A note on p-central groups

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In 1970 Gupta and Rhemtulla introduced the notion of an n-central group which generalises both the notions of abelian and exponent n [7]. Let G be a group and n a natural number. Denote the centre of G by $Z(G)$ and the subgroup of G generated by n^{th}-powers of elements of G by G^n.

Definition A group G is *n-central* if $G^n \leq Z(G)$.

Clearly a group G is n-central if and only if it satisfies the word $[x^n, y] = 1$ for all elements x and y in G. Thus the n-central groups form a variety. (We note that some authors have used the term p-central to mean all elements of order p in a finite p-group are central, this is a very differerent condition.)

Moravec has proved that for G a finitely-generated soluble group of derived length d, then G is n-central if and only if G is isomorphic to the direct product of a finite soluble n-central group of derived length at most d and a free abelian group of finite rank [14, 2.5]. We are interested in p^k-central groups for p a prime number. Clearly a finite p^k-central group is nilpotent and so is a finite p-group modulo an abelian direct factor. Thus we restrict our attention to finite p^k-central p-groups.

Several related concepts have been studied by authors, we recall a few of them. A group is said to be n-abelian if $(xy)^n = x^n y^n$ for all $x, y \in G$. It is easy to see that in an n-abelian group $[x^n, y] = [x, y]^n = [x^n, y^n] = [x, y]^{n^2}$ for all $x, y \in G$. Thus a p^k-abelian p-group is p^k-central. Indeed, n-abelian groups have been classified by Alperin [2]: the variety of n-abelian groups is the join of the varieties of abelian groups, groups of exponent dividing n and groups of exponent dividing $n-1$. More general than an n-central group is an n-Bell group, that is one which satisfies the identity $[x^n, y] = [x, y^n]$ for all $x, y \in G$.

With the exception of recent papers of Moravec [14, 15] and Mann [13], it seems that little work has been done on n-central groups, with results often only occuring as a byproduct of results on other classes of groups. One such example is the result by Kappe and Morse [11, Thm 13] which shows that a metabelian p-group G is p-central if and only if the exponent of the derived group of G divides p and G has nilpotency class at most p. In [14, 1.3] Moravec proves that the assumption that G is a p-group can be dropped. In the same paper Moravec classifies all finitely-generated 2-central groups [14, 2.7] (finite 2-central groups had previously been classified [6]).

It is worth noting that for p odd all p-groups of order at most p^4 are p-central. This is clear for groups of order $\leq p^3$. For groups of order p^4, by the result of Kappe and Morse mentioned above we just need to consider groups of nilpotency class 3, this case is covered in Proposition 2. For $p = 2$ the dihedral group of order 16 gives a group of order 2^4 which is not 2-central. The following presentation gives, for all primes p, a p-group of order p^5 that is not p-central

$$\langle x, y : x^{p^3} = 1 = y^{p^2}, y^{-1}xy = x^{(1+p)} \rangle.$$

Recall, the Nottingham group is a finitely-generated pro-p group in which the p-powers in the group drop quickly down the lower central series, for details see [3]. Thus it is not surprising that certain finite quotients of the Nottingham group give examples of p-central groups, for details see [18]. It is also interesting to note that p-groups with only one non-central conjugacy class size are p-central [10].

In this paper we consider three different aspects of p-central groups. The study of p^k-central groups is a natural setting in which to study the Schur Multiplier of a finite group of exponent p^k. The Schur Multiplier $M(G)$ of a group G is given by the second cohomology group $H^2(G, \mathbb{C}^*)$. When G is finite $M(G)$ is also given by the second integral homology group $H_2(G, \mathbb{Z})$. In Schur's pioneering work at the beginning of the last century he proved that all groups have a covering group: H is a covering group of a group G if H has a subgroup A isomorphic to $M(G)$ which satisfies $A \leq H' \cap Z(H)$ and $G \cong H/A$. So, the covering group of a group of exponent p^k is a p^k-central group. In the next section we study the interplay between the p-power structure and the commutator structure of a p-central group. This leads to the following theorem about the exponent of the Schur multiplier of a finite group of exponent p.

Theorem 1 *Let G be a finite group of exponent p and nilpotency class c. Then the exponent of $M(G)$ is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$.*

This compares favourably with known results of Ellis [5] and Moravec [15] when p is large in comparison to the nilpotency class of the group. We note that a finite non-cyclic group of exponent p has non-trivial mutiplier [12, 3.4.11].

In the second section we consider p-central groups by coclass. A finite p-group of order p^n and nilpotency class c has coclass $n-c$, this invariant was introduced by Leedham-Green and Newman and suggests an interesting way to investigate p-groups. The following theorem is not surprising given the structure of a p-group of coclass r but interesting to note.

Theorem 2 *Let G be a finite p^k-central p-group of coclass r. Then there exists a function $f = f(k, p, r)$ such that the order of G is bounded by p^f.*

An interesting link between Schur Multipliers and coclass is given by Bettina Eick [4]. She proves that for an odd prime p there are at most finitely many p-groups G of coclass r with $|M(G)| \leq s$ for every r and s. She also shows that this does not hold for $p = 2$ by constructing an infinite series of 2-groups with coclass r and trivial Schur Multiplier.

In the final section of this paper we look at the Tate cohomology of p-central groups. Recall a finite p-group G is regular if given $x, y \in G$ there exists $s \in \gamma_2(\langle x, y\rangle)$ such that $(xy)^p = x^p y^p s^p$ [16, 1.2.10]. In [17] Schmid proved that for G a regular p-group, N a non-trivial normal subgroup of G and $Q = G/N$ non-cyclic then the Q-module $A = Z(N)$ has non-trivial cohomology. So, in particular, if G is a non-abelian regular p-group and Φ the Frattini subgroup of G then $H^n(G/\Phi, Z(\Phi)) \neq 0$ for all n, Schmid then asks whether this result holds more generally. Abdollahi has given some cases where the result holds [1], and in the final section we prove the following result.

Theorem 3 *Let G be a finite p-central p-group and N a proper, non-trivial normal subgroup of G that is not maximal. Let $Q = G/N$, then $H^n(Q, Z(N)) \neq 0$ for all n.*

Notation is standard. Given subsets X and Y of a group G, then $[X, Y]$ denotes the group generated by commutators $[x, y] = x^{-1}y^{-1}xy$ where $x \in X$ and $y \in Y$. For n a natural number $[X,_n Y]$ is defined inductively, $[X,_1 Y] = [X, Y]$ and $[X,_n Y] = [[X,_{n-1} Y], Y]$. The lower central series of a group G is denoted by $\gamma_i(G)$ and defined inductively as $G = \gamma_1(G)$ and $\gamma_{i+1}(G) = [\gamma_i(G), G]$ for $i \geq 1$. We also use G' to denote the derived group of

G. The centre of G is denoted by $Z(G)$. For $H \leq G$ we denote the subgroup generated by elements h^{p^i} with $h \in H$ by H^{p^i}.

1 Schur Multipliers

The Schur Multiplier of a group G, denoted $M(G)$, is given by the second cohomology group $H^2(G, \mathbb{C}^*)$ and was introduced by Schur in 1904. For a finite group $M(G)$ can be identified with the second integral homology group $H_2(G, \mathbb{Z})$. The study of Schur Multipliers is closely related to the study of central extensions of groups. A group H is a covering group of G if H has a subgroup $A \cong M(G)$ such that $G \cong H/A$ and $A \leq Z(H) \cap H'$. Schur proved that a covering group always exists, although it need not be unique. For more background on Schur Multipliers see [12]. So, the covering group of a group of exponent p^k is a p^k-central group and information about the derived group of a p^k-central group yields information about the Schur Multiplier of a finite group of exponent p^k. This link has already been explored by Moravec [15].

We focus on p-central groups, and so Schur Multipliers of groups of exponent p. It is known that the derived group of a p-abelian group has exponent p, so identifying when a p-central group is p-abelian is useful.

Lemma 1 *A finite p-group G is p-abelian if and only if it is p-central and regular.*

Proof. Clearly a p-abelian p-group is regular and it is p-central by [8] (or the comment in the introduction). For the opposite direction, note that in a regular p-group $[x^p, y] = 1$ yields $[x, y]^p = 1$ [9, III 10.6(b)] and furthermore $(G')^p = 1$ [16, 1.2.13(i)]. Weichsel showed that G being p-abelian is equivalent to G being regular and satisfying $(G')^p = 1$ [20].□

As a finite p-group of nilpotency class less than p is regular [16, 1.2.11(i)] this yields the following corollary.

Corollary 1 *A finite p-central group of nilpotency class less than p is p-abelian.*

Thus the Schur Multiplier $M(G)$ of a finite group G of exponent p and nilpotency class $\leq p - 2$ has exponent p. But by examining the interplay between the commutator and p-power structure of a p-central group we can do better than this. First we quote a technical lemma.

Lemma 2 *[16, Cor 1.1.32] Let x and y be elements of G, and let p be a prime and r a positive integer. For $a, b \in \langle x, y\rangle$ define $K(a, b)$ to be the normal closure in $\langle x, y\rangle$ of the set of all basic commutators in $\{a, b\}$ of weight at least p^r and of weight at least two in b, together with the p^{r-k+1}th powers of all basic commutators in $\{a, b\}$ of weight less than p^k and of weight at least two in b for $1 \leq k \leq r$. Then*
(i) $(xy)^{p^r} \equiv x^{p^r} y^{p^r} [y, x]^{\binom{p^r}{2}} [y,_2 x]^{\binom{p^r}{3}} \dots [y,_{p^r-1} x] \bmod K(x, y)$.
(ii) $[x^{p^r}, y] \equiv [x, y]^{p^r} [x, y, x]^{\binom{p^r}{2}} \dots [[x, y],_{p^r-1} x] \bmod K(x, [x, y])$.

We isolate the next result to ease the proof of the following Proposition.

Lemma 3 *Let G be a group, $S \subseteq G$ and p a prime. Suppose $L \leq G$ satisfies $(\gamma_2([S, G]))^p \leq L$ and $\gamma_p([S, G]) \leq L$. Further, suppose $[s, g]^p \in L$ for all $s \in S$ and $g \in G$. Then $[S, G]^p \leq L$.*

Proof. This follows inductively from Lemma 2(i). Note that an element of $[S, G]^p$ is of the form $([s_1, g_1] \dots [s_n, g_n])^p$ for some $s_i \in S$ and $g_i \in G$ for $1 \leq i \leq n$. Write $x = ([s_1, g_1] \dots [s_{n-1}, g_{n-1}])^p$ and by induction suppose $x \in L$. Then applying Lemma 2(i) to $(x[s_n, g_n])^p$ and noting the hypotheses of the lemma gives the required result. □

The next result shows how p-powers drop in a finite p-central group.

Proposition 1 *Let G be a finite p-central group and H a subset of G. Define $H_1 = H$ and $H_{i+1} = [H,_i G] \leq G$ for $i \geq 1$. Then $(H_i)^p \leq H_{i+p-1}$ for all $i \geq 2$.*

Proof. Let $i \geq 2$, $x \in H_{i-1}$ and $y \in G$. We begin by showing that $[x, y]^p \in (H_{i+1})^p H_{i+p-1}$. Applying Lemma 2(ii) to $[x^p, y]$ yields

$$1 \equiv [x, y]^p [x, y, x]^{\binom{p}{2}} \dots [x, y,_{p-1} x] \bmod K(x, [x, y]).$$

Note that

$$[x, y, x]^{\binom{p}{2}} \dots [x, y,_{p-2} x]^p \in [H_{i-1}, G, G]^p \leq H_{i+1}^p,$$

and $[x, y,_{p-1} x] \in [H_{i-1},_p G] \leq H_{i+p-1}$. Now consider the normal subgroup $K(x, [x, y])$. First note that $H_i \leq \gamma_i(G)$ and $[H_i, \gamma_j(G)] \leq H_{i+j}$. Thus commutators of weight at least p and of weight at least two in $[x, y]$ lie in H_{2i+p-2}. Similarly p^{th}-powers of commutators of weight less than p and weight of at least two in $[x, y]$ lie in $(H_{2i+1})^p$. Thus $K(x, [x, y]) \leq (H_{2i+1})^p H_{2i+p-1} \leq (H_{i+1})^p H_{i+p-1}$ and consequently $[x, y]^p \in (H_{i+1})^p H_{i+p-1}$.

Applying the previous lemma with $H_{i-1} = S$ and $L = (H_{i+1})^p H_{i+p-1}$. We have

$$(H_i)^p \leq (H_{i+1})^p H_{i+p-1}$$

for $i \geq 2$. Subsituting the above result for H_{i+1} yields

$$(H_i)^p \leq ((H_{i+2})^p H_{i+p})H_{i+p-1} \leq (H_{i+2})^p H_{i+p-1}.$$

Continuing in this manner, and noting G is nilpotent so $(H_{i+k})^p$ is a strictly descending series of subgroups, yields

$$(H_i)^p \leq H_{i+p-1}.\square$$

Corollary 2 *Let G be a finite p-central group then $(\gamma_i(G))^p \leq \gamma_{i+p-1}(G)$ for all $i \geq 2$.*

Using the above proposition we can gain information about the Schur Multiplier of a finite group of exponent p.

Theorem 1 *Let G be a finite group of exponent p and nilpotency class c. Then the exponent $M(G)$ is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$.*

Proof. Suppose H is the covering group of G, then it is sufficient to prove that the exponent of H' is bounded by $p^{\lceil \frac{c}{p-1} \rceil}$. As G has exponent p it follows that H is a p-central group, so we can apply the previous proposition and thus $(H')^p \leq \gamma_{p+1}(H)$. Now, proceed inductively. Since $(H')^{p^k} \leq ((H')^{p^{k-1}})^p$ it follows that $(H')^{p^k} \leq \gamma_{2+k(p-1)}(H)$. As $\gamma_{c+2}(H) = 1$ it follows that $(H')^{p^k} = 1$ when $2 + k(p-1) \geq c + 2$, the result follows. $\square$

This improves known results when p is large compared to c. For example, Ellis has shown that for G a finite p-group of nilpotency class $c \geq 2$, the exponent of $M(G)$ divides $(\exp G)^{\lceil c/2 \rceil}$ [5]. More recently Moravec has bounded the exponent of $M(G)$ by $p^{k\lceil \log_2 c \rceil}$ where k is a function dependent on p and the exponent of G [15].

We should comment that we do not know of a finite p-central group with derived group not of exponent p. Clearly, by Corollary 2, an example would need to have class $\geq p + 1$ and, by a result of Kappe and Morse [11, Thm 13], derived length ≥ 3. Kappe and Morse have also shown that for $p = 2$ and 3 the derived group of a p-central group has exponent p [11, Thms 7 & 9]. This raises an interesting question regarding the exponent of the Schur Multiplier of an exponent p group, does it have to be p?

2 Coclass

Recall the coclass of a finite p-group G of order p^n and nilpotency class c is given by $n - c$. As all finite p-groups have finite coclass the coclass gives a useful invariant for investigating finite p-groups. To study p-groups of coclass 1, also known as p-groups of maximal class, a chain of normal subgroups is introduced:

$$G = P_0 > P_1 > P_2 > \cdots > P_n = \langle 1 \rangle.$$

For $i \geq 2$ the P_i are just the terms of the lower central series and P_1 is a 2-step centralizer, for more details see [16, Chap. 3]. In a p-group of coclass 1 the p-powers drop in a uniform way, this gives us the following dichotomy.

Proposition 2 *Let p be an odd prime and G a finite p-group of order p^n and coclass 1. Then G is p-central if and only if $n \leq p + 1$.*

Proof. That G is p-central if $n \leq p+1$ follows from [16, 3.3.2]. For $n > p+1$ using [16, 3.3.6] we see that $G^p \geq P_1^p = P_p$ and P_p is not central.□

More generally we have the following result.

Theorem 2 *Let G be a finite p^k-central p-group of coclass r. Then there exists a function $f = f(k, p, r)$ such that the order of G is bounded by p^f.*

Proof. Let p be odd, c be the nilpotency class of G and suppose $c \geq (k+1)p^r$. Equivalently $n \geq (k+1)p^r + r$ where p^n is the order of G. Then, by [16, 6.3.9], there exists $m = m(p, r) = (p-1)p^{r-1}$ such that G acts uniserially on $\gamma_m(G)$ and $(\gamma_i(G))^p = \gamma_{i+d}$ for all $i \geq m$ and for some $d = (p-1)p^s$ with $0 \leq s \leq r - 1$. Since G acts uniserially on $\gamma_m(G)$ it follows that $|\gamma_i(G) : \gamma_{i+1}(G)| = p$ for all $i \geq m$ and thus $(\gamma_m(G))^{p^k} = \gamma_{m+kd}$. But $m + kd \leq (k+1)(p-1)p^{r-1} < (k+1)p^r \leq c$ and thus $(\gamma_m(G))^{p^k}$ does not lie in the centre of G. Hence G is not p^k-central.

For $p = 2$ we refer to [16, 6.3.8]. In this case $m(2, r) = 2^{r+2}$ and we suppose $c \geq (3+k)2^{r+1}$, equivalently $n \geq (3+k)2^{r+1} + r$. The result follows with $d = 2^s$ and $0 \leq s \leq r + 1$. So, $m + kd \leq (2+k)2^{r+1} < (3+k)2^{r+1} \leq c$, and G is not p^k-central.

Thus setting $f(k, p, r) = (k+1)p^r + r - 1$ for p odd and $f(k, 2, r) = (3+k)2^{r+1} + r - 1$ works.□

We are not claiming $f(k, p, r)$ is a best bound, indeed it is clear from Proposition 2 that $f(1, p, 1)$ is not best possible.

3 Tate Cohomology

Let G be a finite p-group, N a normal subgroup of G and $A = Z(N)$, the centre of N. Then A is a $Q = G/N$-module and one can investigate the Tate cohomology groups $H^n(Q, A)$. The Q-module A is called cohomologically trivial if $H^n(K, A) = 0$ for all integers n and all subgroups K of Q. By a result of Uchida [19] we know that A is cohomologically trivial if $H^r(Q, A) = 0$ for just one integer r. In [17] Schmid investigates when the cohomology is non-trivial, he proves that if G is a regular p-group and $Q = G/N$ is not cyclic then $H^n(Q, Z(N)) \neq 0$ for all n. So, in particular, if G is a non-abelian regular p-group and Φ the Frattini subgroup of G then $H^n(G/\Phi, Z(\Phi)) \neq 0$ for all n, Schmid then asks whether this holds more generally. Abdollahi addresses this question in [1] (and uses the alternative definition of p-central) and poses the more general question:

Question [1, 1.2] For which finite p-groups G and which normal subgroups N of G do we have $H^n(\frac{G}{N}, Z(N)) \neq 0$ for all integers n?

In this section, using the methods of Schmid and Abdollahi, we prove the following.

Theorem 3 *Let G be a finite p-central p-group and N a proper, non-trivial normal subgroup of G that is not maximal. Let $Q = G/N$, then $H^n(Q, Z(N)) \neq 0$ for all n.*

By Uchida's result we will be able to restrict our attention to $H^0(Q, Z(N))$. Recall, $H^0(Q, A) = A_Q/A^\tau$ where A_Q denotes the fixed points of A under the action of Q and A^τ denotes the image of A under the trace map $\tau = \tau_Q$. The trace map is given by $\tau_Q : a \mapsto a \sum_{x \in Q} x$. For a finite p-central group, G, we see that the trace map is straightforward.

Let A be an abelian normal subgroup of G, $a \in A$ and $x \in G$, then $a^{1+x+\cdots+x^{p-1}} = a^p z$ for some central element z of G. This is clear since $a^{1+x+\cdots+x^{p-1}} = x^{-p}(xa)^p \in Z(G)$ and $a^p \in Z(G)$. The following lemma says slightly more, proving that the central element z in the statement above is the commutator $[a,_{p-1} x]$ and consequently that a is a p-Engel element.

Lemma 4 *Let G be a finite p-central p-group and suppose A is a normal abelian subgroup of G. Let $a \in A$ and $x \in G$ then $a^{1+x+\cdots+x^{p-1}} = a^p[a,_{p-1} x]$ and $[a,_{p-1} x] \in Z(G)$.*

Proof. Apply Lemma 2(i) to $(xa)^p$ and note that $K(x, a) = 1$. Next we show that most of the terms in this expression for $(xa)^p$ vanish. Let $H = \langle A, x \rangle$.

Then $H' = [A, x] = \{[a, x] : a \in A\}$ since A abelian. Now, by applying Lemma 2(ii) to $[a^p, x]$ and noting that all terms vanish except $[a, x]^p$, we see that $[a, x]^p = 1$ and thus H' has exponent p. So, returning to our expression for $(xa)^p$ yields $(xa)^p = x^p a^p z$ where $z = [a,_{p-1} x] \in Z(G)$. □

To prove the theorem we need the following proposition due to Schmid.

Proposition 3 *[17, Prop.1] Suppose $A \neq 0$ is a cohomologically trivial Q-module where A and Q are finite p-groups. Then for every subgroup H of Q, the centralizer $C_Q(A_H) = H$.*

The ideas behind the proof of the theorem follow very closely the ideas of Schmid [17] and Abdollahi [1] but are included for completeness.

Proof of Theorem 3. Suppose for a contradiction $H^n(Q, Z(N)) = 0$ for some integer n. Then, by [19, Theorem 4], it follows that $A = Z(N)$ is a cohomologically trivial Q-module. Thus $H^0(H/N, A) = 0$, where H is a subgroup of G containing N such that $|H : N| = p$. So $A_{H/N} = A^{\tau_{H/N}}$. By Lemma 4, for each $a \in A$, there exists a central element z_a such that $\tau_{H/N}(a) = a^p z_a$. Thus $C_{G/N}(A^{\tau_{H/N}}) = C_{G/N}(A^p) = G/N$ since G is p-central. However, Proposition 3 gives $C_{G/N}(A^p) = C_{G/N}(A_{H/N}) = H/N$. The result follows.□

References

[1] Alireza Abdollahi, Powerful p-groups have non-inner automorphisms of order p and some cohomology, *J. Algebra* **323** (2010) 779-789.

[2] J. L. Alperin, A classification of n-abelian groups, *Canad. J. Math.* **21** (1969) 1238-1244.

[3] Rachel Camina, The Nottingham group, *New horizons in pro-p groups,* 205-221, Progr. Math. **184**, Birkhäuser 2000.

[4] Bettina Eick, Schur multiplicators of finite p-groups with fixed coclass, *Israel Journal of Mathematics* **166**(1) (2008) 157–166.

[5] G. Ellis, On the relation between upper central quotients and lower central series of a group, *Trans. Amer. Math. Soc. 353* **10** (2001) 4219-4234.

[6] T.H. Fay & G.L. Walls, Some remarks on n-potent and n-abelian groups, *J. Indian Math. Soc. (N.S.)* **47** (1983), 217-222.

[7] N.D. Gupta & A.H. Rhemtulla, A note on centre-by-finite-exponent varieties of groups, *J. Austral. Math. Soc.* **11** (1970), 33-36.

[8] Charles Hobby, A characteristic subgroup of a p-group, *Pacific J. Math.* **10** (1960), 853–858.

[9] B. Huppert, *Endliche Gruppen I*, Springer-Verlag, 1967.

[10] I.M.Isaacs, Groups with many equal classes, *Duke Math J.* **37** (1970) 501–506.

[11] L.-C. Kappe & R.F. Morse, Levi-properties in metabelian groups, *Contemp. Math* **109** (1990), 59-72.

[12] Gregory Karpilovsky, *The Schur Multiplier*, LMS Monographs, New Series 2, Oxford University Press, 1987.

[13] Avinoam Mann, The exponents of central factors and commutator groups, *J. Group Theory* **10** (2007), 435-436.

[14] P. Moravec, On power endomorphisms of n-central groups, *J. Group Theory* **9** (2006) 519–536.

[15] P. Moravec, Schur multipliers and power endomorphisms of groups, *J. Algebra* **308** (2007) 12–25.

[16] C.R. Leedham-Green & S. McKay, The Structure of Groups of Prime Power Order, *LMS Monographs* **27** OUP 2002.

[17] Peter Schmid, A cohomological property of regular p-groups, *Mathematische Zeitschrift* **175** (1980) 1-3.

[18] Anitha Thillaisundarum, Topics in p-deficiency and p-groups, PhD Thesis, Cambridge University, 2011.

[19] Kôji Uchida, On Tannaka's Conjecture on the cohomologically trivial modules, it Proc. Japan Acad. **41** (1965), 249–253.

[20] P.M. Weichsel, On p-abelian groups, *Proc. Amer. Math. Soc.* **18** (1967), 736-737.