common when the authors have been studying aspects of the problem related to graphs. Again if we only demand information about the sizes of conjugacy classes and not their multiplicities, the groups G and $G \times A$ will have the same set of conjugacy class sizes whenever A is an abelian group. So we can only state results modulo a direct abelian factor.

Another reason for examining conjugacy classes is their fundamental role in understanding the group ring, especially over $\mathbb{C}$. Recall that if G is a group then the group ring $\mathbb{C}[G]$ is the vector space over $\mathbb{C}$ generated with basis the elements of G. The product is defined for basis elements g, h by using the multiplication in G and then extending by linearity. Thus :-

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{g \in G} b_g g \right) = \left(\sum_{g \in G} \left(\sum_{h,k \in G, hk=g} a_h b_k \right) g \right).$$

For each conjugacy class K of G, define the element $c_K = \sum_{g \in K} g$. Then c_K is in the centre of $\mathbb{C}[G]$ and the elements c_K are a basis for the centre. This gives rise to a complete link to the character theory of G. If $K_1, K_2, \ldots, K_k$ are the conjugacy classes of G, then there are integers $a_{rst} : 1 \leq r, s, t, \leq k$ so that

$$c_{K_r} c_{K_s} = \sum_t a_{rst} c_{K_t}.$$

Knowledge of the a_{rst} is equivalent to knowing the character table. This is a result that goes back to Frobenius, see Lam [Lam98, Theorem 7.6]. So at this level of information we see that knowing character and conjugacy class data are the same.

As we are not mainly concerned with characters and only mention results as comparative results, we refer the reader to [Hup98] or [Isa06], amongst many others, for a discussion of character theory and some of the necessary definitions. We discuss these connections further in Section 7.

We will often refer to *the index* of an element, this is just the size of the conjugacy class containing the element. The benefit of this definition is entirely linguistic. Given an element g in some group G we can talk about the index of g rather than talking about the size of the conjugacy class containing g. So if we are referring to elements we use the term index but if we are talking about conjugacy classes we refer to size.

In Section 2, we introduce the basic definitions and consider some elementary results which are used time and time again (and proved time and time again).

Section 3 considers results which might loosely be described as placing arithmetical conditions on the indices of elements. One of the most famous is Burnside's p^a-theorem. In this, the restriction is that there is an index which is a prime-power. We then look at some conditions which imply solubility. Here the results are quite weak. We see a distinction between demanding conditions on all the indices of elements and only on some. This occurs quite frequently, so for example, if it is known that the indices of all elements are powers of a given prime p then the group is essentially a p-group. Then we can ask what happens if we only ask that p-elements have index a power of p. In this case there are fewer restrictions on the indices but there is a need to recognise the order of elements in the group.

The conjugate type vector is introduced in Section 4. This considers the distinct indices in descending order and looks at various properties of the group which can be deduced from this sequence. Some authors have considered variations. For example, for a fixed prime p they consider the sequence of indices for p'-elements.

In Section 5, the various graphs that can be constructed from the sets of conjugacy class sizes are defined. The properties of the graphs and the relation to the structure of the groups is examined. This has been a very active area in recent years. This has also been the case where character degrees have been used similarly, see the survey by Lewis [Lew08].

The influence of the number of conjugacy classes is examined in Sec-

tion 6 and, as mentioned previously, connections with character theory are considered in Section 7.

There is a large body of work considering the influence of conjugacy class sizes on p-groups. We do not mention these results here, unless they fit in a natural way, but refer the reader to a survey paper on p-groups by A. Mann; one section is devoted to 'Representations and Conjugacy Classes' [Man99].

Notation: The notation we use is standard. Let G be a finite group and x an element of G. The centraliser of $X \subseteq G$ is $\{g \in G : xg = gx \,\forall x \in X\}$ and denoted by $\mathrm{C}_G(X)$. Note that if $X = \{x\}$ we drop the brackets. The conjugacy class of x in G is denoted by x^G. We denote the derived group of G by G' and the Fitting subgroup of G by $\mathrm{Fit}(G)$. If π is a set of primes then π' is the set of primes not in π. An element $g \in G$ is a p-element if the order of g is a p-power. A p'-element is one whose order is prime to p, these elements are also known as p-regular elements. A p-singular element is one which is not a p'-element. The largest normal subgroup of G whose order is a π-number is denoted by $\mathrm{O}_\pi(G)$. Furthermore, $\mathrm{O}^\pi(G)$ is the smallest normal subgroup whose factor group $G/\mathrm{O}^\pi(G)$ is a π-group. In a group G a p-complement is a subgroup H whose index in G is exactly the highest power of p to divide the order of G. We use CFSG to denote the Classification of Finite Simple Groups.

2 Basic definitions and results

Throughout this paper G will be a finite group and p will be a prime.

Baer [Bae53] gave the following definition which we will use:-

Definition 1. *Let G be a finite group and let $x \in G$. The index of x in G is given by $[G : \mathrm{C}_G(x)]$. By the orbit-stabiliser theorem this is the size of the conjugacy class containing x and hence denoted by $|x^G|$.*

Lemma 2. *Let N be a normal subgroup of G. Then*
(i) if $x \in N$, $|x^N|$ divides $|x^G|$.
(ii) if $x \in G$, $|(xN)^{G/N}|$ divides $|x^G|$.

Lemma 3. *[Cam72] Let p be a prime and suppose $|x^G|$ is coprime to p for all p'-elements $x \in G$. Then the Sylow p-subgroup of G is a direct factor of G.*

Proof. Let P be a Sylow p-subgroup of G and let $C = C_G(P)$. We show that PC contains a conjugate of every element in G and thus $G = PC$ by Burnside [Bur55, §26]. The lemma then follows.

If $g \in G$ we can write $g = xy$ where $[x, y] = 1$ and x is a p-element and y is a p'-element. By the hypothesis we have that $C_G(y)$ contains a conjugate of P. So, conjugating as necessary, we can ensure that $P \subset C_G(y)$. There exists $h \in C_G(y)$ so that $x^h \in P$. Hence $(xy)^h = x^h y \in PC$ as required.□

Corollary 4. *Suppose p is a prime which is coprime to $|x^G|$ for all $x \in G$. Then the Sylow p-subgroup of G is an abelian direct factor of G.*

We comment here that this shows a dichotomy with the related theory for character degrees. For if G is a finite group with a normal abelian Sylow p-subgroup then p does not divide any character degree. This is a result of Itô [Itô51]. The converse is also true, though the proof was not completed until 1986 with an application of CFSG [Mic86].

Another useful lemma is the following.

Lemma 5. *Let x and y be two elements of G such that $C_G(x)C_G(y) = G$. Then $(xy)^G = x^G y^G$.*

Proof. Consider $x^g y^h$ with $g, h \in G$. Clearly $x^g y^h$ is conjugate to $x^{gh^{-1}} y$. We can write $gh^{-1} = ab$ where $a \in C_G(x)$ and $b \in C_G(y)$. Thus $x^{gh^{-1}} y = x^{ab} y$ which is conjugate to $x^a y^{b^{-1}} = xy$ as required.□

Corollary 6. *Let x and y be two elements of G such that $|x^G|$ and $|y^G|$ are coprime then $(xy)^G = x^G y^G$.*

This idea was discussed by Tchounikhin in 1930 [Tch30]. In this paper he showed that if there are three indices which are pairwise coprime then the group is not simple. More recently Arad & Fisman proved that if there are two indices which are coprime then the group is not simple [FA87]. Their proof uses CFSG.

We now give some definitions which are useful and provide some unifying themes to our survey.

Definition 7. *Let G be a finite group and $x \in G$. Define*
(i) $\sigma_G(x)$ to be the set of primes dividing $|x^G|$.
(ii) $\sigma^*(G) = \max\{|\sigma_G(x)| : x \in G\}$ and
(iii) $\rho^*(G) = \bigcup_{x \in G} \sigma_G(x)$.

By Corollary 4 it follows that $\rho^*(G)$ is just the set of primes that divide $G/Z(G)$. This set is also known as the set of *eccentric* primes.

The following notation is useful.

Definition 8. *(i) Let* $\mathrm{cs}(G)$ *denote the set of conjugacy class sizes of* G *(equivalently the set of indices of elements of* G*).*
(ii) More generally, for $X \subseteq G$*, let* $\mathrm{cs}(X)$ *denote the set of indices of elements in* X*.*
(ii) Let $\mathrm{cd}(G)$ *denote the set of degrees of irreducible characters of* G*.*

In 1953 Itô introduced the notion of a conjugate type vector:

Definition 9. *[Itô53] The conjugate type vector of a group* G *is the vector* $(n_1, n_2, \ldots, n_r, 1)$ *where* $n_1 > n_2 > \ldots n_r > 1$ *are the distinct indices of elements of* G*. The conjugate rank of* G*,* $\mathrm{crk}(G)$*, is given by* r*.*

Clearly $|\mathrm{cs}(G)| = \mathrm{crk}(G) + 1$.

Definition 10. *(i) A Frobenius group is a group* G *which contains a proper, non-trivial subgroup* H *such that* $H \cap H^g = 1$ *for all* $g \notin H$*. Equivalently* G *is a transitive permutation group in which only the identity element fixes more than one point and there is a non-trivial element which fixes one point.*
(ii) A group G *is called quasi-Frobenius if* $G/Z(G)$ *is Frobenius.*

In a Frobenius group G the subgroup H is called the complement of G. Furthermore, G contains a normal subgroup K, called the kernel, such that G can be written as a product KH, where $\mathrm{C}_G(x) \subseteq K$ for all $1 \neq x \in K$. The orders of H and K are coprime, see [Hup67, V.8].

Let G be quasi-Frobenius and let the pre-image of the kernel and the complement be $\hat{K}$ and $\hat{H}$ respectively. Then, if both $\hat{K}$ and $\hat{H}$ are abelian, the non-trivial conjugacy class sizes of G are $m = |\hat{H}/Z(G)|$ and $n = |\hat{K}/Z(G)|$. Note $\gcd(m, n) = 1$. That a group is quasi-Frobenius often appears as the conclusion of a number of theorems.

3 Arithmetical properties

3.1 Prime-power index

Perhaps the earliest result is that of Sylow.

Theorem 11. *[Syl72] Let G be a finite group all of whose indices are a power of a given prime. Then G has a non-trivial centre.*

The next result, perhaps the most dramatic, is that of Burnside.

Theorem 12. *[Bur04] Let G be a finite group and suppose there exists x in G of prime-power index. Then G is not simple.*

These are two classic results which give information about the group from some arithmetical properties of the set of indices. It is interesting that the first is a very straightforward counting argument using the class equation and the second is a sophisticated proof which depends on character theory.

In 1990 Kazarin proved the following extension to Burnside's result:

Theorem 13. *[Kaz90] Let G be a finite group and let x be an element of G such that $|x^G| = p^a$ for some prime p and integer a. Then $\langle x^G \rangle$ is a soluble subgroup of G.*

Kazarin's proof depends on modular character theory. Using this theorem Camina & Camina [CC98] proved that any element of prime-power index is in the second Fitting subgroup. Flavell [Fla02] proved that an element $x \in G$ is in the second Fitting subgroup if and only if x is in the second Fitting subgroup of $\langle x, y \rangle$ for all $y \in G$. This led to an interesting discussion of how the indices of elements in two-generator subgroups can determine the index of an element in the whole group, see [CSS10].

Baer [Bae53] characterized all finite groups such that every element of prime-power order has prime-power index. He then went on to raise the question of the characterization of those groups whose q-elements, for just one prime q, have prime power index. Camina & Camina [CC98] introduced the following idea based on Baer:

Definition 14. *Let G be a finite group and let q be a prime such that q divides $|G|$. Then G is a q-Baer group, or equivalently has the q-Baer property, if every q-element of G has prime power index.*

They then proved the following theorem:

Theorem 15. *[CC98] Let G be a q-Baer group for some prime q. Then*
(a) G is q-soluble with q-length 1, and

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(b) *there is a unique prime p such that each q-element has p-power index.*
Further, let Q be a Sylow q-subgroup of G, then
(c) *if $p = q$, Q is a direct factor of G, or*
(d) *if $p \neq q$, Q is abelian, $O_p(G)Q$ is normal in G and $G/O_{q'}(G)$ is soluble.*

In 2005, [BK05], Berkovic & Kazarin proved some very similar results but they also proved some new results. One of their results is the following:

Theorem 16. *[BK05] If the index of every p-element of order p (if p is odd) or 4 (if $p = 2$) is a p-power, then G has a normal p-complement.*

Beltrán & Felipe have also proved some similar results, [BF04c].

Another result which guarantees the existence of a normal p-complement is the following:

Theorem 17. *[Cam72] Let G be a group such that p^a is the highest power of the prime p which divides the index of an element of G. Assume that there is a p-element in G whose index is precisely p^a. Then G has a normal p-complement.*

Recall, Lemma 3 says that if all p'-elements have p'-index then the Sylow p-subgroup of G is a direct factor. An interesting strengthening of this result is due to Liu, Wang & Wei:

Theorem 18. *[LWW05] Suppose p is a prime that is coprime to $|x^G|$ for all p'-elements x of G of prime-power order. Then the Sylow p-subgroup of G is a direct factor of G.*

The proof uses the result of [FKS81] which shows that in a transitive permutation group there is an element of prime-power order which acts fixed-point-freely. This result uses CFSG.

In another variation on the theme Dolfi and Lucido say a finite group G has property $P(p, q)$ if every p'-element has q'-index, where p and q are primes. The inspiration for this definition came from ideas in character theory. In particular, a group G has the property $BP(p, q)$ if every p-Brauer character has degree coprime to q. Dolfi & Lucido prove (amongst other things) the following:

Theorem 19. *[DL01] Let p and q be distinct primes and suppose G is a finite group satisfying $P(p, q)$. Then $\mathrm{O}^p(G)$ is q-nilpotent and G has abelian Sylow q-subgroups.*

A significant portion of the paper is taken up with showing that if G is an almost simple group satisfying $P(p, q)$, then $\gcd(|G|, q) = 1$, this uses CFSG.

Interestingly, an example where a given prime almost does not occur is given by:

Theorem 20. *[DMN09] Let G be a finite group having exactly one conjugacy class of size a multiple of a prime p. Then one of the following holds:*
(i) G is a Frobenius group with Frobenius complement of order 2 and Frobenius kernel of order divisible by p;
(ii) G is a doubly transitive Frobenius group whose Frobenius complement has a nontrivial central Sylow p-subgroup;
(iii) p is odd, $G = KH$ where $K = \mathrm{Fit}(G)$ is a q-group, q prime, $H = C_G(P)$ for a Sylow p-subgroup P of G, $K \cap H = Z(K)$ and $G/Z(K)$ is a doubly transitive Frobenius group.

In the paper there is a slightly more detailed version which gives if and only if conditions.

3.2   Solubility

Recognising solubility is clearly an interesting problem. In 1990, Chillag & Herzog proved the following:

Theorem 21. *[CH90] Suppose $|x^G|$ is not divisible by 4 for all $x \in G$. Then G is soluble.*

A proof avoiding CFSG is given in [CC98] and [CW99]. Chillag & Herzog also considered those groups all of whose indices are square-free; these groups were also studied by Cossey & Wang [CW99].

Theorem 22. *[CW99] Suppose that all indices of the group G are square-free. Then G is supersoluble and both $|G/\mathrm{Fit}(G)|$ and G' are cyclic groups with square-free orders. The class of $\mathrm{Fit}(G)$ is at most 2 and G is metabelian.*

Using the results of [FKS81] quoted in Section 3.1, Li strengthened these results as follows:

Theorem 23. *[Li99] Let G be a group and let p be the smallest prime dividing the order of G. Assume that p^2 does not divide the index of any element of q-power order, for q any prime not equal to p. Then G is p-nilpotent. In particular, G is soluble.*

Theorem 24. *[Li99] Suppose that all indices of elements of prime-power order in the group G are square-free. Then G is supersoluble, the derived length of G is bounded by 3, $G/\mathrm{Fit}(G)$ is a direct product of elementary abelian groups and $|\mathrm{Fit}(G)'|$ is a square-free number.*

In [QW09a] Qian & Wang considered the situation when only p'-elements have indices not divisible by p^2. They showed that in this case the highest power of p which can divide the order of any chief factor is p.

Note, if p is the smallest prime that divides the order of a group, then for any other prime q dividing the order of the group q does not divide $p-1$. Using this observation Cossey & Wang [CW99] considered groups G for which there is a prime p dividing the order of G so that if q divides the order of G then q does not divide $p-1$. They prove a result giving the structure of such groups when the index of no element of G is divisible by p^2. A variation on this is given by Liu, Wang and Wei [LWW05]. We note that with these conditions if $p = 2$ then G is soluble by Theorem 21 and if $p > 2$ then G has odd order.

If the prime p divides the order of the group and there exist primes which do not divide $p-1$, we can say something slightly more general:

Lemma 25. *Let G be a soluble group and let p be a prime. Assume that all elements of p'-order have indices not divisible by p^2. Let $\Pi = \{q : q \text{ a prime}, q = p \text{ or } q|(p-1)\}$. Then $G/\mathrm{O}_{p'}(G)$ is a Π-group. Moreover if H is a Hall p'-subgroup of $G/\mathrm{O}_{p'}(G)$ then $|H|$ divides $p-1$.*

Proof. Let G be a minimal counter-example. Note that the hypothesis of the lemma is satisfied by both normal subgroups and quotients. Clearly if $\mathrm{O}_{p'}(G) \neq 1$ we obtain the result by factoring out $\mathrm{O}_{p'}(G)$. So we can assume $\mathrm{O}_{p'}(G) = 1$.

Let $U = \mathrm{O}_p(G)$ and let R be a Sylow r-subgroup of G for some prime r other than p dividing the order of G. By [Hup67, V1.6.5] R acts faithfully

on U. Let $z \in Z(R)$. Then $[U : C_U(z)] = p$. Let $V = C_U(z)$. We now show that V is centralised by R. If not there exists $y \in R$ so that $[V, y] \neq 1$. Let $W = C_U(y)$ and note that $[U : W] = p$. Write $D = V \cap W$, so $[U : D] = p^2$. Let $A = \langle z, y \rangle$, note that A is abelian, so A normalises both V and W. Now U/D is a module for A, and V/D and W/D are submodules. They cannot be isomorphic as A-modules, or else y would centralise V/D, and thus $[V, \langle y \rangle] = [V, \langle y \rangle, \langle y \rangle] \leq [D, \langle y \rangle] \leq [W, \langle y \rangle] = 1$, which is not the case since y does not centralise V. It follows that V/D and W/D are the only one-dimensional A-submodules of U/D.

Now consider the element $x = yz$. Then x does centralize either V/D or W/D, and yet $C_U(x)$ contains D with index p. It follows that $C_U(x)/D$ is an A-submodule of U/D different from both V/D and W/D, and this is a contradiction to the existence of y so V is centralised by R.

Now $[V, R] = 1$ and so if $t \in R$ acts trivially on U/V, then $[U, \langle t \rangle] = [U, \langle t \rangle, \langle t \rangle] \leq [V, \langle t \rangle] = 1$, and thus $t = 1$. Then R acts faithfully on U/V, so $|R|$ divides $p - 1$. As a consequence if r does not divide $p - 1$ then $R = 1$ so G is a Π-group. But this is true for all Sylow q-subgroups of $G/O_p(G)$ and the result follows. $\square$

Note, that in this situation, if $\Pi = \{p\}$, then G has a normal p-complement.

Recall, an A-group is a group with abelian Sylow subgroups. Camina & Camina proved the following result (compare it to Theorem 17):

Theorem 26. *[CC06] Let G be an A-group and suppose 2^a is the highest power of 2 to divide an index of G. Then, if there exists an element $x \in G$ with $|x^G| = 2^a$, G is soluble.*

An easy adaptation of the proof, shows that the result holds if instead of requiring our group to be an A-group, we just require the Sylow 2-subgroup to be abelian. However, if the Sylow 2-subgroup is not abelian the result is false. An example is given as follows. Let V be the 8-dimensional permutation representation of $\mathrm{PSL}(2, 7)$ over $\mathbb{F}_5$. By considering the extension of V by $\mathrm{PSL}(2, 7)$, it can be seen that V has elements of index 8.

Marshall considered soluble A-groups. She proved that there exists a function $g : \mathbb{Z}^+ \mapsto \mathbb{Z}^+$ such that the derived length of a soluble A-group G is bounded by $g(|\mathrm{cs}(G)|)$ [Mar96]. In 1997 Keller gave an explicit function for the bound which is stronger than logarithmic, see his survey article [Kel03].

This can be seen as a contribution towards the following question (recall, $|\mathrm{cs}(G)| = \mathrm{crk}(G) + 1$).

Question 1. *Is it possible to bound the derived length of a soluble group by its conjugate type rank?*

We note the following interesting result by Keller:-

Theorem 27. *[Kel06] Let G be a finite soluble group then the derived length of $G/\mathrm{Fit}(G)$ is bounded by $24\log_2(|\mathrm{cs}(G)|) + 364$.*

We finish this section with a question related to the comments in Section 4.4.

Question 2. *(i) If we know all conjugacy class sizes including multiplicities can we recognise solubility?*
(ii) If we know all conjugacy class sizes can we recognise solubility?

Clearly (ii) is stronger than (i) but we have no idea what the answer might be. One can also look at recognising other classes like supersolubility.

4 Conjugate Type Vectors

4.1 Itô

Recall, the conjugate type vector is a list of the distinct indices in descending order, and the conjugate rank is the number of entries not equal to 1. Itô proved the following:

Theorem 28. *Let G be a finite group,*
(i) [Itô53] with conjugate type vector $(n, 1)$. Then $n = p^a$ for some prime p and G is nilpotent. More exactly, G is a direct product of a p-group and an abelian p'-group. The p-group, P, has an abelian normal subgroup A, such that P/A has exponent p.
(ii) [Itô70a] with conjugate type vector $(n_1, n_2, 1)$. Then G is soluble.

Recently Ishikawa [Ish02] proved that a p-group of conjugate rank 1 has nilpotency class at most 3. Almost immediately a number of authors generalised the result, some to Lie Algebras, [BI03, Man04, Man06, Isa08, Man08,

Man09]. The generalisations involve the subgroup $M(G)$ of finite group G, where $M(G)$ is defined to be the subgroup generated by elements whose indices are 1 and m and m is the smallest non-trivial index. Then Isaacs proved the following:

Theorem 29. *[Isa08] Let G be a finite group which contains a normal abelian subgroup A with $\mathrm{C}_G(A) = A$. Then $M(G)$ is nilpotent of class at most 3.*

In 2009 Guo, Zhao & Shum proved the following generalisation of the rank one case:

Theorem 30. *[GZS09] Let N be a p-soluble normal subgroup of a group G such that N contains a noncentral Sylow r-subgroup, $(r \neq p)$, R of G. If $|x^G| = 1$ or m for every p'-element x of N whose order is divisible by at most two distinct primes, then the p-complements of N are nilpotent.*

Itô's 1952 result follows as a corollary. Zhao and Guo prove another generalisation as follows:

Theorem 31. *[ZG09] Let G be a finite group with a non-central Sylow r-subgroup R and N a normal subgroup of G containing R. If $|x^G| = 1$ or m for every element x of N, then N is nilpotent.*

In 1996 Li proved the following extension of Itô's rank 1 result, again using the results of [FKS81] quoted in Section 3.1:

Theorem 32. *[Li96] Let G be a finite group and let m be a natural number. Assume that if $x \in G$ has prime-power order then x has index 1 or m. Then G is soluble.*

Another variation is given by considering the p'-conjugate type vector.

Definition 33. *The p'-conjugate type vector of a group G is the list of distinct indices of p'-elements in descending order.*

Suppose G is a group with p'-conjugate type vector $(m, 1)$. Alemany, Beltrán & Felipe, in two papers [BF03a, ABF09], proved that $m = p^a q^b$ for primes $p \neq q$ and if a and b are both strictly greater than 0 then $G = PQ \times A$ where P is a Sylow p-subgroup of G, Q is a Sylow q-subgroup of G and A is in the centre of G. We comment that the result follows from the results in [Cam73] and [Cam74]. The second two authors also considered the case where the p'-conjugate type vector is $(m, n, 1)$ where $\gcd(m, n) = 1$, [BF04a].

4.2 Conjugate rank 2

In 1974, A.R. Camina gave a new proof of the solubility of groups of conjugate rank 2, along with more details on the structure of such groups. This depended on work of Schmidt [Sch70] and Rebmann [Reb71] which looked at the lattice of centralizers.

Definition 34. *A group G is an F-group if given any pair x, y with x, $y \notin Z(G)$ then $C_G(x) \not< C_G(y)$.*

In these papers the authors completely classified F-groups. From their results it is possible to see that any conjugate rank 2 group which is an F-group is soluble. Although it is not possible to express the property of being an F-group within the framework of indices, the condition that for no pair of indices is one divisible by the other is sufficient.

In 1974 A.R. Camina considered the situation of conjugate rank 2 groups which are not F-groups. This completed the proof of the solubility of conjugate rank 2 groups.

Theorem 35. *[Cam74] If G has conjugate rank 2 and is not an F-group then G is a direct product of an abelian group and a group whose order is divisible by only two primes (or $\rho^*(G) = 2$).*

This has recently been improved by Dolfi & Jabara. Their proof is independent of [Cam74] but uses [Itô70a]:

Theorem 36. *[DJ09] A finite group G has conjugate rank 2 if and only if, up to an abelian direct factor, either*
(1) G is a p-group for some prime p of conjugate rank 2 or
(2) $G = KL$, with $K \trianglelefteq G$, $\gcd(|K|, |L|) = 1$ and one of the following occurs
    (a) both K and L are abelian, $Z(G) < L$ and G is a quasi-Frobenius group,
    (b) K is abelian, L is a non-abelian p-group, for some prime p and $O_p(G)$ is an abelian subgroup of index p in L and $G/O_p(G)$ is a Frobenius group or
    (c) K is a p-group of conjugate rank 1 for some prime p, L is abelian, $Z(K) = Z(G) \cap K$ and G is quasi-Frobenius.

We note that the results in [BF09] can be deduced from the results of Dolfi & Jabara. This completes the classification of groups of conjugate rank 2.

4.3 Conjugate rank larger than 2

In the early 1970's Itô went on to consider groups of low conjugate rank, 3, 4 and 5 with special reference to the simple groups which can occur [Itô70b, Itô72, Itô73a, Itô73b]. Recently there has been some effort to look at the case of conjugate rank 3. The first papers were those connected with proving nilpotence, see Subsection 4.4. Beltrán and Felipe [BF08b] looked at the structure of soluble groups with conjugate type vector $(mk, m, n, 1)$ where $\gcd(m, n) = 1$ and k divides n. More recently Camina & Camina have shown that if the conjugate rank is larger than 2 and there are two coprime indices amongst any three, then G is soluble. Amongst the results are the following:

Corollary 37. *[CC09] Let G be a finite group with trivial centre and with at most three distinct conjugacy class sizes greater than 1. Then G is either soluble or $PSL(2, 2^a)$.*

Corollary 38. *[CC09] Let G be a finite A-group with at most three distinct conjugacy class sizes greater than 1. Then G is either soluble or $PSL(2, 2^a)$.*

So we ask

Question 3. *Can the conjugate rank 3 groups be classified, especially the non-soluble ones?*

We note the following theorem due to Bianchi, Gillio and Casolo, which followed earlier work [BCM$^+$92], [Man97]:

Theorem 39. *[BGC01] Suppose G is a group with conjugate type vector $(m, n \ldots)$ where m and n are coprime. Suppose $x, y \in G$ with $|x^G| = n$ and $|y^G| = m$ and let $N = C_G(x)$ and $H = C_G(y)$. Then N and H are abelian and G is quasi-Frobenius with kernel $N/Z(G)$ and complement $H/Z(G)$.*

Thus the conjugate type vector is of the form $(m, n, 1)$ (compare with Theorem 44).

4.4 Nilpotency

In 1972 A.R. Camina proved that a finite group with conjugate type vector $(q^b p^a, q^b, p^a, 1)$, where p and q are primes, is nilpotent [Cam72]. Recently Beltrán and Felipe have proved that if G is has conjugate type vector

$(nm, m, n, 1)$ with n and m coprime integers, then G is nilpotent and n and m are prime powers [BF06b, BF08a, BF06a].

Beltrán and Felipe [BF07b] have also shown for G, a finite p-soluble group and m a positive integer not divisible by p, that if the set of conjugacy class sizes of all p'-elements of G is $\{1, m, p^a, mp^a\}$, then G is nilpotent and m is a prime power.

Camina's result led to the question whether you could identify a nilpotent group from its conjugate type vector. More precisely if G and H have the same conjugate type vector and H is nilpotent, does it follow that G is nilpotent? Consideration of the indices of a nilpotent group yields the following property:

Let X be a set of positive integers containing 1. Then X has property $(\star)$ if, for m and n coprime, mn is an element of X if and only if both m and n are elements of X.

Since a nilpotent group, H, is a direct product of its Sylow p-subgroups it follows that the set of indices of H satisfy property $(\star)$. We note the following nice result due to Cossey and Hawkes:

Theorem 40. *[CH00] Let p be a prime and $\mathcal{S}$ a finite set of p-powers containing 1. Then there exists a p-group P of class 2 with the property that $\mathcal{S}$ is the conjugate type vector of P (ordered appropriately).*

Note it is certainly not true that arbitrary sets of numbers can be conjugate type vectors, as the results on graphs of Section 5 indicate. However, given a set X of positive integers containing 1 that satisfies property $(\star)$, then, by Theorem 40, there exists a nilpotent group H for which X is the conjugate type vector of H (ordered appropriately). So, recognising nilpotency is equivalent to asking whether a group whose set of indices satisfy property $(\star)$ is nilpotent?

Although in certain cases it is possible to recognise nilpotency, for example if all the conjugacy classes are square-free [CC98] or if the group is a metabelian A-group [CC06], in general it is not true. The smallest example of a group that shares its conjugate type vector with a nilpotent group, but is not itself nilpotent, has order 160 and conjugate type vector $(20, 10, 5, 4, 2, 1)$. An infinite family of such examples is given in [CC06]. A number of questions are posed in the paper.

Question 4. *Suppose G and H are finite groups with H nilpotent, further suppose G and H have the same conjugate type vector.*
(i) Is it true that G must be soluble?
(ii) If G is not nilpotent, does G have a centre?
(iii) Suppose G is an A-group, then must G be nilpotent?

Note, if you have the additional information of the number of conjugacy classes of each size, then you can recognise nilpotency as Cossey, Hawkes & Mann have proved:

Theorem 41. *[CHM92] Let G and H be finite groups with H nilpotent. Let S_H be the multiset of conjugacy class sizes of elements in H and define S_G similarly. Suppose $S_H = S_G$, then G is nilpotent.*

These ideas have been extended by Mattarei [Mat06]. For a group G and a set of primes π he introduces $S_\pi(G)$ which is the set of all elements of G which have index a π-number. He shows how the structure of G is influenced by the size of this set for the special case that $\pi = p'$ for some prime p.

5   Graphs

In an earlier version of this survey we included a lengthy chapter on graphs associated to conjugacy class sizes. However, since then, Lewis has published an excellent survey concerned with graphs associated to character degrees and conjugacy class sizes [Lew08]. Thus we will just briefly introduce the graphs and mention a few of the more recent results, while still aiming to include a large bibliography.

Let X be a set of positive integers. We associate two graphs to X, the *prime vertex graph* and the *common divisor graph*.

Definition 42. *(i) The common divisor graph of X has vertex set $X^* = X \backslash 1$ (X may or may not contain the element 1) and an edge between $a, b \in X^*$ if a and b are not coprime. We denote the common divisor graph of X by $\Gamma(X)$.*
(ii) The prime vertex graph has vertex set $\rho(X) = \bigcup_{x \in X} \pi(x)$ where $\pi(x)$ denotes the prime divisors of x. There is an edge between $p, q \in \rho(X)$ if pq divides x for some $x \in X$. The prime vertex graph is denoted by $\Delta(X)$.

The connection between these two graphs has been clarified by the recent paper [IP10] which defines a bipartite graph $B(X)$.

Definition 43. *The vertex set of $B(X)$ is given by the disjoint union of the vertex set of $\Gamma(X)$ and the vertex set of $\Delta(X)$, i.e. $X^* \cup \rho(X)$. There is an edge between $p \in \rho(X)$ and $x \in X^*$ if p divides x, i.e. if $p \in \pi(x)$.*

Example. Let $X = \{1, 3, 6, 8\}$:
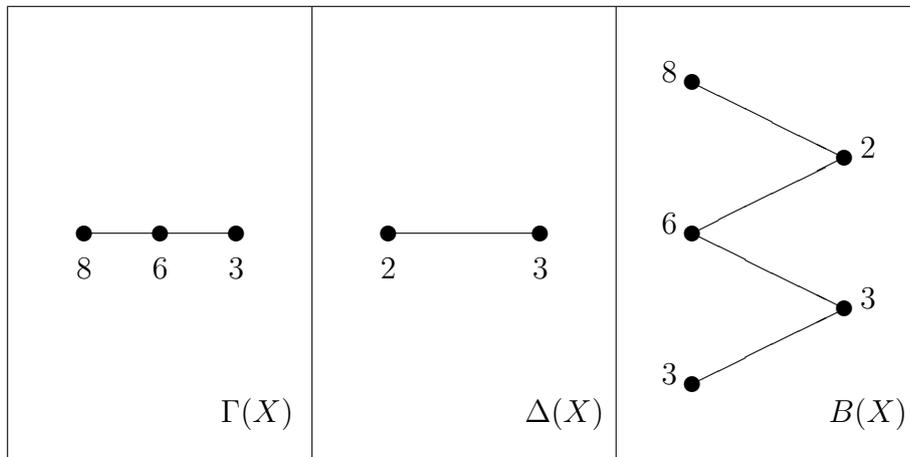
Consideration of $B(X)$ makes it clear that the number of connected components of $\Gamma(X)$ is equal to the number of connected components of $\Delta(X)$. Furthermore the diameter of a connected component of $\Gamma(X)$ differs by at most one from the diameter of the equivalent connected component of $\Delta(X)$.

There are two common choices for the set X: $\mathrm{cs}(G)$ and $\mathrm{cd}(G)$ (recall Definition 8). Much has been written on both these cases but we will concentrate on the case when $X = \mathrm{cs}(G)$. In 1981, Kazarin published a paper on isolated conjugacy classes [Kaz81]. A group G has isolated conjugacy classes if there exist elements $x, y \in G$ with coprime indices such that every element of G has index coprime to either $|x^G|$ or $|y^G|$. This definition goes back to Tchounikhin in 1934 [Tch34]. Kazarin classified all groups with isolated conjugacy classes. He therefore classified all groups such that either $\Gamma(\mathrm{cs}(G))$ has more than one component or $\Gamma(\mathrm{cs}(G))$ is connected and has diameter at least 3:

Theorem 44. *[Kaz81] Let G be a group with isolated conjugacy classes. Let x and y be representatives of the two isolated conjugacy classes with $|x^G| = m_1$ and $|y^G| = n_1$. Then $|G| = mnr$ where r is coprime to both m and n, the only primes which divide m, respectively n, are those which divide m_1, respectively n_1. Further $G = R \times H$ where $|R| = r$ and H is a quasi-Frobenius group where the pre-image of both the kernel and complement are abelian.*

It follows that $\Gamma(cs(G))$ has at most two connected components. If this is the case, we can take (without loss of generality) the conjugate type vector to be $(m_1, n_1, 1)$. Furthermore, if $\Gamma(cs(G))$ is connected its diameter is at most 3. This has also been proved using different techniques in [BHM90] and [CHM93].

In [BHM90], the authors noted that if G is a nonabelian simple group then $\Gamma(cs(G))$ is complete. This follows from work of Fisman & Arad [FA87]. Another nice result is due to Puglisi and Spezia. They proved that if $\Gamma(cs(H))$ is a complete graph for every subgroup H of G, then G is soluble [PS98].

Itô proved an early result about $\Delta(cs(G))$: suppose p and q are two distinct non-adjacent vertices in $\Delta(cs(G))$ then G is either p-nilpotent or q-nilpotent [Itô53, Proposition 5.1]. Dolfi extended this result for the case when G is soluble concluding that in this case both the p-Sylow and q-Sylow subgroups are abelian [Dol95a]. In this paper Dolfi also proved the analogous structural results for $\Delta(cs(G))$: namely, if $\Delta(cs(G))$ is not connected then it has exactly two connected components and they are complete graphs, and if $\Delta(cs(G))$ is connected it has diameter at most 3. These results are also contained in [Alf94]. In a follow up paper [Alf95], Alfandary determines further structural results for $\Delta(cs(G))$ when G is soluble.

Casolo and Dolfi [CD09] have characterised the groups for which $\Delta(cs(G))$ has diameter 3 [CD96b]. In the same paper they show that if a group is not soluble then $\Delta(cs(G))$ is connected and has diameter at most 2. Another nice result of Dolfi is to show that given 3 distinct vertices in $\Delta(cs(G))$ then at least two are connected by an edge [Dol06].

A recent paper gives further results on the bipartite graph $B(cs(G))$ [BDIP09].

Slightly confusingly, another graph has been associated with a finite group G. In this case the vertices are given by the set of non-central conjugacy classes and two vertices C and D are joined if $|C|$ and $|D|$ share a common

divisor [BHM90]. We shall call this graph $\hat{\Gamma}(G)$ and note that it shares many properties with $\Gamma(\mathrm{cs}(G))$.

We note that the following conjecture can be viewed as considering groups G for which $\Gamma(\mathrm{cs}(G)) = \hat{\Gamma}(G)$.

S_3 **conjecture**: Any finite group in which distinct conjugacy classes have distinct sizes is isomorphic to S_3.

This conjecture has been verified for soluble groups by Zhang [Zha94], and Knörr, Lempken and Thielcke [KLT95], independently. More recently Arad, Muzychuk and Oliver have studied the case of insoluble groups [AMO04].

One type of question that is posed when considering these graphs is how the structure of the graph determines the structure of the group. For example, the problem of classifying all groups G such that $\hat{\Gamma}(G)$ has no subgraph K_n (where K_n is the complete graph with n vertices) has been considered in [MQS05] for $n = 4, 5$ and in [FZ03] for $n = 3$. In [CC09], groups G, such that $\Gamma(\mathrm{cs}(G))$ has no triangles have been considered. Such groups have conjugate rank at most 3 and are soluble. This yields the following question:

Question 5. *Let G be a finite group and n a natural number. Is it true, that if $\Gamma(\mathrm{cs}(G))$ has no subgraphs isomorphic to K_n then there is a function of n which bounds the conjugate rank of G?*

Note that this question can also be asked without the language of graphs. The condition translates to requiring that given any set of n distinct indices then there are two which are coprime.

Variants of these graphs have been introduced. For example Beltrán & Felipe have considered a version of $\hat{\Gamma}(G)$ where the vertices are restricted to the p'-conjugacy classes, that is the set of x^G where x is a p'-element. They consider the case when G is p-soluble [BF02, BF03b, BF04b] and have summarised their results in a nicely written survey article [BF07a].

Alternatively, Qian and Wang [QW09b] have considered the conjugacy class sizes of p-singular elements, that is elements whose order is divisible by p. They denote the set of p-singular elements by G_p and consider the graph $\Gamma(\mathrm{cs}(G_p))$. Noting that if p divides $|Z(G)|$ then $\Gamma(\mathrm{cs}(G_p)) = \Gamma(\mathrm{cs}(G))$, they prove that if p divides $|G|$ but not $|Z(G)|$ then $\Gamma(\mathrm{cs}(G_p))$ is connected with diameter at most 3. In the paper the authors also consider groups for which p does not divide m for any $m \in \mathrm{cs}(G_p)$. This leads to a classification of all

finite groups for which every conjugacy class size coincides with the order of a Hall subgroup.

Beltrán has introduced the A-invariant conjugacy graph [Bel03]. Let A and G be finite groups and suppose that A acts on G by automorphisms. Then A acts on the set of conjugacy classes of G. The A-invariant conjugacy graph $\Gamma_A(G)$ has vertices the non-central A-invariant conjugacy classes of G and two vertices are connected by an edge if their cardinalities are not coprime. Consideration of the case when A acts trivially gives that $\Gamma_A(G)$ is a generalisation of $\hat{\Gamma}(G)$. Beltrán noted that the proof that $\hat{\Gamma}(G)$ has at most two connected components given in [BHM90] translates to this more general setting. He then considered the disconnected case and proves the following theorem:

Theorem 45. *[Bel03] Suppose that a group A acts on a group G, where $|A|$ and $|G|$ are coprime, and that $\Gamma_A(G)$ has exactly two connected components. Then G is soluble.*

It is not known whether the result holds when A and G do not have coprime orders.

In a different direction, Isaacs and Praeger introduced a generalisation of $\Gamma(\mathrm{cs}(G))$ known as the IP-graph [IP93].

Definition 46. *Let G be a group acting transitively on a set Ω, and let D denote the set of subdegrees of (G, Ω), that is, the cardinalities of the orbits of the action of a point stabilizer G_α on Ω. Suppose the subdegrees are finite, then the IP-graph of (G, Ω) is the common divisor graph of D.*

That this is a generalisation of $\Gamma(\mathrm{cs}(G))$ can be seen as follows. Let G be a group and $\mathrm{Inn}(G)$ the inner automorphisms of G. Let H be the semidirect product $G \rtimes \mathrm{Inn}(G)$, then H acts transitively on G by sending $x \in G$ to $(xg)^\sigma$ where $g\sigma \in G \rtimes \mathrm{Inn}(G)$. Clearly the orbits of H_1, the stabilizer of the identity, are the conjugacy classes. The authors prove that the IP-graph of (G, Ω) has at most two connected components and that the diameter of a connected component is bounded by 4. However they know of no example of diameter 4, suggesting perhaps that 3 is the correct upper bound.

Question 6. *Does there exist a group G and a set Ω such that the diameter of the IP-graph of (G, Ω) is 4?*

Kaplan has studied the case when the IP-graph is disconnected [Kap97], [Kap99]. Neumann introduced the VIP graph, a variant of the IP graph which does not restrict itself to the case where all subdegrees are finite [Neu93]. More recently the IP graph has been generalised to the setting of naturally valenced schemes [Cam08]. This work was extended by Xu [Xu09].

We would like to introduce a graph using the notion of divisibility. Let X be a set of positive integers, then $D(X)$ the *divisibility graph* is a directed graph. The vertex set of $D(X)$ is given by X^* and there is an edge connecting (a, b) with $a, b \in X^*$ whenever a divides b. We are interested in the properties of this graph when $X = \text{cs}(G)$ for a finite group G. Note that if $D(\text{cs}(G))$ has no edges then G is an F-group.

Question 7. *How many components can $D(\text{cs}(G))$ have?*

It is worth pointing out that whilst many results can be interpreted in the language of graph theory there are many interesting problems that have no such simple description.

6 The number of conjugacy classes

Given a group G of order n with k conjugacy classes what can be said about the relation between n and k? It is trivial to see that $k \leq n$ but can anything be said in the opposite direction? The first to bound n in terms of k was Landau in 1903 [Lan03]. He used a number theoretic approach to the class equation. Brauer [Bra63] was the first to give an explicit bound using Landau's method. He asked, in Problem 3, whether better methods could be found. A similar approach was taken by Newman in [New68] where he improved Landau's result. This gave very general bounds of exponential form. He proved:

Theorem 47. *[New68] Let G be a finite group of order n with k conjugacy classes. Then*
$$k \geq \frac{\log \log n}{\log 4}.$$

There have also been papers which give complete descriptions for small k, the earliest example being Miller in 1919 [Mil19]. The objective here was

to try to classify the isomorphism classes with a given number of conjugacy classes. A number of authors over many years have classified groups with few conjugacy classes [Mil44, Pol68a, Pol68b, VLVL85, VLVL86]. In particular, Vera López & Vera López [VLVL85, VLVL86] examine groups with 13 and 14 conjugacy classes and give lists of such groups. As far as the authors know this is the largest value of k for which this has been attempted.

Cartwright in [Car87] considered soluble groups. He proved:

Theorem 48. *There exist positive constants a and b so that a soluble group of order n has at least $a(\log n)^b$ conjugacy classes.*

There is an estimate for b of 0.00347. But Pyber [Pyb92] has proved the following theorem:

Theorem 49. *Let G be a finite group with k conjugacy classes. Then*

$$k \geq \epsilon \log n / (\log \log n)^8,$$

for some fixed ϵ.

In both of these papers the same methods are used. The problem is reduced to considering restricted classes of groups. Estimates for the number of conjugacy classes of a group are given from the information about the number for quotients and normal subgroups. For example, in the soluble case note that the Fitting subgroup is nilpotent, and the quotient by the Fitting subgroup is a subgroup of a direct product of linear groups.

Pyber's result could be considered an answer to Brauer's question. We note that Keller reduced the power in the denominator from 8 to 7 [Kel]. Also, for nilpotent groups Jaikin-Zapirain has proved a stronger result [JZ].

In 1997 Liebeck & Pyber [LP97] wrote a paper in which they gave upper bounds for the number of conjugacy classes for various classes of groups. Their main result is the following:

Theorem 50. *[LP97, Theorem 1] Let G be a finite simple group of Lie type over a finite field of order q. Let G have rank ℓ and assume that G has k conjugacy classes. Then*

$$k \leq (6q)^\ell.$$

Note that this is the "untwisted" rank.

As a consequence of this they prove

Theorem 51. *[LP97, Theorem 2] Let G be a subgroup of the symmetric group of degree n. Then if G has k conjugacy classes*

$$k \leq 2^{n-1}.$$

This solved a conjecture of Kovács & Robinson [KR93]. They had found a bound of 5^{n-1}, without the classification of finite simple groups. Maróti has reduced the bound to $k \leq 3^{(n-1)/2}$ [Mar05].

An interesting variation has been proved by Jaikin-Zapirain:

Theorem 52. *[JZ05] There exists a function $f(r)$ such that if G is a soluble group with at most r conjugacy classes of size k for any k then $|G| \leq f(r)$.*

7 Comparisons with character theory

7.1 Introduction

Many authors have been inspired by looking at comparisons between character degrees and conjugacy class sizes. As noted in the Introduction, if we know the multiplication constants, then we have a complete connection between the two sets of data. Furthermore, an obvious connection is that the number of conjugacy classes is the same as the number of irreducible characters. A number of authors, including those of this article, have seen this connection and have searched for analogous results.

However, the following elegant construction, given by Fernández-Alcober & Moretó, demonstrates that the link between degrees and indices is somewhat opaque.

Theorem 53. *[FAM01] Given any two integers r and s greater than 1 there exists a p-group G of nilpotency class 2 such that $|cd(G)| = r$ and $|cs(G)| = s$.*

Given this result it is surprising that there are any connections. However, the graphs defined in Section 5 yield interesting comparisons. Recall, we defined the common divisor graph $\Gamma(X)$ and the prime vertex graph $\Delta(X)$

for a set of natural numbers X. We focussed on the case when X is the set of indices of a finite group G, denoted by $\mathrm{cs}(G)$. We commented that much work has been done for the case when X is the set of degrees of irreducible characters, $\mathrm{cd}(G)$. In fact this case was considered first (see [Lew08] for a recent survey of results). What is remarkable is the similarity of the graphs for the two different choices of X. In particular, if G is soluble, then in all cases (both choices of X and graph) the graph has at most two connected components and, if the graph is connected, the diameter of the graph is at most three. However, if G is not soluble, then it is possible for $\Delta(\mathrm{cd}(G))$, and hence also $\Gamma(\mathrm{cd}(G))$, to have three connected components.

Recently Casolo and Dolfi have proved that $\Delta(\mathrm{cd}(G))$ is a subgraph of $\Delta(\mathrm{cs}(G))$. It is easy to see that each prime number that divides an irreducible character degree of G must also divide some class size of G. Casolo and Dolfi [CD09] prove that for distinct primes p and q, if pq divides the degree of some irreducible character of G, then it also divides the size of some conjugacy class of G. Dolfi had previously proved this result for soluble groups [Dol95b].

7.2 Arithmetical considerations

Authors have considered the implications of the arithmetical data of $\mathrm{cd}(G)$ and $\mathrm{cs}(G)$ on the structure of the group. As noted in Section 2, requiring all indices or character degrees to be coprime to a given prime p yields different results, see Corollary 4 and following discussion. In this case, the implications are stronger when the restrictions are placed on $\mathrm{cs}(G)$. However, two theorems proved by John G Thompson in 1970 imply the opposite. The first:

Theorem 54. *[Tho70] Suppose the degree of every non-linear irreducible character is divisible by a prime p. Then G has a normal p-complement.*

Note that very little seems to be known if character degrees are replaced by indices, compare Theorem 17. Certainly the equivalent theorem for indices is not true. For example, if G is $\mathrm{SL}(2,3)$ the indices are $\{1,4,6\}$ but G does not have a normal 2-complement. So we pose the following:

Question 8. *If the index of every non-central element of group G is divisible by a prime p what can be said about G?*

Thompson also proved:

Theorem 55. *[Tho70] If the degrees of irreducible characters of G are linearly ordered by divisibility, then G has a Sylow series.*

In this case the comparison with indices seems very weak. The only comparable result is Theorem 3 in [Cam72], which considers groups with conjugate type vector $(p^a q^b, p^a, 1)$, where p and q are distinct primes.

Question 9. *If the indices of G are linearly ordered by divisibility, what can be said about G?*

In Section 3.2 we noted that if all indices are square-free then G is soluble. If, however, all irreducible character degrees are square-free, G need not be soluble. The smallest example we know is $\mathrm{Alt}(7)$, see for example [CCN+85]. Again we see the stronger result for $\mathrm{cs}(G)$. Another example is given by the two different conclusions drawn when $\{1, p^a, q^b, p^a q^b\}$ is either the set of character degrees or the set of conjugacy class sizes and p and q are distinct primes. In the conjugacy class case we can conclude that $G = P \times Q$ where P is the Sylow p-subgroup and Q the Sylow q-subgroup [Cam72]. This conclusion does not hold in the character case [Lew98].

In Section 4.4 the problem of recognising nilpotency is considered. A character-degree version of Theorem 41 is due to Isaacs [Isa86] and predates the conjugacy class result. As finite nilpotent groups are the direct products of their Sylow p-subgroups, this can be thought of as recognising direct products. Lewis pursues these ideas in [Lew98], where he proves that for p, q and r distinct primes, if $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$ then $G = A \times B$ where $\mathrm{cd}(A) = \{1, p\}$ and $\mathrm{cd}(B) = \{1, q, r\}$. For s another distinct prime, he also proves that if $\mathrm{cd}(G) = \{1, p, q, r, s, pr, ps, qr, qs\}$ then $G = A \times B$ with $\mathrm{cd}(A) = \{1, p, q\}$ and $\mathrm{cd}(B) = \{1, r, s\}$. The equivalent results where character degrees are replaced by conjugacy class sizes are proved in [CC00].

In 2006 Isaacs, Keller, Meierfrankenfeld & Moretó showed that if G is soluble, p a prime divisor of $|G|$ and χ a primitive irreducible character of G. Then the p-part of $\chi(1)$ divides the p-part of $|x^G|^3$ for some $x \in G$. Furthermore, they put forward the following conjecture.

Conjecture [IKMM06] Let χ be a primitive irreducible character of an arbitrary finite group G. Then $\chi(1)$ divides $|x^G|$ for some element $x \in G$.

They checked that the conjecture holds for all irreducible characters (primitive or not) of all groups in the Atlas [CCN+85].

7.3 $k(GV)$-problem

In a slightly different direction Brauer [Bra63] was interested in the number of characters in a given block. Let p be a prime and consider the number of ordinary irreducible characters belonging to the p-block B with defect group D. He asked whether this is less than or equal to $|D|$. For definitions of blocks and related concepts see [Isa06, Chapter 15].

We know that the total number of irreducible characters of a finite group G, is equal to the number of conjugacy classes of G. We denote this number by $k(G)$. Interestingly in the p-soluble case the conjecture reduces to the following problem [Nag62]:

$k(GV)$-**problem** Let G be a finite p'-group for some prime p and let V be a faithful $\mathbb{F}_p$-module for G. Show that the number of conjugacy classes of the semidirect product VG is bounded by $|V|$.

In this situation there is only one p-block so that all characters are in the same p-block with defect group V and Brauer's question is about counting the number of conjugacy classes in a group.

The fundamental ideas for attacking the problem were developed around 1980 by Knörr [Knö84], who proved the conjecture for G supersoluble. Later, Gluck using Knörr's methods proved the conjecture for $|G|$ odd, [Glu84]. These were the first significant partial results. An important step on the way was the paper by Robinson & Thompson, [RT96], which solved the problem for all but finitely many primes.

The problem was finally solved by Gluck, Magaard, Riese & Schmid in 2004, [GMRS04]. This was the final piece of the jigsaw after a long series of papers. A more detailed discussion and details of proofs are in the book by Schmid [Sch07]. One of the consequences of these results is that if G is a p'-group that can be embedded in $\mathrm{GL}(m, p)$ for some integer m then $k(G) \leq p^m - 1$.

7.4   Huppert's conjectures

Let G be a finite group. Let $\bar{\sigma}(G)$ denote the maximal number of primes dividing any one character degree of G and let $\bar{\rho}(G)$ be the set of all primes which divide some character degree of G (compare with Definition 7). In 1985, Huppert [Hup91] conjectured that if G is soluble then $|\bar{\rho}(G)| \leq 2\bar{\sigma}(G)$. This conjecture is still open, the best result so far, for soluble groups, is that $|\bar{\rho}(G)| \leq 3\bar{\sigma}(G) + 2$ [GM87].

He also asked analogously whether $|\rho^*(G)| \leq 2\sigma^*(G)$? Baer's results referred to earlier, Subsection 3.1, show that if $|\sigma^*(G)| = 1$ then $|\rho^*(G)| \leq 2$. In [Man97] Mann considered groups in which all conjugacy class sizes involve at most two primes, that is groups G for which $|\sigma^*(G)| = 2$. Such a group is either soluble, or $G = Z(G) \times S$, where S is isomorphic to either A_5 or $SL(2,8)$ and $\rho^*(G) \leq 4$. This result also appears in [Cas94] where the author is concentrating on Huppert's Conjecture, note that both Baer's and Mann's papers predate Huppert's conjecture. In [Zha98] Zhang shows that $\rho^*(G) \leq 4\sigma^*(G)$ for all soluble groups which generalises the results of some earlier authors, [Fer93, Fer91a, Fer92, Fer91b]. Casolo in [Cas91] proved the following theorem:

Theorem 56. *Let G be a group which is p-nilpotent for at most one prime divisor of $|G|$ (this family includes all nonabelian simple groups), then*

$$|\rho^*(G)| \leq 2\sigma^*(G).$$

So Huppert's conjecture holds for many groups, consequently the following theorem is somewhat surprising:

Theorem 57. *[CD96a] There exist an infinite sequence of finite supersoluble, metabelian groups $\{G_n\}$, such that $|\rho^*(G_n)|/\sigma^*(G_n)$ tends to 3 as $n \to \infty$.*

So Huppert's conjecture is false. However, they do show that $|\rho^*(G)| \leq 4\sigma^*(G) + 2$ for all soluble G. Despite the result being false Casolo & Dolfi have shown that there are linear bounds in both the character and conjugacy class versions of the conjecture for all groups.

Theorem 58. *[CD07] Let G be a finite group. Then*
(a) $|\bar{\rho}(G)| \leq 7\bar{\sigma}(G)$,
(b) $|\rho^*(G)| \leq 5\sigma^*(G)$.

In 2000 Huppert made the following conjecture:

Conjecture [Hup00] If H is any simple nonabelian finite group and G is a finite group such that $\mathrm{cd}(G) = \mathrm{cd}(H)$, then $G \cong H \times A$, where A is abelian.

The complete answer to this question is still to be found in the character case. So we put forward the following question:-

Question 10. *If H is any simple nonabelian finite group and G is a finite group such that $\mathrm{cs}(G) = \mathrm{cs}(H)$, then is $G \cong H \times A$, where A is abelian?*

We note that a slightly more restrictive question, where G is assumed to have trivial centre, has been posed by A.S. Kondratiev in the Kourovka Notebook (12.38). The question is attributed to John G. Thompson. Some progress has been made for particular classes of simple groups [Che96, Che99, IAK02, Vas09].

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