

Part I

Linear waves on Lorentzian manifolds

1 General techniques and basic statements

This section concerns the analysis of the covariant wave equation

$$\square_g \psi = F \tag{1.1}$$

on a Lorentzian manifold $(\mathcal{M}, g)$. Here, ϕ and F are functions from $\mathcal{M}$ to $\mathbb{R}$, also called scalar fields, and $\square_g$ is a differential operator defined by

$$\square_g \psi := \nabla^\mu \nabla_\mu \psi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \psi).$$

If $F \equiv 0$, we call (1.1) the homogeneous (scalar) wave equation on $\mathcal{M}$. Three important examples of inhomogeneous scalar waves arise from:

- adding a potential to the homogeneous wave equation:

$$(\square_g - V)\psi = 0 \implies \square_g \psi = V\psi;$$

- commuting the homogeneous wave equation with vector fields Z :

$$\square_g \psi = 0 \implies \square_g (Z\psi) = [\square_g, Z]\psi + Z(\square_g \psi) = [Z, \square_g]\psi.$$

- nonlinearities: a general quasilinear wave equation

$$\square_{g(\psi, \partial\psi)} \psi = N(\psi, \partial\psi)$$

can be written as an inhomogeneous wave equation

$$\square_{g_0} \psi = F$$

with respect to the “background” metric $g_0 := g(0, 0)$, where $F := N(\psi, \partial\psi) - (\square_{g(\psi, \partial\psi)} - \square_{g_0})\psi$.

The goal of the section is to examples of how, using an entirely geometric language, we can derive analytical properties, like boundedness and decay in appropriate Sobolev (energy) spaces, of solutions to (1.1).

1.1 Symmetries and commutators

We first show that symmetries of the background spacetime interact well with the covariant wave equation (1.1). Recall the following definition:

Definition 1.1. Let X be a vector field on $(\mathcal{M}, g)$. Then, we define its deformation tensor by

$${}^{(X)}\pi := \frac{1}{2} \mathcal{L}_g X \implies {}^{(X)}\pi^{\mu\nu} = \frac{1}{2} (\nabla^\mu X^\nu + \nabla^\nu X^\mu).$$

We say X is Killing if ${}^{(X)}\pi = 0$ and conformally Killing if there exists a scalar field λ such that ${}^{(X)}\pi = \lambda g$.

Lemma 1.2. Let $(\mathcal{M}, g)$ be 4-dimensional, and let Z be a vector field on $\mathcal{M}$. Assume ψ satisfies (1.1) for some given F . Then, we have

$$\square_g (Z\psi) = Z(F) + 2{}^{(Z)}\pi^{\mu\nu} \nabla_\mu \nabla_\nu \psi + 2\nabla_\mu {}^{(Z)}\pi^{\mu\nu} \nabla_\nu \psi - \nabla_\nu {}^{(Z)}\pi^\mu_\mu \nabla^\nu \psi.$$

In particular: if Z is Killing, we have

$$\square_g (Z\psi) = Z(\square_g \psi) = Z(F);$$

if Z is conformally Killing, with ${}^{(Z)}\pi = \lambda g$, we have

$$\square_g (Z\psi) = (Z + 2\lambda)(\square_g \psi) - 2\nabla_\alpha \lambda \nabla^\alpha \psi = Z(F) + 2\lambda F + (2 - n - 1)\nabla_\alpha \lambda \nabla^\alpha \psi.$$

Proof. We compute

$$\begin{aligned} [\square_g, Z]\psi &= \nabla^\mu \nabla_\mu (Z^\nu \nabla_\nu \psi) - Z^\nu \nabla_\nu \nabla^\mu \nabla_\mu \psi = \nabla^\mu (Z^\nu \nabla_\mu \nabla_\nu \psi + \nabla_\mu Z^\nu \nabla_\nu \psi) - Z^\nu \nabla_\nu \nabla^\mu \nabla_\mu \psi \\ &= \nabla^\mu Z^\nu (\nabla_\mu \nabla_\nu \psi + \nabla_\nu \nabla_\mu \psi) + \nabla^\mu \nabla_\mu Z^\nu \nabla_\nu \psi + Z^\nu (\nabla^\mu \nabla_\mu \nabla_\nu \psi - \nabla_\nu \nabla^\mu \nabla_\mu \psi) \\ &= 2{}^{(Z)}\pi^{\mu\nu} \nabla_\mu \nabla_\nu \psi + \nabla^\mu \nabla_\mu Z^\nu \nabla_\nu \psi + Z^\nu [\nabla_\mu, \nabla_\nu] \nabla^\mu \psi \end{aligned}$$

using the fact that $[\nabla_\mu, \nabla_\nu]\psi = 0$ for a scalar field. Now note that

$$[\nabla_\mu, \nabla_\nu]\nabla^\mu\psi = R_{\lambda\nu}\nabla^\lambda\psi,$$

and, moreover, since ${}^{(Z)}\pi_\mu^\mu = \nabla^\alpha X_\alpha$, we have

$$2\nabla^{\mu(Z)}\pi_{\mu\nu} = \nabla^\mu\nabla_\mu Z_\nu + \nabla_\nu\nabla_\mu X^\mu + [\nabla_\mu, \nabla_\nu]X^\mu = \nabla^\mu\nabla_\mu Z_\nu + \nabla_\nu{}^{(Z)}\pi_\mu^\mu + R_{\lambda\nu}X^\lambda$$

Combining these identities, we obtain

$$[\square_g, Z]\psi = 2{}^{(Z)}\pi^{\mu\nu}\nabla_\mu\nabla_\nu\psi + 2\nabla^{\mu(Z)}\pi_{\mu\nu}\nabla^\nu\psi - \nabla_\nu{}^{(Z)}\pi_\mu^\mu\nabla^\nu\psi, \quad (1.2)$$

since $R_{\lambda\nu}(X^\nu\nabla^\lambda\psi - X^\lambda\nabla^\nu\psi) = 0$ by the symmetries of the Ricci tensor. $\square$

1.2 Energy-momentum tensor and multipliers

The homogeneous wave equation comes from a Lagrangian structure:

$$0 = \square_g\psi\nabla_\nu\psi = \nabla^\mu T_{\mu\nu}[\psi], \quad (1.3)$$

where the energy-momentum tensor

$$T_{\mu\nu}[\psi] := \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\psi\nabla_\alpha\psi \quad (1.4)$$

leads to the action

$$\mathcal{L}[\psi] := \int_{\mathcal{M}} g_{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi \, \text{dvol}_g.$$

We will make use of this Lagrangian structure heavily through this notes. However, as we have already seen, it is sometimes useful to think in terms of rescaled versions of ψ , such as the radiation field. Thus, let us introduce another function β on $\mathcal{M}$, and define the rescaling $\phi := \beta^{-1}\psi$.

Lemma 1.3. *Let F be given and assume ψ solves the wave equation (1.1). Then ϕ satisfies*

$$\nabla^\mu(\beta^2\nabla_\mu\phi) - U\phi = \beta F, \quad U = -\beta^{-1}\square_g\beta.$$

Proof. We compute

$$\begin{aligned} \square_g\psi &= \nabla^\mu\nabla_\mu\psi = \nabla^\mu(\beta\nabla_\mu(\beta^{-1}\psi)) - \nabla^\mu(\beta\nabla_\mu\beta^{-1}\psi) \\ &= \beta^{-1}\nabla^\mu(\beta^2\nabla_\mu(\beta^{-1}\psi)) - (\beta^{-1}\nabla^\mu\beta)\beta\nabla_\mu(\beta^{-1}\psi) + \nabla^\mu(\beta^{-1}\psi\nabla_\mu\beta) \\ &= \beta^{-1}\nabla^\mu(\beta^2\nabla_\mu\phi) - \nabla^\mu\beta\nabla_\mu\phi + \nabla^\mu(\phi\nabla_\mu\beta) \\ &= \beta^{-1}\nabla^\mu(\beta^2\nabla_\mu\phi) + \square_g\beta\phi = \beta^{-1}[\nabla^\mu(\beta^2\nabla_\mu\phi) - U\phi], \end{aligned}$$

which concludes the proof. $\square$

Let us now generalize the definition in (1.4).

Definition 1.4. Take a scalar field ψ on $\mathcal{M}$. We define its twisted energy-momentum tensor by

$$\begin{aligned} \tilde{T}_{\mu\nu}[\psi] &:= \beta\nabla_\mu(\beta^{-1}\psi)\beta\nabla_\nu(\beta^{-1}\psi) - \frac{1}{2}g_{\mu\nu}[\beta\nabla^\alpha(\beta^{-1}\psi)\beta\nabla_\alpha(\beta^{-1}\psi) + U\beta^{-2}\psi^2] \\ &= \beta^2\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}(\nabla^\alpha\phi\nabla_\alpha\phi + U\phi^2)\right]. \end{aligned} \quad (1.5)$$

Next, we generalize the divergence identity (1.3):

Lemma 1.5. *Let F be given and assume ψ solves (1.1). Then, its twisted energy-momentum tensor satisfies*

$$\nabla^\mu\tilde{T}_{\mu\nu}[\psi] = \beta F\nabla_\nu\phi + \tilde{S}_\nu[\psi], \quad (1.6)$$

where we have set

$$\tilde{S}_\nu[\psi] := -\beta\nabla_\nu\beta\nabla^\alpha\phi\nabla_\alpha\phi - \frac{1}{2}\nabla_\nu(U\beta^2)\phi^2.$$

Proof. We notice that

$$\tilde{T}_{\mu\nu}[\psi] = \beta^2 T_{\mu\nu}[\phi] - \frac{1}{2} g_{\mu\nu} U \phi^2 \beta^2,$$

and thus, using (1.3), we compute

$$\begin{aligned} \nabla^\mu \tilde{T}_{\mu\nu}[\psi] &= \beta^2 \square_g (\beta^{-1} \psi) \nabla_\nu \phi + \nabla^\mu (\beta^2) T_{\mu\nu}[\phi] - \frac{1}{2} \nabla_\nu (U \beta^2) \phi^2 - \frac{1}{2} \nabla_\nu (\phi^2) U \beta^2 \\ &= \beta^2 [\square_g \beta^{-1} \psi + 2 \nabla^\mu \beta^{-1} \nabla_\mu \psi + \beta^{-1} \square_g \psi] \nabla_\nu \phi + 2 \beta \nabla^\mu \beta \nabla_\mu \phi \nabla_\nu \phi - \beta \nabla_\nu \beta \nabla^\alpha \phi \nabla_\alpha \phi \\ &\quad - \frac{1}{2} \nabla_\nu (U \beta^2) \phi^2 - \phi \nabla_\nu \phi U \beta^2 \\ &= \beta F \nabla_\nu \phi + \tilde{S}_\nu[\psi] + [\beta^2 (\beta \square_g \beta^{-1} - U) \phi + 2 \beta \nabla^\mu \beta^{-1} \beta \nabla_\mu \psi + 2 \beta \nabla^\mu \beta \nabla_\mu \phi] \nabla_\nu \phi \\ &= \beta F \nabla_\nu \phi + \tilde{S}_\nu[\psi] + \beta^2 (\beta \square_g \beta^{-1} + 2 \nabla^\mu \beta \nabla_\mu \beta^{-1} - U) \phi \nabla_\nu \phi. \end{aligned}$$

The term in round brackets vanishes; indeed,

$$\begin{aligned} \beta \square_g \beta^{-1} &= \beta \nabla^\mu \nabla_\mu \beta^{-1} = \nabla^\mu (\beta \nabla_\mu \beta^{-1}) - \nabla^\mu \beta \nabla_\mu \beta^{-1} = -\nabla^\mu (\beta^{-1} \nabla_\mu \beta) - \nabla^\mu \beta \nabla_\mu \beta^{-1} \\ &= -\beta^{-1} \nabla^\mu \nabla_\mu \beta - 2 \nabla^\mu \beta \nabla_\mu \beta^{-1} = -\beta^{-1} \square_g \beta - 2 \nabla^\mu \beta \nabla_\mu \beta^{-1} = U - 2 \nabla^\mu \beta \nabla_\mu \beta^{-1}. \end{aligned}$$

This concludes the proof. $\square$

Remark 1.6. In practice, by multiplying (1.1) by $\beta \nabla_\nu \phi$ and integrating by parts, one can identify the appropriate energy-momentum tensor $\tilde{T}[\psi]$ and one-form $\tilde{S}[\psi]$ which allow for identity (1.6) to hold.

The energy-momentum tensor for a scalar field naturally induces a notion of current associated to vector field on $\mathcal{M}$. We jump ahead to introduce the most general form thereof that we will consider in these notes:

Definition 1.7. Take a scalar field ψ and a vector field X on $\mathcal{M}$. We define the twisted X -current of ψ as the one-form

$$\tilde{J}_\mu^X[\psi] := X^\nu \tilde{T}_{\mu\nu}[\psi]. \quad (1.7)$$

Lemma 1.8. Let F be given and assume ψ solves (1.1). For any vector field X , the twisted X -current of ψ satisfies

$$\nabla^\mu \tilde{J}_\mu^X[\psi] = \tilde{K}^X[\psi] + \tilde{\mathcal{E}}^X[\psi], \quad (1.8)$$

where we have set

$$\begin{aligned} \tilde{K}^X[\psi] &:= {}^{(X)}\pi^{\mu\nu} \left(\tilde{T}_{\mu\nu}[\psi] - \beta \phi^2 \nabla_\mu \nabla_\nu \beta \right) - \beta \left(\nabla_\nu {}^{(X)}\pi^{\mu\nu} \nabla_\mu \beta - \frac{1}{2} \nabla_\nu {}^{(X)}\pi_\mu^\mu \nabla^\nu \beta \right) \phi^2 \\ &\quad - \beta X(\beta) \left(\nabla^\alpha \phi \nabla_\alpha \phi + \frac{1}{2} U \phi^2 \right) + \frac{1}{2} \beta \square_g X(\beta) \phi^2, \end{aligned} \quad (1.9)$$

$$\tilde{\mathcal{E}}^X[\psi] := \beta F X(\phi). \quad (1.10)$$

In particular, if X is Killing and $X(\beta) = 0$, $\tilde{K}^X[\psi] = 0$.

Proof. We compute

$$\nabla^\mu \tilde{J}_\mu^X[\psi] = \nabla^\mu \left(X^\nu \tilde{T}_{\mu\nu}[\psi] \right) = \nabla^\mu X^\nu \tilde{T}_{\mu\nu}[\psi] + X^\nu \nabla^\mu \tilde{T}_{\mu\nu}[\psi] = {}^{(X)}\pi^{\mu\nu} \tilde{T}_{\mu\nu}[\psi] + X^\nu \nabla^\mu \tilde{T}_{\mu\nu}[\psi],$$

and then use (1.6) to obtain (1.8) with

$$\tilde{K}^X[\psi] = {}^{(X)}\pi^{\mu\nu} \tilde{T}_{\mu\nu}[\psi] + X^\nu \tilde{S}_\nu[\psi] = {}^{(X)}\pi^{\mu\nu} \tilde{T}_{\mu\nu}[\psi] - \beta X(\beta) \nabla^\alpha \phi \nabla_\alpha \phi - \frac{1}{2} X(U \beta^2) \phi^2.$$

To simplify the last term, we recall that

$$X(U \beta^2) = -X(\beta \square_g \beta) = X(\beta) U \beta - \beta \square_g X(\beta) + \beta [\square_g, X] \beta,$$

and we apply (1.2), i.e.

$$[\square_g, X] \beta = 2 {}^{(X)}\pi^{\mu\nu} \nabla_\mu \nabla_\nu \beta + 2 \nabla^\mu {}^{(X)}\pi_{\mu\nu} \nabla^\nu \beta - \nabla_\nu {}^{(X)}\pi_\mu^\mu \nabla^\nu \beta,$$

to rewrite the commutator term. $\square$

Remark 1.9. Note that the one-form $\tilde{J}^X[\psi]$ and the scalar $\tilde{K}^X[\psi]$ only depend on the 1-jet of ψ , i.e. ψ and its first derivatives. For solutions of the homogeneous wave equation, $\tilde{\mathcal{E}}^X[\psi] \equiv 0$ and thus (1.8) imply that $\tilde{J}^X[\psi]$ is a compatible current in the sense of [5].

For later applications, it will also be convenient to define an auxiliary current:

Definition 1.10. Take a scalar field ψ and a vector field X on $\mathcal{M}$. We define the w -Lagrangian current of ψ as the one-form

$$\tilde{J}_\mu^{\text{aux},w}[\psi] := \beta^2 \left[w\phi \nabla_\mu \phi - \frac{1}{2} \phi^2 \nabla_\mu w \right]. \quad (1.11)$$

Lemma 1.11. Let F be given and assume ψ solves (1.1). For any scalar field w , the w -Lagrangian current of ψ satisfies

$$\nabla^\mu \tilde{J}_\mu^{\text{aux},w}[\psi] = \tilde{K}^{\text{aux},w}[\psi] + \tilde{\mathcal{E}}^{\text{aux},w}[\psi], \quad (1.12)$$

where we have set

$$\tilde{K}^{\text{aux},w}[\psi] := \beta^2 (wU\phi^2 + \nabla^\mu \phi \nabla_\mu \phi) - \frac{1}{2} \nabla^\mu (\beta^2 \nabla_\mu w) \quad (1.13)$$

$$\tilde{\mathcal{E}}^{\text{aux},w}[\psi] := \beta F w \phi. \quad (1.14)$$

Proof. We compute

$$\begin{aligned} \nabla^\mu \tilde{J}_\mu^{\text{aux},w}[\psi] &= \nabla^\mu (\beta^2) \left[w\phi \nabla_\mu \phi - \frac{1}{2} \phi^2 \nabla_\mu w \right] + \beta^2 \left[w \nabla^\mu \phi \nabla_\mu \phi + w\phi \square_g \phi - \frac{1}{2} \phi^2 \square_g w \right] \\ &= \nabla^\mu (\beta^2) \left[w\phi \nabla_\mu \phi - \frac{1}{2} \phi^2 \nabla_\mu w \right] + \beta^2 w \nabla^\mu \phi \nabla_\mu \phi + w\beta^2 \phi (\square_g \beta^{-1} \psi + 2 \nabla^\mu \beta^{-1} \nabla_\mu \psi + \beta^{-1} \square_g \psi) \\ &\quad - \frac{1}{2} \phi^2 \nabla^\mu (\beta^2 \nabla_\mu w) + \frac{1}{2} \phi^2 \nabla^\mu (\beta^2) \nabla_\mu w \\ &= \tilde{K}^{\text{aux},w}[\psi] + \tilde{\mathcal{E}}^{\text{aux},w}[\psi] + w\beta^2 \phi [-U\phi + \beta \square_g \beta^{-1} \phi - 2\beta^{-1} \nabla^\mu \beta \beta^{-1} \nabla_\mu \psi + 2\beta^{-1} \nabla^\mu \beta \nabla_\mu \phi], \end{aligned}$$

where the term in square brackets vanishes by the same argument as in the previous lemma. $\square$

Remark 1.12. In practice, by multiplying (1.1) by $\beta w \phi$ and integrating by parts, one can identify the appropriate one-form $\tilde{J}_\mu^{\text{aux},w}[\psi]$ and scalar $\tilde{K}^{\text{aux},w}[\psi]$ which allow for identity (1.12) to hold.

1.3 Stokes' theorem and volume forms

Finally, we can integrate the identities in Lemmas 1.8 and 1.11. Let $\mathcal{R} \subset \mathcal{M}$ be a smooth, time oriented, subset of $\mathcal{M}$ with boundary

$$\partial \mathcal{R} \equiv \mathcal{T} \sqcup \mathcal{S} \sqcup \mathcal{N}$$

where $\mathcal{T}$ is a timelike hypersurface (or unions thereof), $\mathcal{S}$ is a spacelike hypersurface (or unions thereof) and $\mathcal{N}$ is a null hypersurface (or unions thereof).

Proposition 1.13 (Stokes' theorem). Let F be given, and ψ be a solutions to (1.1). Let $\mathcal{R} \subset \mathcal{M}$ be a smooth, time oriented, subset of $\mathcal{M}$ with boundary $\partial \mathcal{R}$. Then, we have

$$\begin{aligned} \int_{\mathcal{T}} \tilde{J}_\mu^\circ[\psi] n_{\mathcal{T}}^\mu \text{dvol}_{\mathcal{S}} + \int_{\mathcal{T}} \tilde{J}_\mu^\circ[\psi] n_{\mathcal{S}}^\mu \text{dvol}_{\mathcal{S}} + \int_{\mathcal{N}} \tilde{J}_\mu^\circ[\psi] n_{\mathcal{N}}^\mu \text{dvol}_{\mathcal{N}} \\ = \int_{\partial \mathcal{R}} \tilde{J}_\mu^\circ[\psi] n_{\partial \mathcal{R}}^\mu \text{dvol}_{\partial \mathcal{R}} = \int_{\mathcal{R}} \left(\tilde{K}^\circ[\psi] + \tilde{\mathcal{E}}^\circ[\psi] \right) \text{dvol}_{\mathcal{R}}, \end{aligned} \quad (1.15)$$

where $\circ$ can be replaced by any vector field X , or by the notation aux,w for a scalar field w , and $n_{\mathcal{T}} \text{dvol}_{\mathcal{T}}$, $n_{\mathcal{S}} \text{dvol}_{\mathcal{S}}$ and $n_{\mathcal{N}} \text{dvol}_{\mathcal{N}}$ are chosen as follows:

- $n_{\mathcal{T}}$ is a spacelike vector field satisfying $g(X, n_{\mathcal{T}}) = 0$ for all vector fields X on $\mathcal{T}$, which is normalized so that $g(n_{\mathcal{T}}, n_{\mathcal{T}}) = 1$ and outward-pointing with respect to $\text{int}(\mathcal{R})$; $\text{dvol}_{\mathcal{T}} = \text{dvol}_{g|_{\mathcal{T}}}$;
- $n_{\mathcal{S}}$ is a timelike vector field satisfying $g(X, n_{\mathcal{S}}) = 0$ for all vector fields X on $\mathcal{S}$, which is normalized so that $g(n_{\mathcal{S}}, n_{\mathcal{S}}) = -1$ and future-directed; $\text{dvol}_{\mathcal{S}} = \text{dvol}_{g|_{\mathcal{S}}}$;
- we pick $\underline{n}_{\mathcal{N}}$ to be a null generator of $\mathcal{N}$, i.e. to be a future-directed null vector field satisfying $g(X, \underline{n}_{\mathcal{N}}) = 0$ for all vector fields X on $\mathcal{N}$ and tangential to the null geodesics on $\mathcal{M}$; then, $n_{\mathcal{N}}$ is another future-oriented null vector field normalized so that $g(\underline{n}_{\mathcal{N}}, n_{\mathcal{N}}) = -1$ and let $\text{dvol}_{\mathcal{N}} = -n_{\mathcal{N}} \lrcorner \text{dvol}_g$.

Remark 1.14. The choice of signs in Proposition 1.13 implies that the integral over future-directed (past-directed) causal boundary components comes with a positive (negative) sign and the integral over spacelike boundary components has the usual sign convention.