1 Introduction

In this survey we consider the influence of the sizes of conjugacy classes on finite groups. Over the last 30 years there have been many papers on the topic and it would seem to be a good idea to try to bring some of the key results together in one place. This is especially relevant as some authors seem unaware of others writing in the field as well as some of the older results which seem to get reproved quite regularly. It is hoped that in writing this, less time will be spent in reproving old results, enabling more progress to be made on some of the more interesting problems.

How much information can one expect to obtain from the sizes of conjugacy classes? Sylow in 1872 examined what happened if there was information about the sizes of all conjugacy classes, whereas in 1904 Burnside showed that strong results could be obtained if there was particular information about the size of just one conjugacy class. Landau in 1903 bounded the order of the group in terms of the number of conjugacy classes whilst in 1919 Miller gave a detailed analysis of groups with very few conjugacy classes. Very little then seems to have been done until 1953 when both Baer
and Itô published papers on this topic but with different conditions on the sizes.

By looking at these early results it can been seen that much will depend on how much information is given and it is important to be explicit. For example if one knows that there is only one conjugacy class size then the group is abelian, but this can be any abelian group. However if you know the collection of conjugacy class sizes, that is the multiplicities, then the order of the group is also known. However it would still not be possible to identify the group. Some authors have considered the situation where the multiplicities of the conjugacy class sizes are used if the size is not 1. This is particularly true when the authors have been studying aspects of the problem related to graphs. Again if we only demand information about the sizes of conjugacy classes and not their multiplicities, the group $G$ and $G \times A$ will have the same set whenever $A$ is an abelian group. So we can only state results modulo a direct abelian factor.

Another reason for examining conjugacy classes is their fundamental role in understanding the group ring, especially over $\mathbb{C}$. Recall that if $G$ is a group then the group ring $\mathbb{C}[G]$ is the vector space over $\mathbb{C}$ generated with basis the elements of $G$. For each conjugacy class $K$ of $G$ define the element $c_K = \sum_{g \in K} g$. Then $c_K$ is in the centre of $\mathbb{C}[G]$ and the elements $c_K$ are a basis for the centre. This gives rise to a complete link to the character theory of $G$. If $K_1, K_2, \ldots, K_k$ are the conjugacy classes of $G$ then there are integers $a_{rst} : 1 \leq r, s, t \leq k$ so that

$$c_{K_r}c_{K_s} = \sum_t a_{rst}c_{K_t}.$$  

Knowledge of the $a_{rst}$ is equivalent to knowing the character table, this is a result that goes back to Frobenius, see Lam [Lam98, Theorem 7.6]. So at this level of information we see that knowing character and conjugacy class data are the same.

As we are not mainly concerned with characters, and only mention results as comparative results we refer the reader to [Hup98] or [Isa06], amongst many others, for a discussion of character theory and some of the necessary definitions. We discuss these connections further in Section 7.

We will often refer to the index of an element, this is just the size of the conjugacy class containing the element. The benefit of this definition is
entirely linguistic. Given an element \( g \) in some group \( G \) we can talk about the index of \( g \) rather than talking about the size of the conjugacy class containing \( g \). So if we are referring to elements we use the term index but if we are talking about conjugacy classes we refer to size.

In Section 2 we introduce the basic definitions and consider some elementary results which are used time and time again, and proved time and time again in many papers.

Section 3 considers results which might loosely be described as placing arithmetical conditions on the indices of elements. One of the most famous is Burnside’s \( p^a \) theorem. In this the restriction is that there is an index which is a prime-power. We then look at some conditions which imply solubility, here the results are quite weak. But we see a distinction between demanding conditions on all the indices of elements and only on some. This occurs quite frequently, so for example, if we know that the indices of all elements are powers of a given prime \( p \) then the group is essentially a \( p \)-group. But then we can ask what happens if we only ask that \( p \)-elements have index a power of \( p \). Here we are demanding fewer restrictions on the indices but that we can recognise the order of elements in the group.

The conjugate type vector is introduced in Section 4. This considers the indices in descending order and looks at various properties of the group which can be deduced from this sequence. Some authors have considered variations. For example for a fixed prime \( p \) they consider the sequence of indices for \( p' \)-elements.

There is a large body of work considering the influence of conjugacy class sizes on \( p \)-groups. We do not mention these results here, unless they fit in a natural way, but refer the reader to a survey paper on \( p \)-groups by A. Mann; one section is devoted to ‘Representations and Conjugacy Classes’ [Man99].

In Section 5 various graphs that can be constructed from the sets of conjugacy class sizes are defined. The properties of the graphs and the relation to the structure of the groups is examined. This has been a very active area in recent years. This has also been the case where character degrees have been used similarly, see the survey by Lewis [Lew08].

Examining the influence of the number of conjugacy classes is considered in Section 6 and, as mentioned previously, connections with character theory are considered in Section 7.
Notation: The notation we use is standard. Let $G$ be a finite group and $x$ an element of $G$. The centraliser of $X \subseteq G$ is $\{g \in G : xg = gx \forall x \in X\}$ and denoted by $C_G(X)$, note if $X = \{x\}$ we drop the brackets. The conjugacy class of $x$ in $G$ is denoted by $x^G$. The index of $x$ will be denoted by $\text{Ind}_G(x)$. We denote the derived group of $G$ by $G'$ and the Fitting subgroup of $G$ by $\text{Fit}(G)$. We comment, given a prime $p$, that the terms $p$-regular and $p$-singular are often used. Note that an element being $p$-regular is the same as saying that the element is a $p'$-element, that is that its order is not divisible by $p$. A $p$-singular element is one which is not a $p'$-element. Also if $\pi$ is a set of primes then $\pi'$ is the set of primes not in $\pi$. $O_\pi(G)$ is the largest normal subgroup of $G$ whose order is a $\pi$-number. Also $O^\pi(G)$ is the smallest normal subgroup whose factor group $G/O^\pi(G)$ is a $\pi$-group. In a group $G$ a $p$-complement is a subgroup $H$ whose index in $G$ is exactly the highest power of $p$ to divide the order of $G$. We use CFSG to denote the Classification of Finite Simple Groups.

2 Basic Definitions and Results

Baer [Bae53] gave the following definition which we will use:-

Definition 1 Let $G$ be a finite group and let $x \in G$. The index of $x$ in $G$ is given by $[G : C_G(x)]$ and is denoted by $\text{Ind}_G(x)$.

Note, $\text{Ind}_G(x)$ is the size of the conjugacy class of $x$, by the orbit-stabiliser theorem.

Lemma 1 Let $N$ be a normal subgroup of $G$. Then
(i) if $x \in N$, $\text{Ind}_N(x)$ divides $\text{Ind}_G(x)$.
(ii) if $x \in G$, $\text{Ind}_{G/N}(xN)$ divides $\text{Ind}_G(x)$.

Lemma 2 [Cam72] If $p$ is a prime which does not divide $\text{Ind}_G(x)$ for all $x$ of $p'$-order then the Sylow $p$-subgroup of $G$ is a direct factor of $G$.

Proof. Let $P$ be a Sylow $p$-subgroup of $G$ and let $C = C_G(P)$. We now show that $PC$ contains a conjugate of every element in $G$. The lemma then follows by an old argument of Burnside [Bur55, §26].
If \( g \in G \) we can write \( g = xy \) where \([x, y] = 1\) and \( x \) is a \( p \)-element and \( y \) is a \( p' \)-element. By the hypothesis we have that \( C_G(y) \) contains a conjugate of \( P \). So, conjugating as necessary, we can ensure that \( P \subset C_G(y) \). There exists \( h \in C_G(y) \) so that \( x^h \in P \). Hence \((xy)^h = x^hy \in PC\) as required.

**Corollary 1** If \( p \) is a prime which does not divide \( \text{Ind}_G(x) \) for all \( x \) then the Sylow \( p \)-subgroup of \( G \) is an abelian direct factor of \( G \).

We comment here that this shows a dichotomy with the related theory for character degrees. For if \( G \) is a finite group with a normal abelian Sylow \( p \)-subgroup then \( p \) does not divide any character degree. This is a result of Itô [Itô51], the converse is also true, though the proof was not completed until 1986 with an application of CFSG [Mic86].

Another useful lemma is the following.

**Lemma 3** Let \( x \) and \( y \) be two elements of \( G \) such that \( C_G(x)C_G(y) = G \). Then \((xy)^G = x^Gy^G\).

**Proof.** Consider \( x^gy^h \) with \( g, h \in G \). Clearly \( x^gy^h \) is conjugate to \( x^{gh^{-1}}y \). We can write \( gh^{-1} = ab \) where \( a \in C_G(x) \) and \( b \in C_G(y) \). Thus \( x^{gh^{-1}}y = x^{ab}y \) which is conjugate to \( x^ay^{b^{-1}} = xy \) as required.

**Corollary 2** Let \( x \) and \( y \) be two elements of \( G \) such that \( \text{Ind}_G(x) \) and \( \text{Ind}_G(y) \) are coprime then \((xy)^G = x^Gy^G\).

This idea was discussed by Tchounikhin in 1930 [Tch30]. In this paper he shows that if there are three indices which are pairwise coprime then the group is not simple.

We now give some definitions which are useful and provide some unifying themes to our survey.

**Definition 2** Let \( G \) be a finite group and \( g \in G \). Define

(i) \( \sigma_G(g) \) to be the set of primes dividing \( \text{Ind}_G(g) \).

(ii) \( \sigma^*(G) = \max\{|\sigma_G(g)| : g \in G\} \) and

(iii) \( \rho^*(G) = \bigcup_{g \in G} \sigma_G(g) \).

5
We note that $\rho^*(G)$ is just the set of primes that divide $G/Z(G)$ by Corollary 1, this set is also known as the set of eccentric primes.

In 1953 Itô introduced the notion of a conjugate type vector:

**Definition 3** [Itô53] The conjugate type vector of a group $G$ is the vector $(n_1, n_2, \ldots, n_r, 1)$ where $n_1 > n_2 > \ldots n_r > 1$ are the distinct indices of elements of $G$. The conjugate rank of $G$, $\text{crk}(G)$, is given by $r$.

**Definition 4** (i) A group $G$ is called Frobenius if $G$ can be written as a product of two groups of coprime order $K$ and $C$ where $K$ is normal and $C_G(x) \subseteq K$ for all $1 \neq x \in K$. $K$ is called the kernel of $G$ and $C$ the complement. For more structure see [Hup67, V.8].

(ii) A group $G$ is called quasi-Frobenius if $G/Z(G)$ is Frobenius.

Let $G$ be quasi-Frobenius and let the pre-image of the kernel and the complement be $K$ and $C$ respectively. Then, if both $K$ and $C$ are abelian, the non-trivial conjugacy class sizes of $G$ are $m = |C/Z(G)|$ and $n = |K/Z(G)|$. Note $\gcd(m, n) = 1$.

### 3 Arithmetical Properties

#### 3.1 Prime-power index

Perhaps the earliest results are those of Sylow [Syl72] which says that a group, all of whose indices are a power of a given prime, has a non-trivial centre and of Burnside [Bur04] which says that if one index is a power of a prime then the group is not simple. These are two classic results which give information about the group from some arithmetical properties of the set of indices.

In 1990 Kazarin proved the following extension to Burnside’s result:

**Theorem 1** [Kaz90] Let $G$ be a finite group and let $x$ be an element of $G$ such that $\text{Ind}_G(x) = p^a$ for some prime $p$ and integer $a$. Then $\langle x^G \rangle$ is a soluble subgroup of $G$. 

6
Whilst the proof of Burnside’s result depends on ordinary character theory, Kazarin’s result depends on modular character theory. Using Theorem 1 Camina & Camina [CC98] were able to prove that any element of prime-power index is in the second Fitting subgroup. Flavell [Fla02] has shown that whether an element is in the second Fitting subgroup is determined by the behaviour of two generator subgroups. This led to an interesting discussion of how the indices of elements in two generator subgroups can determine the index of an element in the whole group, see [CSS10].

Baer [Bae53] characterized all finite groups such that every element of prime-power order has prime-power index. He then went on to raise the question of the characterization of those groups whose \( q \)-elements, for just one prime \( q \), have prime power index. Camina & Camina [CC98] introduced the following idea based on Baer:

**Definition 5** Let \( G \) be a finite group and let \( q \) be a prime such that \( q \) divides \( |G| \). Then \( G \) is a \( q \)-Baer group, or equivalently has the \( q \)-Baer property, if every \( q \)-element of \( G \) has prime power index.

They then proved the following theorem:

**Theorem 2** [CC98] Let \( G \) be a \( q \)-Baer group for some prime \( q \). Then
(a) \( G \) is \( q \)-soluble with \( q \)-length 1, and
(b) there is a unique prime \( p \) such that each \( q \)-element has \( p \)-power index. Further, let \( Q \) be a Sylow \( q \)-subgroup of \( G \), then
(c) if \( p = q \), \( Q \) is a direct factor of \( G \), or
(d) if \( p \neq q \), \( Q \) is abelian, \( O_p(G)Q \) is normal in \( G \) and \( G/O_q(G) \) is soluble.

Berkovic & Kazarin prove some very similar results but also some different ones. One of their results is the following:

**Theorem 3** [BK05] If the index of every \( p \)-element of order \( p \) (if \( p \) is odd) or 4 (if \( p = 2 \)) is a \( p \)-power, then \( G \) has a normal \( p \)-complement.

Beltrán & Felipe have also proved some similar results, [BF04c].

Recall, Lemma 2 says that if all \( p' \)-elements have \( p' \)-index then the Sylow \( p \)-subgroup of \( G \) is a direct factor. An interesting strengthening of this result is due to Liu, Wang & Wei:
Theorem 4 [LWW05] If \( p \) is a prime which does not divide \( \text{Ind}_G(x) \) for all \( p' \)-elements \( x \) of prime-power order then the Sylow \( p \)-subgroup of \( G \) is a direct factor of \( G \).

The proof uses the result of [FKS81] which shows that in a transitive permutation group there is an element of prime-power order which acts fixed-point-freely, this result uses CFSG.

In another variation on the theme Dolfi and Lucido say a finite group \( G \) has property \( P(p, q) \) if every \( p' \)-element has \( q' \)-index. The inspiration for this definition came from ideas in character theory. In particular, a group \( G \) has the property \( BP(p, q) \) if every \( p \)-Brauer character has degree prime to \( q \). Dolfi & Lucido prove (amongst other things) the following:

Theorem 5 [DL01] Let \( G \) be a finite group satisfying \( P(p, q) \) with \( p \neq q \). Then \( \text{O}^p(G) \) is \( q \)-nilpotent and \( G \) has abelian Sylow \( q \)-subgroups.

A significant portion of the paper is taken up with showing that if \( G \) is an almost simple group satisfying \( P(p, q) \), then \( \gcd(|G|, q) = 1 \), this uses CFSG.

Interestingly an example where a given prime almost does not occur is given by:

Theorem 6 [DMN09] Let \( G \) be a finite group having exactly one conjugacy class of size a multiple of a prime \( p \). Then one of the following holds:

(i) \( G \) is a Frobenius group with Frobenius complement of order 2 and Frobenius kernel of order divisible by \( p \);
(ii) \( G \) is a doubly transitive Frobenius group whose Frobenius complement has a nontrivial central Sylow \( p \)-subgroup;
(iii) \( p \) is odd, \( G = KH \) where \( K = \text{Fit}(G) \) is a \( q \)-group, \( q \) prime, \( H = C_G(P) \) for a Sylow \( p \)-subgroup \( P \) of \( G \), \( K \cap H = Z(K) \) and \( G/Z(K) \) is a doubly transitive Frobenius group.

In the paper there is a slightly more detailed version which gives if and only if conditions.

3.2 Solubility

Recognising solubility is clearly an interesting problem. In 1990 Chillag & Herzog proved the following:
Theorem 7 [CH90] Suppose 4 does not divide \( \text{Ind}_G(x) \) for all \( x \in G \). Then \( G \) is soluble.

A proof avoiding CFSG is given in [CC98] and [CW99]. Chillag & Herzog also considered those groups all of whose indices are square-free; these groups were also studied by Cossey & Wang [CW99].

Theorem 8 [CW99] Suppose that all indices of the group \( G \) are square-free. Then \( G \) is supersoluble and both \( |G/\text{Fit}(G)| \) and \( G' \) are cyclic groups with square-free orders. The class of \( \text{Fit}(G) \) is at most 2 and \( G \) is metabelian.

Using the results of [FKS81] quoted in Section 3.1 Li strengthens these results as follows:

Theorem 9 [Li99] Let \( G \) be a group and let \( p \) be the smallest prime dividing the order of \( G \). Assume that \( p^2 \) does not divide the index of any element of \( q \)-power order, for \( q \) any prime not equal to \( p \). Then \( G \) is \( p \)-nilpotent. In particular, \( G \) is soluble.

Theorem 10 [Li99] Suppose that all indices of elements of prime-power order in the group \( G \) are square-free. Then \( G \) is supersoluble, the derived length of \( G \) is bounded by 3, \( G/\text{Fit}(G) \) is a direct product of elementary abelian groups and \( |\text{Fit}(G)'| \) is a square-free number.

In [QW09a] the authors consider what happens if only \( p' \)-elements have indices not divisible by \( p^2 \). They show that in this case the highest power of \( p \) which can divide the order of any chief factor is \( p \).

Note, if \( p \) is the smallest prime that divides the order of a group then for any other prime \( q \) dividing the order of the group \( q \) does not divide \( p - 1 \). Using this observation Cossey & Wang [CW99] considered groups \( G \) for which there is a prime \( p \) dividing the order of \( G \) so that if \( q \) divides the order of \( G \) then \( q \) does not divide \( p - 1 \). They prove a result giving the structure of such groups when the index of no element of \( G \) is divisible by \( p^2 \). A variation on this is given by Liu, Wang and Wei [LWW05]. We note that with these conditions if \( p = 2 \) then \( G \) is soluble by Theorem 7 and if \( p > 2 \) then \( G \) has odd order.

If the prime \( p \) divides the order of the group and there exists primes which do not divide \( p - 1 \) we can say something slightly more general:
Lemma 4 Let \( G \) be a soluble group and let \( p \) be a prime. Assume that all elements of \( p' \)-order have indices not divisible by \( p^2 \). Let \( \Pi = \{ q : q \text{ a prime, } q \neq p \text{ and } q \text{ not dividing } p-1 \} \). Then \( G/O_{p'}(G) \) is a \( \Pi' \)-group.

Proof. Let \( G \) be a minimal counter-example. If \( \Pi \) is empty there is nothing to prove. Note that this hypothesis goes over to normal subgroups and quotients. Clearly if \( O_{p'}(G) \neq 1 \) we obtain the result by factoring out \( O_{p'}(G) \). So we can assume \( O_{p'}(G) = 1 \).

Assume that \( G \) has a proper normal subgroup \( N \), from the above comment \( O_{p'}(N) = 1 \). So \( N \) is a \( \Pi' \)-group. If \( N \) is a maximal normal subgroup it has prime index, say \( q \), then \( q \in \Pi \). If there were two distinct maximal normal subgroups then \( G \) would be a \( \Pi' \)-group and so there is nothing to prove. So \( N \) is the unique maximal normal subgroup of \( G \).

Let \( M \) be a minimal normal subgroup. Let \( x \) be an element of order \( q \), which is not in \( N \) as \( q \in \Pi \) and \( N \) is a \( \Pi' \)-group, and consider the action of \( x \) on \( M \). Since \( M \) is normal \( [M : C_M(x)] = 1 \) or \( p \). Since \( q \in \Pi \) the action of \( x \) is trivial on \( M/C_M(x) \), so \( x \) centralises \( M \). Since \( x \notin N \) and \( x \in C_G(M) \) it follows that \( M \leq Z(G) \). But \( O_{p'}(G) = 1 \) so \( O_{p'}(G/M) = 1 \) and \( G/M \) is a \( \Pi' \)-group which contradicts the assertion that \( [G : N] = q \).

Note that if \( G \) is a \( \{ p \} \cup \Pi \)-group then \( G \) has a normal \( p \)-complement.

Recall, an \( A \)-group is a group with abelian Sylow subgroups. Camina & Camina proved the following result:

Theorem 11 [CC06] Let \( G \) be an \( A \)-group and suppose \( 2^a \) is the highest power of \( 2 \) to divide an index of \( G \). Then, if there exists an element \( x \in G \) with \( \text{Ind}_G(x) = 2^a \), it follows that \( G \) is soluble.

An easy adaptation of the proof, shows that the result holds if instead of requiring our group to be an \( A \)-group, we just require the Sylow 2-subgroup to be abelian. However, if the Sylow 2-subgroup is not abelian the result is false. For example, the simple group \( \text{PSL}(2,7) \) has a permutation representation of degree 8. Let \( V \) be a permutation module of degree 8 over a finite field of characteristic 5. By considering the extension of \( V \) by \( \text{PSL}(2,7) \) it can be seen that \( V \) has elements of index 8.

Marshall considered soluble \( A \)-groups. Let the conjugate rank of \( G \), \( \text{crk}(G) \), be greater than one. She proved that there exists a function \( g : \mathbb{Z}^+ \mapsto \mathbb{Z}^+ \)
such that the derived length of a soluble $A$-group $G$ is bounded by $g(\crk(G) + 1)$ [Mar96]. In 1997 Keller gave an explicit function for the bound which is stronger than logarithmic, see his survey article [Kel03]. This can be seen as a contribution towards the following question.

**Question 1** Is it possible to bound the derived length of a soluble group by its conjugate type rank?

We note the following interesting result by Keller:-

**Theorem 12** [Kel06] Let $G$ be a finite soluble group then the derived length of $G/\text{Fit}(G)$ is bounded by $24 \log_2(\crk(G) + 1) + 364$.

We finish this section with a question related to the comments in Section 4.4.

**Question 2** (i) If we know all conjugacy class sizes including multiplicities can we recognise solubility?
(ii) If we know all conjugacy class sizes can we recognise solubility?

Clearly (ii) is stronger than (i) but we have no idea what the answer might be. One can also look at recognising other classes like supersolubility.

4 Conjugate Type Vectors

4.1 Itô

Recall, the conjugate type vector is a list of the distinct conjugacy class sizes in descending order, and the conjugate rank is the number of entries not equal to 1. Itô proved the following:

**Theorem 13** Let $G$ be a finite group,
(i) [Itô53] with conjugate type vector $(n, 1)$. Then $n = p^a$ for some prime $p$ and $G$ is nilpotent. More exactly, $G$ is a direct product of a $p$-group and an abelian $p'$-group. The $p$-group, $P$, has an abelian normal subgroup $A$, such that $P/A$ has exponent $p$.
(ii) [Itô70a] with conjugate type vector $(n_1, n_2, 1)$. Then $G$ is soluble.
Recently Ishikawa [Ish02] proved that a $p$-group of conjugate rank 1 has nilpotency class at most 3. Almost immediately a number of authors generalised the result, some to Lie Algebras, [BI03, Man04, Man06, Isa08, Man08, Man09]. The generalisations involve the subgroup $M(G)$ of finite group $G$, where $M(G)$ is defined to be the subgroup generated by elements whose indices are 1 and $m$ and $m$ is the smallest non-trivial index. Then Isaacs proved the following:

**Theorem 14** [Isa08] Let $G$ be a finite group which contains a normal abelian subgroup $A$ with $C_G(A) = A$. Then $M(G)$ is nilpotent of class at most 3.

In 2009 Guo, Zhao & Shum proved the following generalisation of the rank one case:

**Theorem 15** [GZS09] Let $N$ be a $p$-soluble normal subgroup of a group $G$ such that $N$ contains a noncentral Sylow $r$-subgroup, $(r \neq p)$, $R$ of $G$. If $|x^G| = 1$ or $m$ for every $p'$-element $x$ of $N$ whose order is divisible by at most two distinct primes, then the $p$-complements of $N$ are nilpotent.

Itô's 1952 result follows as a corollary. Zhao and Guo give another generalisation as follows:

**Theorem 16** [ZG09] Let $G$ be a finite group with a non-central Sylow $r$-subgroup $R$ and $N$ a normal subgroup of $G$ containing $R$. If $|x^G| = 1$ or $m$ for every element $x$ of $N$, then $N$ is nilpotent.

In 1996 Li proved the following extension of Itô's rank 1 result, again using the argument quoted in Section 3.1:

**Theorem 17** [Li96] Let $G$ be a finite group and let $m$ be a natural number. Assume that if $x \in G$ has prime-power order then $x$ has index 1 or $m$. Then $G$ is soluble.

Another variation is given by considering the $p'$-conjugate type vector.

**Definition 6** The $p'$-conjugate type vector of a group $G$ is the list of distinct indices of $p'$-elements in descending order.
Suppose $G$ is a group with $p'$-conjugate type vector $(m, 1)$. Three authors, [BF03a, ABF09], have shown that $m = p^aq^b$ for primes $p \neq q$ and if $a$ and $b$ are both strictly greater than 0 then $G = PQ \times A$ where $P$ is a Sylow $p$-subgroup of $G$, $Q$ is a Sylow $q$-subgroup of $G$ and $A$ is in the centre of $G$. We comment that the result follows from the results in [Cam73] and [Cam74]. The first two authors also considered the case where the $p'$-conjugate type vector is $(m, n, 1)$ where $\gcd(m, n) = 1$, [BF04a].

4.2 Conjugate rank 2

In 1974, A.R. Camina gave a new proof of the solubility of groups of conjugate rank 2, along with more details on the structure of such groups. This depended on work of Schmidt [Sch70] and Rebmann [Reb71] which looked at the lattice of centralizers.

**Definition 7** A group $G$ is an $F$-group if given any pair $x, y$ with $x, y \notin Z(G)$ then $C_G(x) \nsubseteq C_G(y)$.

In these papers the authors completely classify $F$-groups. It is not possible to express the condition within the framework of indices. However the condition that for no pair of indices is one divisible by the other implies that the group is an $F$-group.

A.R. Camina’s theorem says what happens if this does not occur:

**Theorem 18** [Cam74] If $G$ has conjugate rank 2 and is not an $F$-group then $G$ is a direct product of an abelian group and a group whose order is divisible by only two primes (or $\rho^*(G) = 2$).

This has recently been improved by Dolfi & Jabara. Their proof is independent of [Cam74] but uses [Itô70a]:

**Theorem 19** [DJ09] A finite group $G$ has conjugate rank 2 if and only if, up to an abelian direct factor, either

1. $G$ is a $p$-group for some prime $p$ or
2. $G = KL$, with $K \trianglelefteq G$, $\gcd(|K|, |L|) = 1$ and one of the following occurs
(a) both $K$ and $L$ are abelian, $Z(G) < L$ and $G$ is a quasi-Frobenius group,

(b) $K$ is abelian, $L$ is a non-abelian $p$-group, for some prime $p$ and $O_p(G)$ is an abelian subgroup of index $p$ in $L$ and $G/O_p(G)$ is a Frobenius group or

(c) $K$ is a $p$-group of conjugate rank 1 for some prime $p$, $L$ is abelian, $Z(K) = Z(G) \cap K$ and $G$ is quasi-Frobenius.

We note that the results in [BF09] can be deduced from the results of Dolfi & Jabara. This completes the classification of groups of conjugate rank 2.

4.3 Conjugate rank larger than 2

Itô went on in the early 1970’s to consider groups of low conjugate rank, 3, 4 and 5 with special reference to the simple groups which can occur, [Itô70b, Itô72, Itô73a, Itô73b]. Recently there has been some effort to look at the case of conjugate rank 3. The first papers were those connected with proving nilpotence, see Subsection 4.4. Beltrán and Felipe [BF08b] looked at the structure of soluble groups with conjugate type vector $(mk, m, n, 1)$ where $\gcd(m, n) = 1$ and $k$ divides $n$. More recently Camina & Camina have shown that if the conjugate rank is larger than 2 and there are two coprime indices amongst any three, then $G$ is soluble. Amongst the results are the following:

**Corollary 3** [CC09] Let $G$ be a finite group with trivial centre and with at most three distinct conjugacy class sizes greater than 1. Then $G$ is either soluble or $PSL(2, 2^n)$.

**Corollary 4** [CC09] Let $G$ be a finite $A$-group with at most three distinct conjugacy class sizes greater than 1. Then $G$ is either soluble or $PSL(2, 2^n)$.

So we ask

**Question 3** Can the rank 3 groups be classified, especially the non-soluble ones?

We note the following theorem due to Bianchi, Gillio and Casolo, which followed earlier work [BCM+92], [Man97]:

14
Theorem 20 [BGC01] Suppose $G$ is a group with conjugate type vector $(m, n, \ldots)$ where $m$ and $n$ are coprime. Suppose $x, y \in G$ with $\text{Ind}_G(x) = n$ and $\text{Ind}_G(y) = m$ and let $N = C_G(x)$ and $H = C_G(y)$. Then $N$ and $H$ are abelian and $G$ is quasi-Frobenius with kernel $N/Z(G)$ and complement $H/Z(G)$.

Thus the conjugate type vector is of the form $(m, n, 1)$ (compare with Theorem 23).

4.4 Nilpotency

In 1972 A.R. Camina proved that a finite group with conjugate type vector $(q^b p^a, q^b, p^a, 1)$, where $p$ and $q$ are primes, is nilpotent [Cam72]. Recently Beltrán and Felipe have proved that if $G$ has conjugate type vector $(nm, m, n, 1)$ with $n$ and $m$ coprime integers, then $G$ is nilpotent and $n$ and $m$ are prime powers [BF06b, BF08a, BF06a].

Beltrán and Felipe [BF07b] have also shown for $G$ a finite $p$-soluble group and $m$ a positive integer not divisible by $p$ that if the set of conjugacy class sizes of all $p'$-elements of $G$ is $\{1, m, p^a, mp^a\}$, then $G$ is nilpotent and $m$ is a prime power.

Camina's result led to the question whether you could identify a nilpotent group from its conjugate type vector. More precisely if $G$ and $H$ have the same conjugate type vector and $H$ is nilpotent, does it follow that $G$ is nilpotent? Notice that a nilpotent group satisfies the following property: if $m$ and $n$ are coprime integers, there are elements of index $m$ and of index $n$ if and only if there is an element of index $mn$. This follows from the fact that a nilpotent group is a direct product of its Sylow $p$-subgroups. We note the following nice result due to Cossey and Hawkes:

Theorem 21 [CH00] Let $p$ be a prime and $S$ a finite set of $p$-powers containing 1. Then there exists a $p$-group $P$ of class 2 with the property that $S$ is the conjugate type vector of $P$ (ordered appropriately).

(Note it is certainly not true that arbitrary sets of numbers can be conjugate type vectors, as the results on graphs of Section 5 indicate.) Using Cossey and Hawkes result it follows that recognising nilpotency is equivalent to ask-
ing whether a group whose indices satisfy the property indicated above is nilpotent.

Although in certain cases you can recognise nilpotency, for example if all the conjugacy classes are square-free [CC98], or if the group is a metabelian A-group [CC06], in general it is not true. The smallest example of a group who shares its conjugate type vector with a nilpotent group, but is not itself nilpotent, has order 160 and conjugate type vector (20, 10, 5, 4, 2, 1). An infinite family of such examples is given in [CC06]. A number of questions are posed in the paper.

**Question 4** Suppose \( G \) and \( H \) are finite groups with \( H \) nilpotent, further suppose \( G \) and \( H \) have the same conjugate type vector.

(i) Is it true that \( G \) must be soluble?

(ii) If \( G \) is not nilpotent, does \( G \) have a centre?

(iii) Suppose \( G \) is an A-group, then must \( G \) be nilpotent?

Note, if you have the additional information of the number of conjugacy classes of each size, then you can recognise nilpotency as Cossey, Hawkes & Mann have proved:

**Theorem 22** [CHM92] Let \( G \) and \( H \) be finite groups with \( H \) nilpotent. Let \( S_H \) be the multiset of conjugacy class sizes of elements in \( H \) and define \( S_G \) similarly. Suppose \( S_H = S_G \), then \( G \) is nilpotent.

These ideas have been extended by Mattarei [Mat06].

5 Graphs

In an earlier version of this survey we included a lengthy chapter on graphs associated to conjugacy class sizes. However, since then Lewis has published an excellent survey concerned with graphs associated to character degrees and conjugacy class sizes [Lew08]. Thus we will just briefly introduce the graphs and mention a few of the more recent results, while still aiming to include a large bibliography.

Let \( X \) be a set of positive integers. We associate two graphs to \( X \), the prime vertex graph and the common divisor graph.
**Definition 8** (i) The common divisor graph of \(X\) has vertex set \(X^* = X \setminus 1\) (\(X\) may or may not contain the element 1) and an edge between \(a, b \in X^*\) if \(a\) and \(b\) are not coprime. We denote the common divisor graph of \(X\) by \(\Gamma(X)\).

(ii) The prime vertex graph has vertex set \(\rho(X) = \bigcup_{x \in X} \pi(x)\) where \(\pi(x)\) denotes the prime divisors of \(x\). There is an edge between \(p, q \in \rho(X)\) if \(pq\) divides \(x\) for some \(x \in X\). The prime vertex graph is denoted by \(\Delta(X)\).

The connection between these two graphs has been clarified by the recent paper [IP09] which defines a bipartite graph \(B(X)\).

**Definition 9** The vertex set of \(B(X)\) is given by the disjoint union of the vertex set of \(\Gamma(X)\) and the vertex set of \(\Delta(X)\), i.e. \(X^* \cup \rho(X)\). There is an edge between \(p \in \rho(X)\) and \(x \in X^*\) if \(p\) divides \(x\), i.e. if \(p \in \pi(x)\).

Consideration of \(B(X)\) makes it clear that the number of connected components of \(\Gamma(X)\) is equal to the number of connected components of \(\Delta(X)\). Furthermore the diameter of a connected component of \(\Gamma(X)\) differs by at most one from the diameter of the equivalent connected component of \(\Delta(X)\).

There are two common choices for the set \(X\):

**Definition 10** (i) \(cs(G)\) the set of conjugacy class sizes of \(G\) (equivalently the set of indices of elements of \(G\)), and (ii) \(cd(G)\) the set of degrees of irreducible characters of \(G\).

Much has been written on both these cases but we will concentrate on the case when \(X = cs(G)\). In 1981 Kazarin wrote a paper on isolated conjugacy classes [Kaz81]. A group \(G\) has isolated conjugacy classes if there exist elements \(x, y \in G\) with coprime indices such that every element of \(G\) has index coprime to either \(\text{Ind}_G(x)\) or \(\text{Ind}_G(y)\). Kazarin classified all groups with isolated conjugacy classes. He therefore classified all groups such that either \(\Gamma(cs(G))\) has more than one component or \(\Gamma(cs(G))\) is connected and has diameter at least 3:

**Theorem 23** [Kaz81] Let \(G\) be a group with isolated conjugacy classes. Let \(x\) and \(y\) be representatives of the two isolated conjugacy classes with \(\text{Ind}_G(x) = m_1\) and \(\text{Ind}_G(y) = n_1\). Then \(|G| = mnr\) where \(r\) is coprime to both \(m\) and \(n\) and
the only primes which divide $m$, respectively $n$, are those which divide $m_1$, respectively $n_1$. Further $G = R \times H$ where $|R| = r$ and $H$ is a quasi-Frobenius group where the pre-image of both the kernel and complement are abelian.

It follows that $\Gamma(\text{cs}(G))$ has at most two connected components. If this is the case we can take (without loss of generality) the conjugate type vector to be $(m_1, n_1, 1)$. Furthermore, if $\Gamma(\text{cs}(G))$ is connected its diameter is at most 3. This has also been proved using different techniques in [BHM90] and [CHM93].

In [BHM90] the authors note that if $G$ is a nonabelian simple group then $\Gamma(\text{cs}(G))$ is complete, this follows from work of Fisman & Arad [FA87]. Another nice result is due to Puglisi and Spezia. They prove that if $\Gamma(\text{cs}(H))$ is a complete graph for every subgroup $H$ of $G$, then $G$ is soluble [PS98].

Itô proved an early result about $\Delta(\text{cs}(G))$: suppose $p$ and $q$ are two distinct non-adjacent vertices in $\Delta(\text{cs}(G))$ then $G$ is either $p$-nilpotent or $q$-nilpotent [Itô53, Proposition 5.1]. Dolfi extended this result for the case when $G$ is soluble concluding that in this case both the $p$-Sylow and $q$-Sylow subgroups are abelian [Dol95a]. In this paper Dolfi also proves the analogous structural results for $\Delta(\text{cs}(G))$: namely, if $\Delta(\text{cs}(G))$ is not connected then it has exactly two connected components and they are complete graphs, and if $\Delta(\text{cs}(G))$ is connected it has diameter at most 3. These results are also contained in [Alf94]. Alfandary determines further structural results for $\Delta(\text{cs}(G))$ when $G$ is soluble in a follow up paper [Alf95].

Casolo and Dolfi [CD09] have characterised the groups for which $\Delta(\text{cs}(G))$ has diameter 3 [CD96b]. In the same paper they show that if a group is not soluble then $\Delta(\text{cs}(G))$ is connected and has diameter at most 2. Another nice result of Dolfi is to show that given 3 distinct vertices in $\Delta(\text{cs}(G))$ then at least two are connected by an edge [Dol06].

A recent paper gives further results on the bipartite graph $B(\text{cs}(G))$ [BDIP09].

Slightly confusingly, another graph has been associated to a finite group $G$. In this case the vertices are given by the set of non-central conjugacy classes and two vertices $C$ and $D$ are joined if $|C|$ and $|D|$ share a common divisor [BHM90]. We shall call this graph $\widehat{\Gamma}(G)$ and note that it shares many properties with $\Gamma(\text{cs}(G))$.

We note that the following conjecture can be viewed as considering groups
For which $\Gamma(\text{cs}(G)) = \hat{\Gamma}(G)$.

**$S_3$ conjecture:** Any finite group in which distinct conjugacy classes have distinct sizes is isomorphic to $S_3$.

This conjecture has been verified for soluble groups by Zhang [Zha94], and Knörr, Lempken and Thielcke [KLT95], independently. More recently Arad, Muzychuk and Oliver have studied the case of insoluble groups [AMO04].

One type of question that is posed when considering these graphs is how the structure of the graph determines the structure of the group? For example the problem of classifying all groups $G$ such that $\hat{\Gamma}(G)$ has no subgraph $K_n$ (where $K_n$ is the complete graph with $n$ vertices) has been considered in [MQS05] for $n = 4, 5$ and in [FZ03] for $n = 3$. In [CC09] groups $G$ such that $\Gamma(\text{cs}(G))$ has no triangles have been considered, such groups have conjugate rank at most 3 and are soluble. This yields the following question.

**Question 5** Let $G$ be a finite group and $n$ a natural number. Is it true that if $\Gamma(\text{cs}(G))$ has no subgraphs isomorphic to $K_n$ then there is a function of $n$ which bounds the conjugate rank of $G$?

Note that this question can also be asked without the language of graphs. The condition translates to requiring that given any set of $n$ distinct indices then there are two which are coprime.

Variants of these graphs have been introduced. For example Beltrán & Felipe have considered a version of $\hat{\Gamma}(G)$ where the vertices are restricted to the $p'$-conjugacy classes, that is the set of $x^G$ where $x$ is a $p'$-element. They consider the case when $G$ is $p$-soluble [BF02, BF03b, BF04b] and have summarised their results in a nicely written survey article [BF07a].

Alternatively, Qian and Wang [QW09b] have considered the conjugacy class sizes of $p$-singular elements, that is elements whose order is divisible by $p$. They denote the set of $p$-singular elements by $G_p$ and consider the graph $\Gamma(\text{cs}(G_p))$. Noting that if $p$ divides $|Z(G)|$ then $\Gamma(\text{cs}(G_p)) = \Gamma(\text{cs}(G))$ they prove that if $p$ divides $|G|$ but not $|Z(G)|$ then $\Gamma(\text{cs}(G_p))$ is connected with diameter at most 3. In the paper the authors also consider groups for which $p$ does not divide $m$ for any $m \in \text{cs}(G_p)$, this leads to a classification of all finite groups for which every conjugacy class size coincides with a Hall number.

Beltrán has introduced the $A$-invariant conjugacy graph [Bel03]. Let $A$
and $G$ be finite groups and suppose that $A$ acts on $G$ by automorphisms. Then $A$ acts on the set of conjugacy classes of $G$. The $A$-invariant conjugacy graph $\Gamma_A(G)$ has vertices the non-central $A$-invariant conjugacy classes of $G$ and two vertices are connected by an edge if their cardinalities are not coprime. Consideration of the case when $A$ acts trivially gives that $\Gamma_A(G)$ is a generalisation of $\hat{\Gamma}(G)$. Beltrán notes that the proof that $\hat{\Gamma}(G)$ has at most two connected components given in [BHM90] translates to this more general setting. He then considers the disconnected case and proves the following theorem:

**Theorem 24** [Bel03] Suppose that a group $A$ acts coprimely on a group $G$ and that $\Gamma_A(G)$ has exactly two connected components. Then $G$ is solvable.

It is not known whether the result holds when $A$ and $G$ do not have coprime orders.

In a different direction Isaacs and Praeger introduced a generalisation of $\Gamma(\text{cs}(G))$ known as the IP-graph [IP93].

**Definition 11** Let $G$ be a group acting transitively on a set $\Omega$, and let $D$ denote the set of subdegrees of $(G, \Omega)$, that is, the cardinalities of the orbits of the action of a point stabilizer $G_\alpha$ on $\Omega$. Suppose the subdegrees are finite, then the IP-graph of $(G, \Omega)$ is the common divisor graph of $D$.

That this is a generalisation of $\Gamma(\text{cs}(G))$ can be seen as follows. Let $G$ be a group and $\text{Inn}(G)$ the inner automorphisms of $G$. Let $H$ be the semidirect product $G \rtimes \text{Inn}(G)$, then $H$ acts transitively on $G$ by sending $x \in G$ to $(xg)^\sigma$ where $g \sigma \in G \rtimes \text{Inn}(G)$. Clearly the orbits of $H_1$, the stabilizer of the identity, are the conjugacy classes. The authors prove that the IP-graph of $(G, \Omega)$ has at most two connected components and that the diameter of a connected component is bounded by 4. However they know of no example of diameter 4, suggesting perhaps that 3 is the correct upper bound.

**Question 6** Does there exist a group $G$ and a set $\Omega$ such that the diameter of the IP-graph of $(G, \Omega)$ is 4?

Kaplan has studied the case when the IP-graph is disconnected [Kap97], [Kap99]. Neumann introduced the VIP graph, a variant of the IP graph which
does not restrict itself to the case where all subdegrees are finite [Neu93].
More recently the IP graph has been generalised to the setting of naturally
valenced schemes [Cam08]. This work was extended by Xu [Xu09].

We would like to introduce a graph using the notion of divisibility. Let
$X$ be a set of positive integers, then $D(X)$ the divisibility graph is a directed
graph. The vertex set of $D(X)$ is given by $X^*$ and there is an edge connecting
$(a, b)$ with $a, b \in X^*$ whenever $a$ divides $b$. We are interested in the properties
of this graph when $X = \text{cs}(G)$ for a finite group $G$. Note that if $D(\text{cs}(G))$
has no edges then $G$ is an $F$-group.

**Question 7** *How many components can $D(\text{cs}(G))$ have?*

It is worth pointing out that whilst many results can be interpreted in the
language of graph theory there are many interesting problems that have no
such simple description.

## 6 The number of conjugacy classes

Given a group $G$ of order $n$ with $k$ conjugacy classes what can be said about
the relation between $n$ and $k$? It is trivial to see that $k \leq n$ but can any-
thing be said in the opposite direction? The first to bound $n$ in terms of $k$
was Landau in 1903 [Lan03]. He used a number theoretic approach to the
class equation. Brauer [Bra63] was the first to give an explicit bound using
Landau’s method. He asked, in Problem 3, whether better methods could be
found. A similar approach was taken by Newman in [New68] where he im-
proved Landau’s result. This gave very general bounds of exponential form.
He proved:

**Theorem 25** [New68] Let $G$ be a finite group of order $n$ with $k$ conjugacy
classes. Then

$$k \geq \frac{\log \log n}{\log 4}.$$ 

Also there have been papers which give complete descriptions for small
$k$, the earliest example being Miller in 1919 [Mil19]. The objective here is
to try to classify the isomorphism classes with a given number of conjugacy
classes. A number of authors over many years have classified groups with few
conjugacy classes [Mil44, Pol68a, Pol68b, VLVL85, VLVL86]. In particular, Vera López & Vera López [VLVL85, VLVL86] examine groups with 13 and 14 conjugacy classes and give lists of such groups. As far as the authors know this is the largest value of $k$ for which this has been attempted.

Cartwright in [Car87] considered soluble groups. He proves:

**Theorem 26** There exist positive constants $a$ and $b$ so that a soluble group of order $n$ has at least $a(\log n)^b$ conjugacy classes.

There is an estimate for $b$ of 0.00347. But Pyber [Pyb92] has proved the following theorem:

**Theorem 27** Let $G$ be a finite group with $k$ conjugacy classes. Then

$$k \geq \epsilon \log n / (\log \log n)^8,$$

for some fixed $\epsilon$.

This could be considered an answer to Brauer’s question. We note that Keller has reduced the power in the denominator from 8 to 7 [Kel]. Also, for nilpotent groups Jaikin-Zapirain has proved a stronger result [JZ].

In 1997 Liebeck & Pyber [LP97] wrote a paper in which they gave upper bounds for the number of conjugacy classes for various classes of groups. Their main result is the following:

**Theorem 28** [LP97, Theorem 1] Let $G$ be a finite simple group of Lie type over a finite field of order $q$. Let $G$ have rank $\ell$ and assume that $G$ has $k$ conjugacy classes. Then

$$k \leq (6q)^\ell.$$

Note that this is the “untwisted” rank.

As a consequence of this they prove

**Theorem 29** [LP97, Theorem 2] Let $G$ be a subgroup of the symmetric group of degree $n$. Then if $G$ has $k$ conjugacy classes

$$k \leq 2^{n-1}.$$
This solved a conjecture of Kovács & Robinson [KR93]. They had found a bound of $5^{n-1}$, without the classification of finite simple groups. Maróti has reduced the bound to $k \leq 3^{(n-1)/2}$ [Mar05].

An interesting variation has been proved by Jaikin-Zapirain:

**Theorem 30** [JZ05] There exists a function $f(r)$ such that if $G$ is a soluble group with at most $r$ conjugacy classes of size $k$ for any $k$ then $|G| \leq f(r)$.

## 7 Comparisons with Character Theory

### 7.1 Introduction

The most obvious connection with character theory is that the number of conjugacy classes is the same as the number of irreducible characters. A number of authors, including those of this article, have seen a connection between character degrees and conjugacy class sizes and have searched for analogous results. As we commented in the Introduction, if we know the multiplication constants then we have a complete connection between the two sets of data. However if we only know the degrees and the indices the link seems less clear.

### 7.2 The differences

Perhaps the first obvious difference is that the order of the group is given by the sum of the sizes of the conjugacy classes, but the sum of the squares of the degrees of the irreducible characters. This might help to explain the following dichotomies. As previously mentioned in Section 2, if $p$ is coprime to all indices of elements of $G$ then the Sylow $p$-subgroup of $G$ is an abelian direct factor of $G$. However, Itô proved that if $G$ has a normal abelian Sylow $p$-subgroup then $p$ is coprime to all character degrees of $G$. Also, in Section 3.2 we noted that if all indices are square-free then $G$ is soluble. However, if all irreducible character degrees are square-free $G$ need not be soluble, the smallest example we know is $\text{Alt}(7)$, see for example [CCN+85]. Another example is given by the two different conclusions drawn when $\{1, p^a, q^b, p^aq^b\}$ is either the set of character degrees or the set of conjugacy class sizes and $p$
and \( q \) are distinct primes. In the conjugacy class case we can conclude that \( G = P \times Q \) where \( P \) is the Sylow \( p \)-subgroup and \( Q \) the Sylow \( q \)-subgroup [Cam72]. This conclusion does not hold in the character case [Lew98].

One might also ask whether there are connections between the sizes of \( \text{cs}(G) \) and \( \text{cd}(G) \)? The answer is no. In an elegant construction Fernández-Alcober & Moretó prove the following theorem:-

**Theorem 31** [FAM01] Given any two integers \( r \) and \( s \) greater than 1 there exists a \( p \)-group \( G \) of nilpotency class 2 such that \( |\text{cd}(G)| = r \) and \( |\text{cs}(G)| = s \).

### 7.3 The similarities

In Section 5 we defined the common divisor graph \( \Gamma(X) \) and the prime vertex graph \( \Delta(X) \) for a set of natural numbers \( X \). We focussed on the case when \( X \) is the set of indices of a finite group \( G \), denoted by \( \text{cs}(G) \), but commented that much work has been done for the case when \( X \) is the set of degrees of irreducible characters, \( \text{cd}(G) \), in fact this case was considered first (see [Lew08] for a recent survey of results). What is remarkable is the similarity of the graphs for the two different choices of \( X \). In particular, if \( G \) is soluble then in all cases (both choices of \( X \) and graph) the graph has at most two connected components and if the graph is connected the diameter of the graph is at most three. However, if \( G \) is not soluble then it is possible for \( \Delta(\text{cd}(G)) \), and hence also \( \Gamma(\text{cd}(G)) \), to have three connected components.

Recently Casolo and Dolfi have proved that \( \Delta(\text{cd}(G)) \) is a subgraph of \( \Delta(\text{cs}(G)) \). It is easy to see that each prime number that divides an irreducible character degree of \( G \) must also divide some class size of \( G \). Casolo and Dolfi [CD09] prove that for distinct primes \( p \) and \( q \) if \( pq \) divides the degree of some irreducible character of \( G \), then it also divides the size of some conjugacy class of \( G \). Dolfi had previously proved this result for soluble groups [Dol95b].

Authors have considered the implications of the arithmetical data of \( \text{cd}(G) \) on the structure of the group. For example in [Lew98] Lewis proves that for \( p, q \) and \( r \) distinct primes, if \( \text{cd}(G) = \{1, p, q, r, pq, pr\} \) then \( G = A \times B \) where \( \text{cd}(A) = \{1, p\} \) and \( \text{cd}(B) = \{1, q, r\} \). For \( s \) another distinct prime he also proves that if \( \text{cd}(G) = \{1, p, q, r, s, pr, ps, qr, qs\} \) then \( G = A \times B \) with \( \text{cd}(A) = \{1, p, q\} \) and \( \text{cd}(B) = \{1, r, s\} \). The equivalent results where character degrees are replaced by conjugacy class sizes are proved
In 2006 Isaacs, Keller, Meierfrankenfeld & Moretó showed that if $G$ is soluble, $p$ a prime divisor of $|G|$ and $\chi$ a primitive irreducible character of $G$. Then the $p$-part of $\chi(1)$ divides the $p$-part of $(\text{Ind}_G(g))^3$ for some $g \in G$. Furthermore, they put forward the following conjecture.

**Conjecture [IKMM06]** Let $\chi$ be a primitive irreducible character of an arbitrary finite group $G$. Then $\chi(1)$ divides $\text{Ind}_G(g)$ for some element $g \in G$.

They checked that the conjecture holds for all irreducible characters (primitive or not) of all groups in the Atlas [CCN+85].

### 7.4 $k(GV)$-problem

In a slightly different direction Brauer [Bra63] was interested in the number of characters in a given block. Let $p$ be a prime and consider the number of ordinary irreducible characters belonging to the $p$-block $B$ with defect group $D$. He asked whether this is less than or equal to $|D|$. For definitions of blocks and related concepts see [Isa06, Chapter 15].

We know that the total number of irreducible characters is the number of conjugacy classes. Interestingly in the $p$-soluble case the conjecture reduces to the following problem [Nag62]:

**$k(GV)$-problem** Let $G$ be a finite $p'$-group for some prime $p$ and let $V$ be a faithful $\mathbb{F}_p$-module for $G$. Show that the number of conjugacy classes of the semidirect product $VG$ is bounded by $|V|$.

In this situation there is only one $p$-block so that all characters are in the same $p$-block with defect group $V$ and Brauer’s question is about counting the number of conjugacy classes in a group.

The fundamental ideas for attacking the problem were developed around 1980 by Knörr [Knö84], who proved the conjecture for $G$ supersoluble and later for $|G|$ odd [Glu84], the first significant partial results. An important step on the way was the paper by Robinson & Thompson, [RT96], which solved the problem for all but finitely many primes.

The problem was finally solved by Gluck, Magaard, Riese & Schmid in 2004, [GMRS04]. This was the final piece of the jigsaw after a long series of papers. A more detailed discussion and details of proofs are in the book.
by Schmid [Sch07]. One of the consequences of these results is that if $G$ is a $p'$-group that can be embedded in $GL(m, p)$ for some integer $m$ then $k(G) \leq p^m - 1$.

### 7.5 Huppert’s Conjectures

Let $G$ be a finite soluble group. Let $\bar{\sigma}(G)$ denote the number of primes dividing any one character degree of $G$ and let $\bar{\rho}(G)$ be the set of all primes which divide some character degree of $G$. In 1985, Huppert [Hup91] conjectured that if $G$ is soluble then $|\bar{\rho}(G)| \leq 2\bar{\sigma}(G)$. This conjecture is still open, the best result so far, for soluble groups, is that $|\bar{\rho}(G)| \leq 3\bar{\sigma}(G) + 2$ [GM87].

He also asked analogously whether $|\rho^*(G)| \leq 2\sigma^*(G)$? Baer’s results referred to earlier, Subsection 3.1, show that if $|\sigma^*(G)| = 1$ then $|\rho^*(G)| \leq 2$. In [Man97] Mann considered groups in which all conjugacy class sizes involve at most 2 primes, that is groups $G$ for which $|\sigma^*(G)| = 2$. Such a group is either soluble, or $G = Z(G) \times S$, where $S$ is isomorphic to either $A_5$ or $SL(2, 8)$ and $\rho^*(G) \leq 4$. This result also appears in [Cas94] where the author is concentrating on Huppert’s Conjecture, note that both Baer’s and Mann’s papers predate Huppert’s conjecture. In [Zha98] Zhang shows that $\rho^*(G) \leq 4\sigma^*(G)$ for all soluble groups which generalises the results of some earlier authors, [Fer93, Fer91a, Fer92, Fer91b]. Casolo in [Cas91] proved the following theorem:

**Theorem 32** Let $G$ be a group which is $p$-nilpotent for at most one prime divisor of $|G|$ (this family includes all nonabelian simple groups), then $$\rho^*(G) \leq 2\sigma^*(G).$$

So Huppert’s conjecture holds for many groups, consequently the following theorem is somewhat surprising:

**Theorem 33** [CD96a] There exist an infinite sequence of finite supersoluble, metabelian groups $\{G_n\}$, such that $|\rho^*(G_n)|/\sigma^*(G_n)$ tends to 3 as $n \to \infty$.

So Huppert’s conjecture is false. However, they do show that $|\rho^*(G)| \leq 4\sigma^*(G) + 2$ for all soluble $G$. Despite the result being false Casolo & Dolfi have shown that there are linear bounds in both the character and conjugacy class versions of the conjecture for all groups.

26
Theorem 34 [CD07] Let $G$ be a finite group. Then

(a) $|\bar{\rho}(G)| \leq 7\bar{\sigma}(G)$,
(b) $|\rho^*(G)| \leq 5\sigma^*(G)$.

In 2000 Huppert made the following conjecture:

Conjecture [Hup00] If $H$ is any simple nonabelian finite group and $G$ is a finite group such that $\text{cd}(G) = \text{cd}(H)$, then $G \cong H \times A$, where $A$ is abelian.

The complete answer to this question is still to be found in the character case. So we put forward the following question:-

Question 8 If $H$ is any simple nonabelian finite group and $G$ is a finite group such that $\text{cs}(G) = \text{cs}(H)$, then is $G \cong H \times A$, where $A$ is abelian?

References


32


Thomas Michael Keller. Finite groups have even more conjugacy classes. Israel J. Math. to appear.


