Weak turbulence on Schwarzschild-AdS spacetime

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Nonlinear aspects of general relativity

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Introduction: Anti-de Sitter spacetime

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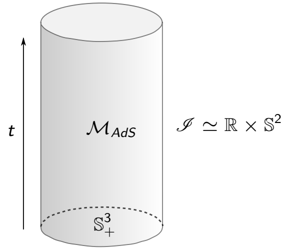
Simplest solution of the **vacuum** Einstein equations with **negative** cosmological constant $\Lambda = -3$: *Anti-de Sitter* spacetime $(\mathbb{R}^{3+1}, g_{AdS})$,

$$g_{AdS} = -\left(1 + r^2\right) dt^2 + \left(1 + r^2\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2}.$$

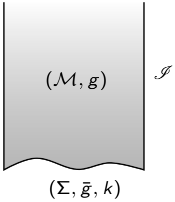
It can be conformally identified with the cylinder $\left(\mathbb{R} \times \mathbb{S}^3_+, -dt^2 + g_{\mathbb{S}^3}\right)$.



- Conformal boundary $\mathscr{I}$ at $r = +\infty$: Of timelike character.
- The appropriate framework to study asymptotically AdS solutions to the Einstein equations: **Initial-boundary value problem**.
- Confinement: By assuming **reflecting** conditions on $\mathscr{I}$.

The initial-boundary value problem

The **initial-boundary value problem** for the *vacuum* Einstein equations in the asymptoticaly AdS setting:



- Initial data $(\Sigma^3, \bar{g}, k)$ satisfying the *constraint equations*

$$R[\bar{g}] + (\mathrm{tr} k)^2 - |k|^2 = -6,$$
$$\mathrm{div}(k - \mathrm{tr} k \cdot \bar{g}) = 0.$$

- Conformal boundary conditions on $\mathscr{I}$, plus compatibility conditions at the "corner" $\partial\Sigma = \mathscr{I} \cap \Sigma$.

Existence and geometric **uniqueness** of solutions to the vacuum IBVP: FRIEDRICH (1995).

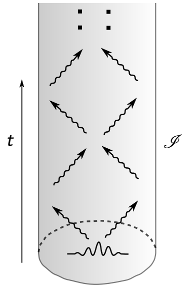
- Unique *reflecting* boundary condition in this class: $g|_{\mathscr{I}} \sim g_{\mathbb{R}\times\mathbb{S}^2}$.
    - Geometric uniqueness for the IBVP in the case of a regular boundary (i.e. not at conformal infinity) is trickier: FOURNODAVLOS–SMULEVICI

Evolution under confinement

Evolution under confinement

The well-posedness of the initial-boundary value problem allows the study of the long time dynamics of asymptotically AdS solutions.

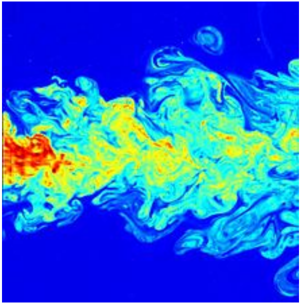
Question: *What are the stability properties of small initial perturbations of AdS under reflecting boundary conditions at $\mathscr{I}$?*



- In the case of the linear toy model $\Box_g \phi - \mu\phi = 0$: The scalar field ϕ does *not* decay as $t \to \infty$ when reflecting conditions are assumed on $\mathscr{I}$.
- Non-linear effects can accumulate over long timescales, possibly precipitating cascading effects.

The phenomenon of turbulence

The phenomenon of turbulence



Turbulence is ubiquitous in confined systems governed by non-linear dynamics:

- Archetype: Incompressible fluids with $Re = \frac{U \cdot L}{\nu} \gg 1$

$$\partial_t u + u \cdot \nabla u - \nu \Delta u = -\nabla p + F.$$

What behaviour qualifies as turbulence?

- Cascade of energy towards increasingly smaller scales in some irreversible way.
- Emergence of self-similar patterns.

Weak turbulence: Arising from small initial data.

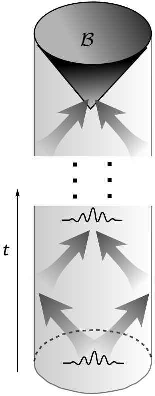
- Higher order norm inflation for cubic NLS on $\mathbb{T}^2$: Colliander–Keel–Staffilani–Takaoka–Tao
- Cubic Szegö equation on $\mathbb{S}^1$: Gerard–Grellier

Question. *Does turbulence appear in the equations of general relativity in the asymptotically AdS setting?*

The AdS instability conjecture

The AdS instability conjecture

In 2006, DAFERMOS–HOLZEGEL conjectured the following scenario:



AdS instability conjecture

Assuming a **reflecting** boundary condition on $\mathscr{I}$ for the vacuum equations, there exist arbitrarily small perturbations of the AdS initial data which lead to the formation of a **black hole** region after sufficiently long time. In particular, $(\mathcal{M}_{AdS}, g_{AdS})$ is *non-linearly unstable*.

- Black hole formation: Concentration of energy at small scales:
  - Hoop conjecture: $\frac{\text{Energy}}{\text{Diameter}} \gtrsim 1$ for a trapped surface.
  - LUK–M. '22: Growth of a scale-invariant norm is necessary for trapped surface formation.

- In the case of **maximally dissipative** boundary conditions: Linearized perturbations decay at a superpolynomial rate: HOLZEGEL–LUK–SMULEVICI–WARNICK.

AdS Instability: The spherically symmetric case

AdS Instability: The spherically symmetric case

We can first try to understand the AdS instability conjecture in the spherically symmetric setting of the **Einstein–scalar field system:**

$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - 3 g_{\mu\nu} = 8\pi T_{\mu\nu}[\phi], \\ \Box_g \phi - 2\alpha\phi = 0. \end{cases}$$

Asymptotic expansion for ϕ near $\mathscr{I}$:

$$\phi = r^{-\lambda_-} \underbrace{\phi^-_{\mathscr{I}}}_{\text{Dirichlet data}} + r^{-\lambda_+} \underbrace{\phi^+_{\mathscr{I}}}_{\text{Neumann data}} + \mathcal{O}(r^{-2-\lambda_-}), \quad \lambda_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} + 2\alpha}.$$

- **Conformally coupled** case: $\alpha = -1$.

Well-posedness in *spherical symmetry*:

- FRIEDRICH: Homogeneous Dirichlet boundary conditions in the conformally coupled case.
- HOLZEGEL–SMULEVICI: Homogeneous Dirichlet bc's when $\alpha > -\frac{9}{8}$.
- HOLZEGEL–WARNICK: More general bc's (including Neumann) for $\alpha \in (-\frac{9}{8}, -\frac{5}{8})$.

AdS Instability: The spherically symmetric case

AdS Instability: The spherically symmetric case

First numerical and heuristic study of the instability of AdS in the setting of the spherically symmetric Einstein–scalar field system with **Dirichlet** conditions at $\mathscr{I}$: BIZON–ROSTWOROWSKI (2011).

- Proposed instability mechanism: Perturbative analysis of the effective scalar field equation

$$\Box_{(AdS)}\phi \simeq \frac{2m[\phi]}{r^3}\phi = \mathcal{N}[\phi, \phi, \phi]$$

suggests that energy is transferred to high frequency modes of

$$\phi(t, x) = \sum a_k(t) e^{i\omega_k t} E_k(x)$$

through **resonant** interactions (when $\omega_k = \omega_l - \omega_m + \omega_n$):

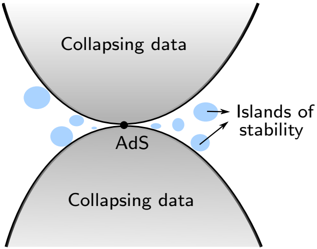
$$\ddot{a}_k + 2i\omega_k \dot{a}_k = \sum_{m,n,l} e^{i(\omega_l - \omega_m + \omega_n - \omega_k)t} \mathbb{P}_{k;l,m,n} \mathcal{N}[a_l, \bar{a}_m, a_n]$$

AdS Instability: Further questions

AdS Instability: Further questions

Subsequent numerical and heuristic works:
Buchel–Lehner–Liebling, Dias–Horowitz–Marolf–Santos, Balasubramanian–Buchel–Green–Lehner–Liebling, Bizon–Maliborski, Craps–Evnin–Vanhoof, Dimitrakopoulos–Freivogel–Lippert–Yang...

Questions explored:



- Effects of changing the Klein-Gordon mass and the bc's? What happens when the spectrum is no longer resonant?
- Do *all* perturbations of AdS spacetime collapse into black holes? Are there "islands of stability", i.e. sets "of positive measure" in the moduli space of initial data close to AdS giving rise to non-collapsing solutions?
  - Chatzikaleas–Smulevici: Construction of quasiperiodic perturbations of the trivial initial data for semilinear toy models; stable at exponential timescales.

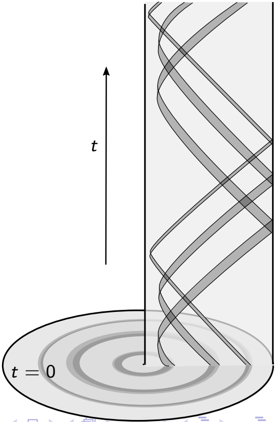
AdS Instability: An alternative approach

AdS Instability: An alternative approach

The resonant mode mixing mechanism: relevant for the first stage of the instability, where perturbation theory is still valid.

An alternative approach for a rigorous proof of the conjecture: Interaction of wave packets in *physical space* & *monotonicity* properties of the equations.

- **Einstein–massless Vlasov system** for small initial perturbations with respect to a carefully designed scale invariant norm (M., 2018).
- *Conformally coupled* **Einstein–scalar field system** with either Dirichlet or Neumann boundary conditions for small initial perturbations with respect to Christodoulou's BV norm (M., *forthcoming*).
  - The short pulse method introduces an upper bound on the regularity of the initial data norm.



Evolution beyond black hole formation

Evolution beyond black hole formation

Once a black hole is formed, what are its long time dynamics? Does the dispersion of energy through the black hole horizon make the exterior asymptotically stationary?

Related to the stability problem for AdS black holes!

- In the spherically symmetric setting: Spacetime settles down to the Schwarzschild-AdS exterior metric

$$g_M = -\left(1 - \frac{2M}{r} + r^2\right)dt^2 + \left(1 - \frac{2M}{r} + r^2\right)^{-1} dr^2 + r^2 g_{\mathbb{S}^2},$$

  Non-linear stability of Schwarzschild–AdS spacetime in spherical symmetry: Holzegel–Smulevici.

  - Exponentially fast decay of perturbations in this case.

- Outside spherical symmetry the situation becomes more interesting.

Linear waves on Schwarzschild–AdS

Linear waves on Schwarzschild–AdS

The conformally coupled wave equation

$$\Box_{g_M}\phi + 2\phi = 0$$

on Schwarzschild–AdS is separable, i.e. admits solutions of the form

$$\phi(t,r,\theta,\varphi) = \frac{e^{-i\omega t}}{r} R_{\omega,\ell}(r) P_{\ell,m}(\theta,\varphi).$$

Radial ODE:

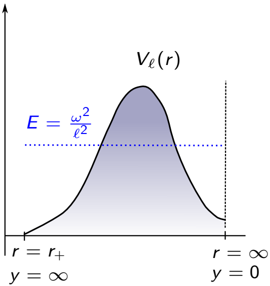
$$R''(y) + \Big(\omega^2 - \ell(\ell+1)V_\ell(y)\Big)R(y) = 0,$$

where

$$y(r) = \int_r^\infty \left(1 - \frac{2M}{\bar r} + \bar r^2\right)^{-1} d\bar r \sim \frac{1}{r}$$

and

- $V_\ell(r) = 1 + \frac{1}{r^2} - \frac{2M}{r^3} + O\left(\frac{M}{\ell^2 r}\right)$
- BC's: $R|_{y=0} = 0$, ϕ smooth at $r = r_+$.

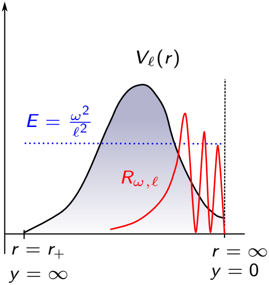


Linear waves on Schwarzschild–AdS

Linear waves on Schwarzschild–AdS

The existence of trapped null geodesics near $y = 0$ leads to slow decay:

- Construction of $O(e^{-c\ell})$ real quasimodes: Dirichlet conditions at $r = r_0, +\infty$ for some r_0 in the forbidden region.
    - Discrete set of eigenvalues $\omega = \omega(n, \ell)$.
- Quasinormal modes wt. $Im(\omega) = O(e^{-c\ell})$: Gannot ('12).



- Holzegel–Smulevici ('11–'13) *Sharp* decay rate

$$|\phi|^2 \lesssim \frac{1}{(\log t)^{2k}} \mathcal{E}^{(k)}[\phi](0).$$

The stability question for Schwarzschild–AdS

The stability question for Schwarzschild–AdS

What does the existence of long-lived quasinormal modes concentrated along stably trapped null geodesics tell us about non-linear perturbations?

- Argument **against** stability: Exponential lifespan of linear perturbations might lead to non-linear instability.
- Argument **for** stability (Dias–Horowitz–Marolf–Santos): The frequencies of the **lowest lying** (i.e. $n = 1$) quasinormal modes are **not** resonant:

$$Re(\omega_j)(1, \ell) - \omega_j^{(AdS)}(1, \ell) \sim \ell^{-\frac{1}{2}}.$$

  - In the absence of resonant interactions, a Nekhoroshev-type estimate & logarithmic decay could in principle imply global stability.

Question: Do small non-spherically symmetric perturbations of the Schwarzschild–AdS exterior eventually create new trapped surfaces?

Weak turbulence on Schwarzschild–AdS

Weak turbulence on Schwarzschild–AdS

Toy model: Let us consider the quasilinear scalar wave equation

$$\Box_{g_M}\phi + 2\phi = \phi|\partial_t\phi|^2 + |\phi|^2\partial_t^2\phi \qquad (1)$$

on the exterior of Schwarzschild–AdS with mass parameter $M > 0$.

- Similar non-linear structure to the Einstein equations in wave coordinates:

$$\Box_g g = g^{-1}\cdot\partial g\cdot\partial g$$

- Conserved H^1 energy through the foliation $\{t = const\}$:

$$\mathcal{E}^{(\partial_t)}[\phi](\tau) + \int_{t=\tau} |r\phi|^2|\partial_t\phi|^2 \,\mathrm{dvol}_{\Sigma_\tau}$$
$$\sim \int_{t=\tau}\Big((1+|r\phi|^2)|\partial_t(r\phi)|^2 + |\partial_y(r\phi)|^2 + |\nabla_{\mathbb{S}^2}(r\phi)|^2\Big)dy d\theta_{\mathbb{S}^2}.$$

- Like the Einstein vacumm equations, (1) is $H^{\frac{3}{2}}$ critical (and, hence, supercritical with respect to $\mathcal{E}^{(\partial_t)}[\phi](\cdot)$).

Weak turbulence on Schwarzschild–AdS

Weak turbulence on Schwarzschild–AdS

Theorem (Kehle–M., forthcoming)

For any $\epsilon \ll 1$ and any $s \geq s_0 > 1$, there exists an **open and dense** set of Schwarzschild-AdS mass parameters M and solutions ϕ to

$$\Box_{g_M}\phi + 2\phi = \phi|\partial_t\phi|^2 + |\phi|^2\partial_t^2\phi$$

with $r\phi|_{r=\infty} = 0$ such that

$$\|(\phi,\dot{\phi})|_{t=0}\|_{H^s\times H^{s-1}} < \epsilon \quad \text{and} \quad \|(\phi,\dot{\phi})|_{t=T}\|_{H^s\times H^{s-1}} > \frac{1}{\epsilon}$$

for some $T \gg 1$.

Remarks:

- The scalar field ϕ has to be *complex valued*.
- The proof also applies in the case of other types of nonlinearities, e.g. satisfying the null condition or carrying degenerating $\frac{1}{r^N}$ weights.
- Surprisingly, the proof does not work for $\Box_g\phi + 2\phi = |\phi|^{2N}\phi$.

Sketch of the proof: The setup

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Let us split ϕ as

$$\phi = \phi_0 + \phi_- + \phi_+ + \sum_{\substack{k=(n,\ell):\, n \lesssim \ell^{\delta_0} \\ k \notin \{0,+,-\}}} \phi_k + \psi,$$

where the functions ϕ_0, $\phi_\pm$ and ϕ_k are formed in terms of **normal** modes for the linear wave equation

$$\phi_k(t,x) = a_k(t) \cdot \frac{e^{-i\omega_{n,\ell} t}}{r} R_{n,\ell}(r) P_{\ell,m}(\theta,\varphi)$$

with a mirror at $r = r_0 > r_{\mathcal{H}^+}$ and satisfy

$$\Box_{g_M} \phi_k + 2\phi_k = \sum_{k_1,k_2,k_3 \in \{0,\pm\}} \mathbb{P}_k \Big(\phi_{k_1} \partial_t \bar{\phi}_{k_2} \partial_t \phi_{k_3} + \phi_{k_1} \bar{\phi}_{k_2} \partial_t^2 \phi_{k_3} \Big)$$

or, equivalently, the a_k's satisfy the 2^{nd}-order ODE system:

$$\ddot{a}_k - 2i\omega_k \dot{a}_k = \sum_{k_1,k_2,k_3 \in \{0,\pm\}} e^{i(\omega_{k_1} - \omega_{k_2} + \omega_{k_3} - \omega_k)t}\, \mathbb{P}_{k;k_1,k_2,k_3} \cdot \big(\omega_{k_2}\omega_{k_3} - \omega_{k_3}^2\big)\, a_{k_1} \bar{a}_{k_2} a_{k_3}.$$

Sketch of the proof: The setup

Sketch of the proof: The setup

Our aim is to show that, by choosing the initial data for ϕ and the mass parameter M appropriately:

- The frequencies of ϕ_0, ϕ_+ and ϕ_- lie on an arithmetic progression and are "resonantly separated" from the rest of the frequencies of the ϕ_k's.
- The ϕ_k's remain small for $t \in [0, T]$.
- The system of ODEs for the "dominant" terms a_0 and $a_\pm$ and $a_k(t)$ exhibits tranfer of energy from small to large frequencies in the time interval $t \in [0, T]$.
- The error term ψ remains under control for $t \in [0, T]$.

Sketch of the proof: Spectral analysis

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Our first step is to analyze the spectrum of the *linear* wave operator on Schwarzschild-AdS.

When $M = 0$, the AdS frequencies for $n \ll \ell$ can be explicitly calculated:

$$\omega_{AdS}(n, l) = \pm(2n + \ell) + O(e^{-c\ell}).$$

Key ingredient: Perturbative formula when $M > 0$ for $n \lesssim \log(\ell)$, $\ell \gg 1$:

$$\omega(n, \ell) = \omega_{AdS}(n, \ell) + \frac{f_1(n)}{\ell^{\frac{1}{2}}} M + \frac{f_2(n)}{\ell} M^2 + O_n(\ell^{-\frac{3}{2}}).$$

Sketch of the proof: Spectral analysis

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Lemma

Let $\xi \gg 1$ be a frequency scale. There exists an open and dense set of mass parameters $M > 0$ and modes ϕ_0, ϕ_-, ϕ_+ with frequencies satisfying the following properties:

- $|\omega_0|, |\omega_\pm| \sim \xi, \quad \frac{|\omega_\pm|}{\omega_0} \gg 1.$
- $|m_0| = \ell l_0$, $|m_\pm| = \ell l_\pm$ and $n_0, n_\pm \ll \log \xi$.
- Resonant conditions:

$$\begin{cases} \omega_- - 2\omega_0 + \omega_+ = O(\xi^{-\infty}), \\ m_- - 2m_0 + m_+ = 0. \end{cases}$$

Moreover, every other mode ϕ_k which would be $O(\xi^{-\infty})$-resonant with ϕ_0, $\phi_\pm$ when $M = 0$ satisfies

$$\min_{k_1, k_2, k_3 \in \{0, \pm\}} \left| \omega_k - \left(\omega_{k_1} - \omega_{k_2} + \omega_{k_3}\right) \right| > \frac{1}{\xi^2}.$$

Sketch of the proof: The subdominant modes

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Using the separation properties of the spectrum and standard stationary phase analysis: We can show that the ϕ_k's, $k \notin \{0, \pm\}$, remain small relative to ϕ_0, ϕ_{pm}.

- When $\log(\ell) \lesssim n \lesssim \ell^{\delta_0}$: The analysis using the perturbative expansion of the spectrum fails, but we make use of weaker monotonicity properties of $\omega(n, \ell)$.

Sketch of the proof: The dominant modes

Sketch of the proof: The dominant modes

Let us focus on the 3×3 ODE system satisfied by the complex amplitudes a_0, $a_\pm$ of the dominant modes.

- Choosing the initial data for a_0, $a_\pm$ appropriately so that the 1^{st} order slowly-oscillating approximation holds:

$$\dot{X} \simeq \mathcal{F}(X), \quad X = \begin{bmatrix} a_0 \\ a_- \\ a_+ \end{bmatrix}.$$

- At $t = 0$: $\frac{|a_\pm|}{|a_0|} \ll 1$.

Our aim is to show that $\limsup_{t \to T} |a_\pm|(t) \gg |a_{pm}|(0)$ for $T \sim \xi^{2s+c_0}$.

- The system for $a_0, a_\pm$ satisfies a conservation law involving all three of $a_0, a_\pm$ (corresponding to the conservation of the ∂_t-energy of the original system) and a **hidden** conservation law involving only $a_\pm$.
- The latter conservation law is non-definite in our case, but positive definite in the case of the "simpler" algebraic non-linear wave equation

$$\Box_g \phi + 2\phi = |\phi|^{2N} \phi.$$

Sketch of the proof: The error terms

Sketch of the proof: The error terms

Going back to the expression

$$\phi = \phi_0 + \phi_- + \phi_+ + \sum_{\substack{k=(n,\ell):\, n \lesssim \ell^{\delta_0} \\ k \notin \{0,+,-\}}} \phi_k + \psi,$$

it remains to show that $\|\psi\|_{L^\infty[0,T]H^s} \ll |a_\pm|\xi^s$.

The error term satisfies a quasilinear equation of the form

$$\Box_{g_M}\psi + 2\psi + \mathcal{N}(\tilde{\phi};\psi) = \mathcal{Q}(\tilde{\phi}),$$

where

$$\tilde{\phi} \doteq \phi_0 + \phi_- + \phi_+ + \sum_{\substack{k=(n,\ell):\, n \lesssim \ell^{\delta_0} \\ k \notin \{0,+,-\}}} \phi_k.$$

- The time dependent coefficients in the linearized part can in principle lead to growth: The ODE $y'' + \left(\xi^2 + \|\partial\tilde{\phi}\|_\infty \sin(\xi t)\right)y = 0$ can exhibit exponential growth within the timescale $t \in [0, T]$.

Sketch of the proof: The error terms

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The vector field $V = \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi}$ satisfies the following conditions:

- It commutes well with $\tilde{\phi}$ ($\|V\tilde{\phi}\| \sim \|\tilde{\phi}\|$ compared to $\|\partial_t \tilde{\phi}\| \sim \xi \|\tilde{\phi}\|$).
- It is causal near $r = \infty$.

We can perform enery estimates using V as a commutator and ∂_t as a multiplier.

- Logarithmic losses: The resulting energy norm satisfies

$$\mathcal{E}[\psi](\tau) \lesssim (\log \xi)^{\frac{1}{2}} \int_0^{\tau} K(\tilde{\phi})(t) \cdot \mathcal{E}[\psi](t)\, dt + O(\xi^{-N})$$

with $\int_0^T K(\tilde{\phi})(t)\, dt \lesssim 1$.

Open questions

Open questions

- What is the structure of the non-linearity responsible for the instability?
- Can we also infer that $\|\phi\|_{H^{s'}}$ grows for $s' < s$ in our model?
    - Trapped surface formation would correspond to the growth of $\|\phi\|_{H^{\frac{3}{2}}}$.
- Can we find initial data in H^s for which the solution ϕ either blows up in finite time or $\limsup_{t \to +\infty} \|\phi(t)\|_{H^s} = +\infty$?
- What about the vacuum Einstein equations?

Thank you for your attention!