2023-10-12 @PCTS: Gluing small black holes along timelike geodesics.

① Motivation & Context: rigorous construction of "many-black-hole" spacetimes (M,g): $\mathrm{Ric}(g)=0$ (or $\mathrm{Ric}(g)-\frac{1}{2}R_g\, g = T$).

(a) Explicit spacetime metrics:

- Majumdar–Papapetrou (1945/47) — solutions of Einstein–Maxwell

$$\begin{cases} g = -\Omega^{-2}\,dt^2 + \Omega^2\,dx^2 \\ A = \Omega^{-1}\,dt, \end{cases} \qquad (x=(x_1,x_2,x_3))$$

$$\Delta\Omega = 0 \rightsquigarrow \Omega = 1 + \sum_{i=1}^{N} \frac{m_i}{|x-x_i|}.$$

[$x=x_i$ is a horizon with area $4\pi m_i^2$;
$N=1 \Rightarrow$ extremal Reissner–Nordström;
$N\geq 2 \Rightarrow N$ extremally charged RN BHs.]

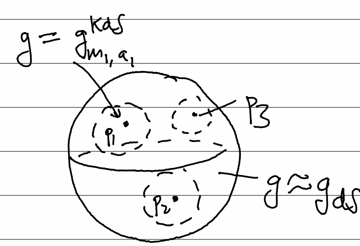
- Kastor–Traschen (1993): $\Lambda>0$ version. [All very rigid.]

Initial data of g [& of matter fields, if present] at spacelike hypersurface $X\subset M$: $(\gamma,k) =$ (1st 2nd fundamental form); constraint equations.

(b) Explicit initial data:

- Misner (1963): $k=0$, $\gamma=\Omega^4\,dx^2$, $\Delta\Omega=0$, $\Omega \xrightarrow{|x|\to\infty} 1$; quotient by group action
- Brill–Lindquist (1963): Einstein–Maxwell version

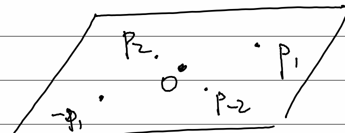
More flexible method: gluing.



(a') • many-black-hole de Sitter spacetimes (H. '21),
$(\mathbb{R}_t \times \mathbb{S}^3,\ g_{dS} = -dt^2 + (\cosh t)^2 g_{\mathbb{S}^3})$

(b') • initial data constructions based on seminal work by Corvino '00.
(Isenberg–Mazzeo–Pollack '02, Chruściel–Delay '03, Carlotto–Schoen '16, Aretakis–Czimek–Rodnianski '21, Anderson–Corvino–Pasqualotto '23)

- Chruściel–Delay '03:
  - small black holes into $\mathbb{R}^3$
  - parity condition



- H. '22: gluing rescaled AF initial data $(\hat\gamma, \hat k)$ into the neighborhood of a point $p \in X$ in an initial data set (X, γ, k) (generic or subx $k(dS)$).

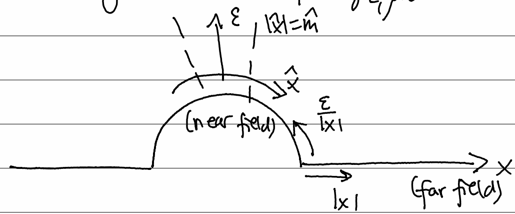
Special case $\hat k = k = 0$, $\hat\gamma = -(1-\frac{2m}{\hat r})^{-1} d\hat r^2 + \hat r^2 g_{S^2}$ (Schwartschild mass $\hat m$), geodesic normal coordinates $x \in \mathbb{R}^3$ around p:

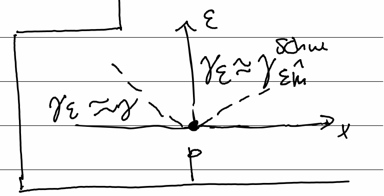
Thm. $\exists \gamma_\varepsilon, \varepsilon > 0$: (i) γ_ε satisfies the constraint equations on $X \setminus \{|x| < \varepsilon \hat m\}$

(ii) $\gamma_\varepsilon \to \gamma$ in $C^\infty(X \setminus \{p\})$ (and $\gamma_\varepsilon = \gamma$ dist $\gtrsim 1$ from p).

(iii) $\gamma_{\varepsilon,\mu\nu}(\varepsilon \hat x) \longrightarrow \hat\gamma_{\mu\nu}(\hat x)$ for bounded $\hat x$ $\left(\gamma_\varepsilon(x) \simeq -(1-\frac{2\varepsilon\hat m}{|x|}) \ldots\right) = \gamma^{\mathrm{Schw.}}_{\varepsilon\hat m}$

(iv) L^∞_g-smoothness of $\gamma_{\varepsilon,\mu\nu}$ in $|x|, \frac{x}{|x|}, \frac{\varepsilon}{|x|}$





(c) Spacetime evolution of initial data sets from gluing?

- Chruściel–Mazzeo '03: evolution of CD data has disconnected black hole region

[· otherwise: only the obvious short-time existence. Can lead to dramatic results — e.g. disproof of 3rd law of BH thermodynamics, Kehle–Unger '23.]

2. Gluing directly on the spacetime level.

[Spacetime gluing appears more difficult than I.D. gluing?! But have direct control of metric in various asymptotic regimes! And can free ourselves from taking the IVP too seriously all the time — it serves some purposes, but is not always the best perspective!]

Theorem (H. '23) (Formal spacetime gluing.)

Let · (M, g) be globally hyperbolic, $\mathrm{Ric}(g) = 0$ (or w/ $\Lambda \in \mathbb{R}$).

· C = maximally extended timelike geodesic

[· $X \subset M$ Cauchy surface]

[· g has no nonzero Killing fields near $C \cap X$]

- $\hat{m} > 0$, $|\hat{a}| < \hat{m}$.

Let (t,x) = Fermi normal coordinates around C.

Then: $\exists\, g_\varepsilon$ on $M \setminus \{|x| < \varepsilon \hat{m}\}$ s.t.:

(i) $\mathrm{Ric}_{\mu\nu}(g_\varepsilon) = O(\varepsilon^\infty)$ (with all derivatives) [and $=0$ to all orders at X $\Rightarrow$ constraint eqn's satisfied.]

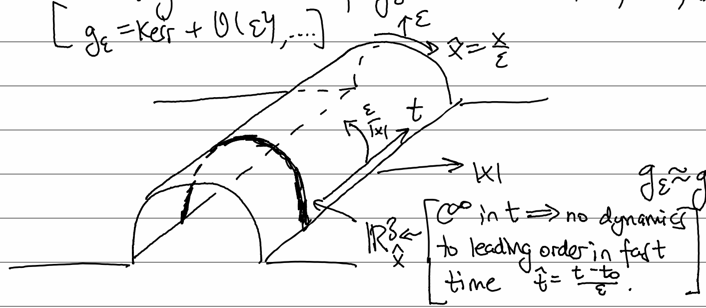
(ii) $g_\varepsilon \to g$ in $C^\infty(M \setminus C)$

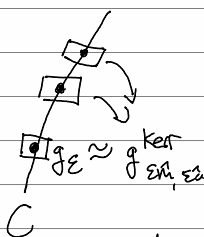
(iii) $(g_\varepsilon)_{\mu\nu}(t, \varepsilon\hat{x}) \longrightarrow (g^{\mathrm{Kerr}}_{\hat{m},\hat{a}})_{\mu\nu}(\hat{x})$ for bounded $\hat{x}, t$

$\left(g_\varepsilon(t,x) \approx g^{\mathrm{Kerr}}_{\varepsilon\hat{m},\varepsilon\hat{a}}(x/\varepsilon) \text{ for } |x| \lesssim \varepsilon \right)$

(iv) log-smoothness of g_ε in t, $|x|$, $\frac{x}{|x|}$, $\frac{\varepsilon}{|x|}$.

[$g_\varepsilon = \mathrm{Kerr} + O(\varepsilon^4, \dots)$]

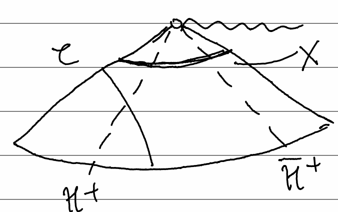




Remarks (1) Gralla–Wald '08: base rigorous derivation of 1st order gravitational self-force on family of metrics satisfying (iv). [We justify their setup in vacuum to all orders in perturbation theory.]

(2) matched asymptotic expansions (Burke '71; D'Eath, Kates, Thorne–Hartle, ...) $\Rightarrow$ $\hat{m}$ const, $\hat{a}$ parallel along C.

(3) (M,g) = subextremal Kerr–de Sitter, C crossing $\mathcal{H}^+$ $\Rightarrow$ BH merger



X: initial data of g_ε are $\approx$ KdS data, $O(\varepsilon^\infty)$ violation of constraints.

H–Vasy '18 applies, producing

$\tilde{g}_\varepsilon = g^{\mathrm{KdS}}_{m(\varepsilon), a(\varepsilon)} + O(e^{-\alpha t})$,

$\mathrm{Ric}(\tilde{g}_\varepsilon) - \Lambda \tilde{g}_\varepsilon = O(\varepsilon^\infty e^{-\alpha t})$.

(4) Conjecture: $\exists\, h_\varepsilon = O(\varepsilon^\infty)$ s.t. $\mathrm{Ric}(g_\varepsilon + h_\varepsilon) = 0$? [Formal perturbation theory approximates a true solution.]

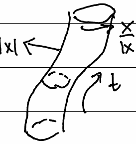
[Can then do analogue of (2) in Kerr using K–S, Giorgi, Chen.]

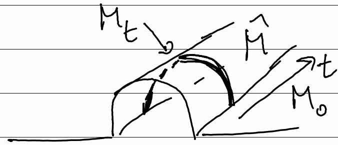
(5) Yang '14: gluing for Einstein–Klein–Gordon, with scalar field potential scaled s.t. KG soliton is very small; metric remains uniformly C^1 as $\varepsilon \searrow 0$, even near C.

③ Comments on the proof.

(i) Work on $\tilde{M} = [[0,1)_\varepsilon \times M; \{0\}\times\mathcal{C}]$
$= \mathrm{m.w.c.}$;

$$\tilde{M}_{\varepsilon_0} = \tilde{M} \cap \{\varepsilon = \varepsilon_0\} = \begin{cases} M, & \varepsilon_0 > 0 \\ M_0 \cup \hat{M}, & \varepsilon_0 = 0 \end{cases}$$

M_0: 



Seek $\tilde{g} = \tilde{g}_{\mu\nu}\, dz^\mu dz^\nu$ $(z = (t,x)$ on $M)$
where $\tilde{g}_{\mu\nu}$ = log-smooth on $\tilde{M}$.
Then $g_\varepsilon = \tilde{g}|_{\tilde{M}_\varepsilon}$.

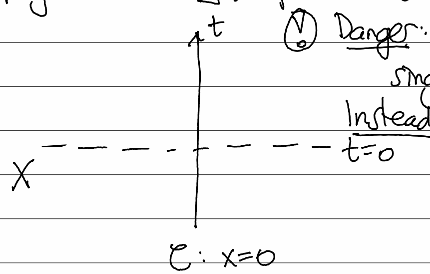
Ansatz: any $\tilde{g}_0$ with $\tilde{g}_0|_{M_0} = g$, $(\tilde{g}_0)_{\mu\nu}|_{\hat{M}_t} = (g^{\mathrm{Kerr}}_{m,\mathfrak{a}})_{\mu\nu}$..

(ii) Construct joint (gen.) Taylor series of $\tilde{g}$ on $\tilde{M}$.
Equations for correction terms?
<u>Want</u> $\mathrm{Ric}(\tilde{g}_0 + h) \approx D_{\tilde{g}_0}\mathrm{Ric}(h) + \mathrm{Ric}(\tilde{g}_0) = 0$,
so: $D_{\tilde{g}_0}\mathrm{Ric}(h) = -\mathrm{Ric}(\tilde{g}_0)$.

<u>@ M_0</u>: $D_g \mathrm{Ric}(h) = (\text{error}) \leftarrow$ log-smooth on M_0: C^∞ on $M\setminus\mathcal{C}$,
$r^\alpha \log^k r$ $(\alpha \geq -1)$ @ $r = |x| = 0$.
$= \mathbb{R}_t \times [0,1)_{|x|} \times \mathbb{S}^2_{x/|x|}$

Seek h = log-smooth on M_0.

Toy model: $\Box h = f = r^{-1}$ on Minkowski.



(!) <u>Danger</u>: solution via IVP leads to singularities on light cone from $X \cap \mathcal{C}$.
<u>Instead</u>: solve in Taylor series @ $r=0$, then solve
$D_g \mathrm{Ric}(h') =$ remaining error
$= \mathcal{O}(r^\infty)$
$= C^\infty$ on M
using IVP. (Need to solve lin. constraints @ $X \rightsquigarrow$ Chruściel–Delay methods as MH '22)

<u>@ $\hat{M} = \mathbb{R}_t \times \overline{\mathbb{R}^3_{\hat{x}}}$</u>:
Local time near $\hat{M}_{t_0}$:
$\hat{t} = \frac{t - t_0}{\varepsilon}$
$\hat{x} = \frac{x}{\varepsilon}$
[rescaling (t,x) around $(t_0, 0)$]

Smoothness in t: $a(t) = a(t_0) + \mathcal{O}(\varepsilon)$: quasistationary / adiabatic behavior of small BH.

$\Rightarrow$ Solve $D_{g^{\mathrm{Kerr}}}\mathrm{Ric}(h(t_0, \hat{x})) = (\text{error})$ for each t_0 separately.

Danger: at fixed t_0, solving $D_{g_{\mathrm{Kerr}}} \mathrm{Ric}(h(\hat{x})) = f(\hat{x})$ using IVP

produces $h = h(\hat{t}, \hat{x})$: fast dynamics, radiation @

$(\hat{t}, \hat{x})$-null infinity $\rightsquigarrow$ oscillations for

$|\hat{t} - \hat{x}| \overset{<}{\sim} 1$, i.e. $|t - x| \lesssim \varepsilon$

$\not\!$ smoothness [as $\varepsilon \searrow 0$ in LC from $(t,x) = (0,0)$]

Instead: lin. Einstein eqn. around Kerr at 0 energy: Häfner–H.–Vasy '21, Andersson–Häfner–Whiting '22

$\Rightarrow$ 7-dim. cokernel; eliminate by modulating

- mass (1 dim)
- angular momentum vector (3 dim.)
- center of mass (3 dim)

at earlier orders in the Taylor expansion.