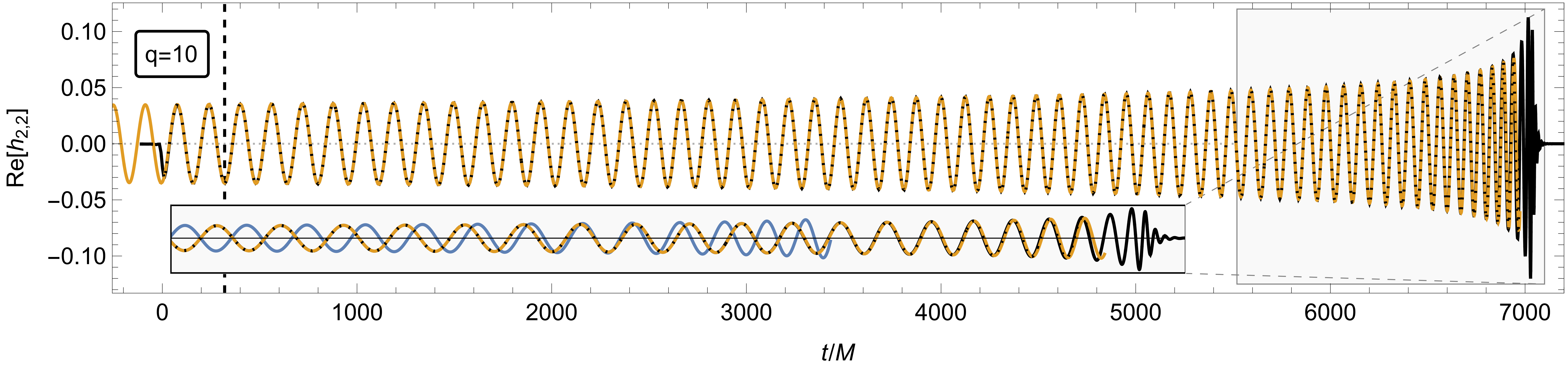


Gravitational waveforms for compact binaries from second-order self-force theory

Barry Wardell
University College Dublin

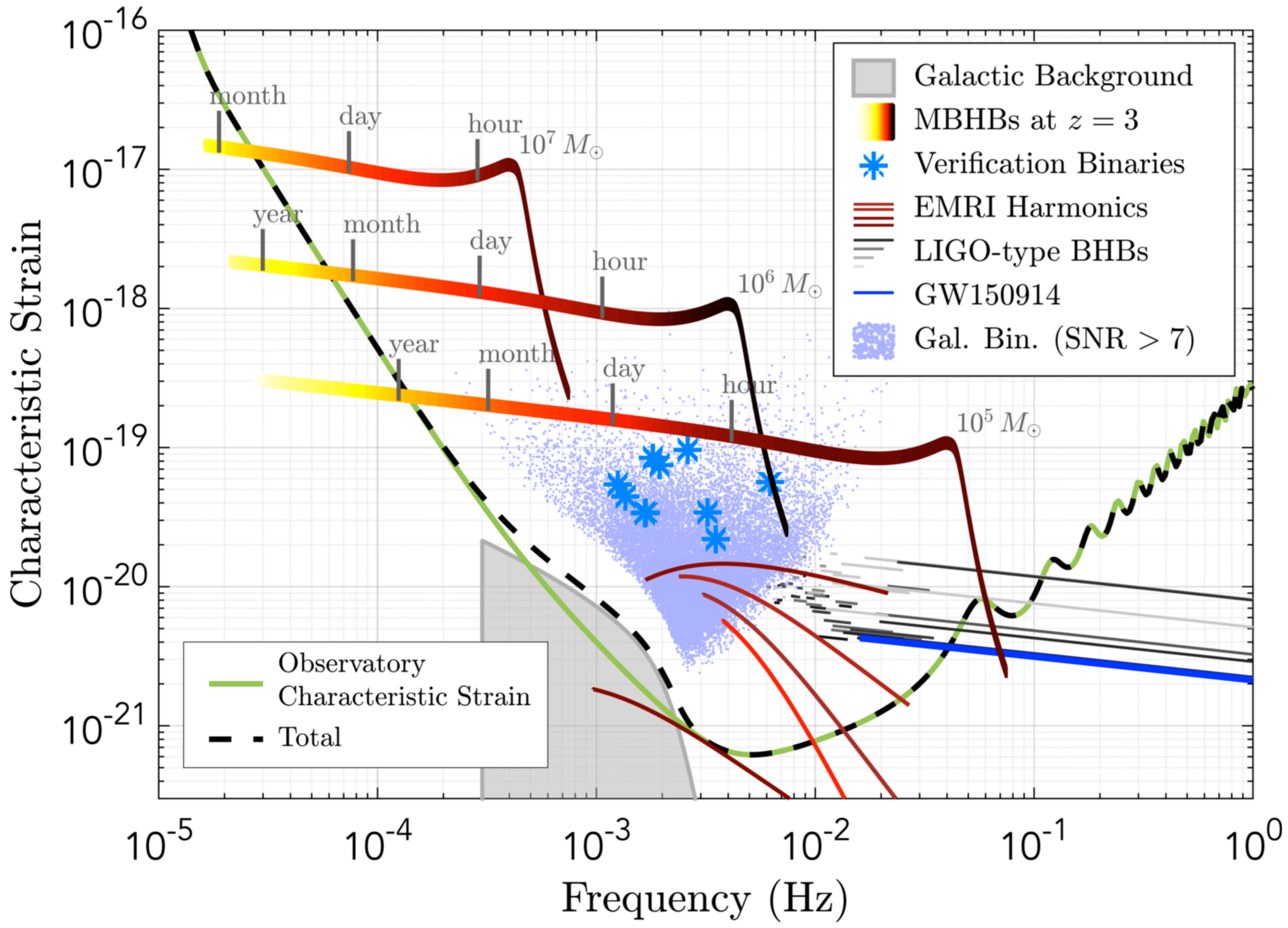
Phys. Rev. Lett. 124, 021101 / Phys. Rev. Lett. 127, 151102 / Phys. Rev. Lett. 130, 241402

Second **O**rder **S**elf-force Collaboration / **A**symmetric **B**inaries **C**ollaboration

Nonlinear Aspects of General Relativity - Princeton University - 12th October 2023

Self-force: Why?

Observing gravitational waves with LISA



Detect and estimate parameters for extreme mass-ratio inspirals (EMRIs) using LISA

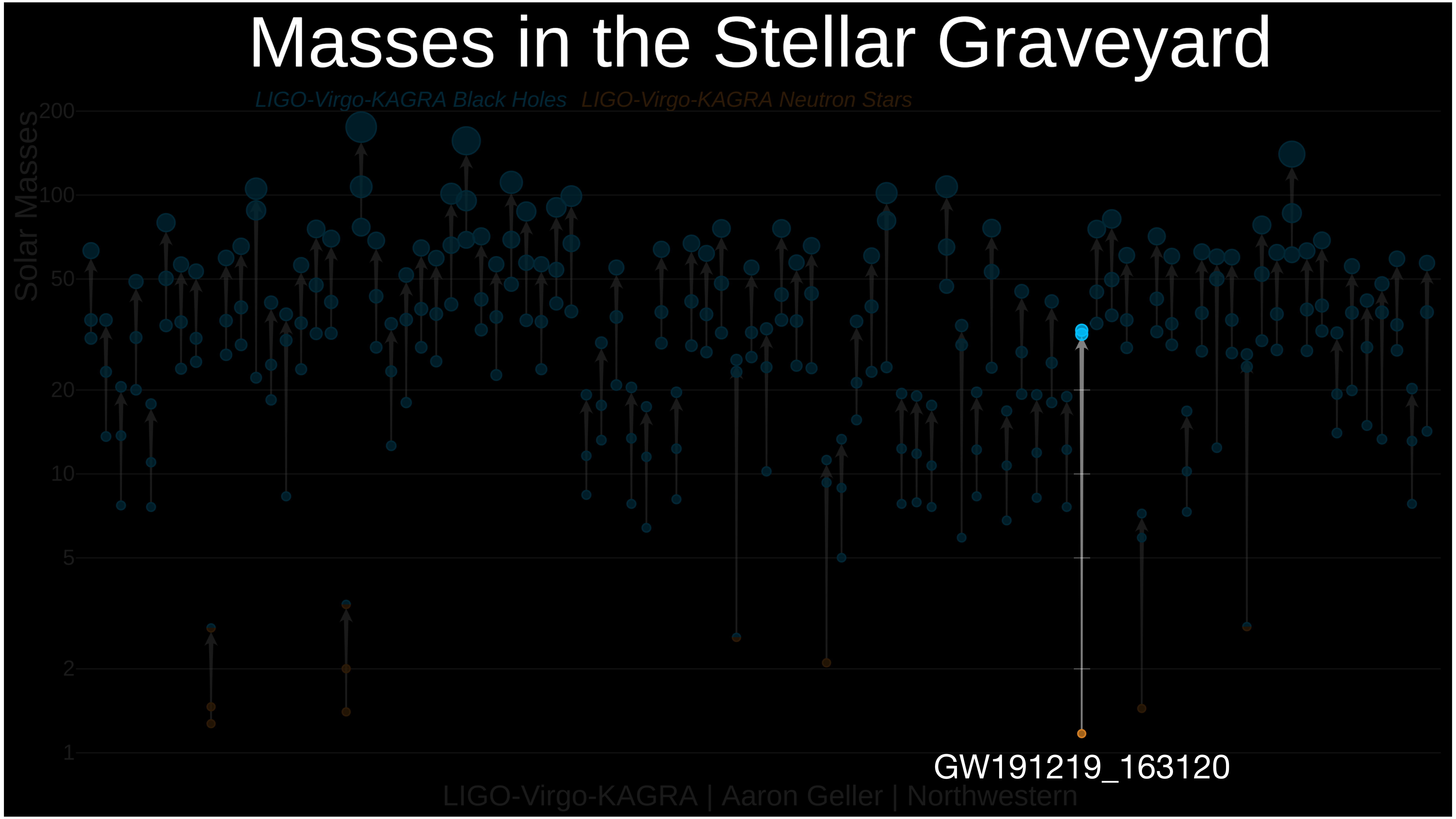
Extreme mass ratio inspirals

LISA
CONSORTIUM

- Binary black hole systems with a **large mass ratio** $q = m_1/m_2 \sim 10^6$.
- Many (>10,000) possibly inclined and eccentric orbits observable by LISA.
- LISA parameter estimation will require $\ll$ 1 radian phase accuracy in the waveform.

"Provide an accurate model of EMRI waveforms for a BH CO, within GR, across the astrophysically relevant parameter space. The model should allow generic MBH and CO spin magnitudes and orientations, generic orbital inclination, and generic eccentricity in the relevant range. It should be phase-accurate to within a fraction of a radian over the entire in-band portion of the inspiral."
[LISA Data Analysis Work Package 1.2]

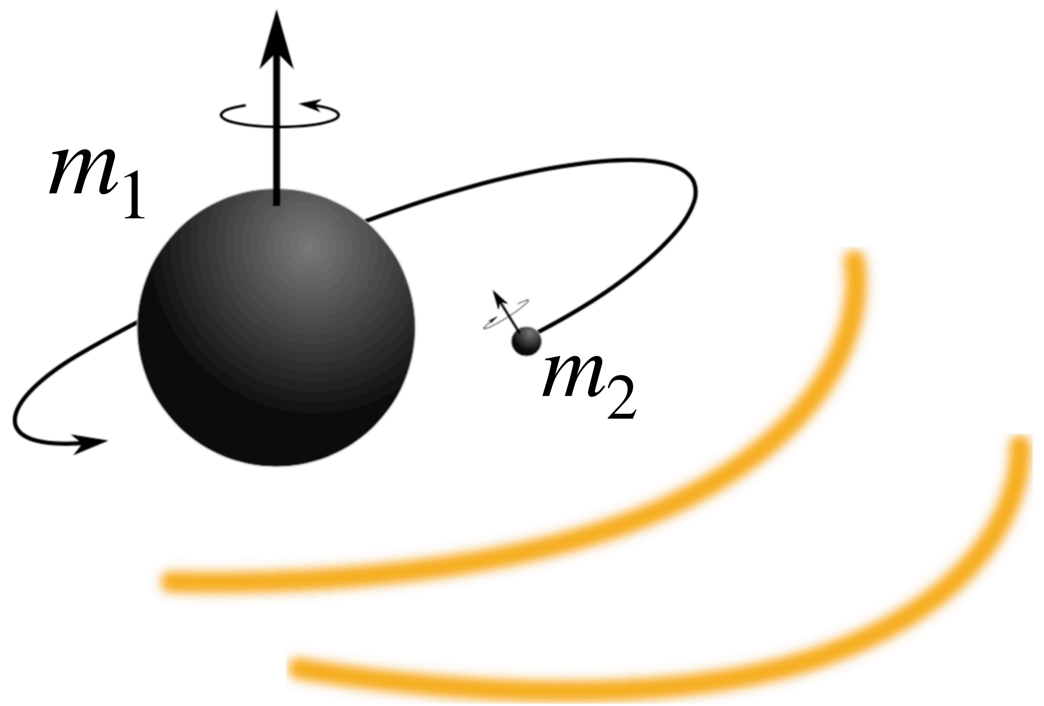
Observing gravitational waves with LIGO



"The mass ratio of GW191219_163120's source is inferred to be $0.038^{+0.005}_{-0.004}$, which is **extremely challenging for waveform modeling**, and thus there may be **systematic uncertainties** in results for this candidate."

Self-force: How?

Gravitational self-force



Expand exact binary spacetime about that of the primary

Schwarzschild/Kerr black hole

$$g^{\text{exact}}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h^{(1)}_{\alpha\beta} + \epsilon^2 h^{(2)}_{\alpha\beta} + \mathcal{O}(\epsilon^3)$$

$$\epsilon = \frac{m_2}{m_1}$$

Substitute expansion into the Einstein equation

$$G_{\mu\nu}[\text{g}] = 8\pi T_{\mu\nu}$$

Expand out in powers of ϵ

$$\epsilon^0 : \qquad G_{\alpha\beta}[g] = 0$$

$$\epsilon^1 : \qquad G^1_{\alpha\beta}[h^{(1)R}] = 8\pi T_{\alpha\beta} - G^1_{\alpha\beta}[h^{(1)S}]$$

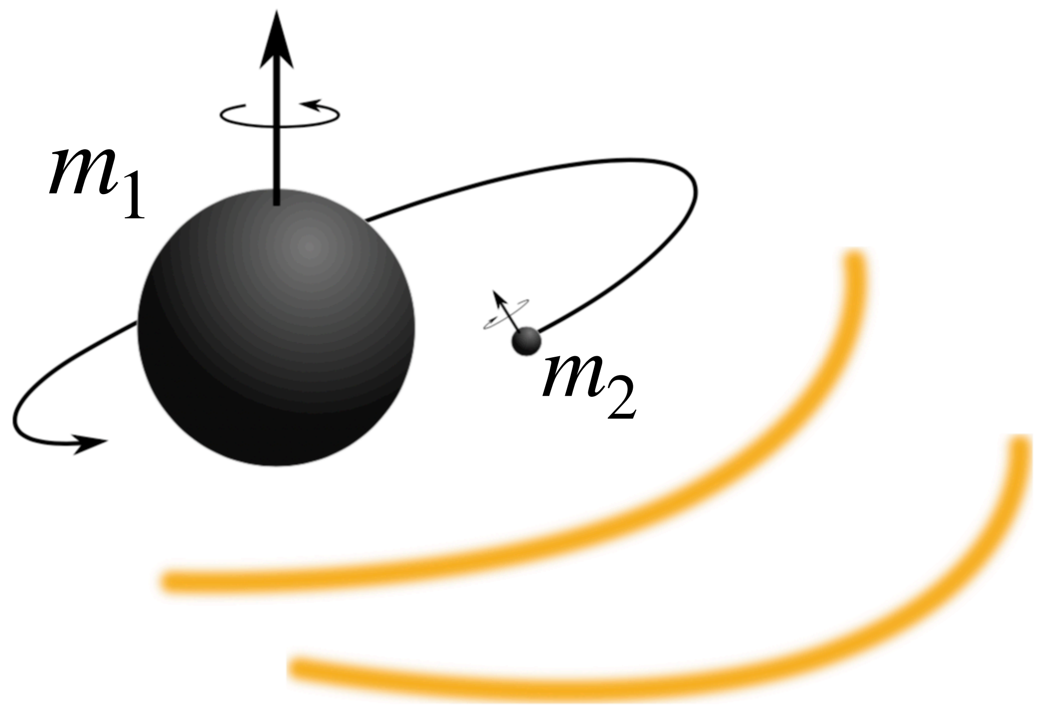
$$\epsilon^2 : \qquad G^1_{\alpha\beta}[h^{(2)R}] = - G^2_{\alpha\beta}[h^{(1)}, h^{(1)}] - G^1_{\alpha\beta}[h^{(2)S}] + \partial_{\tilde{t}} h^{(1)}$$

$$\partial_{\tilde{t}} h^1 = \dot{\Omega} \partial_\Omega h^1$$

This is hard.

Perform two-timescale expansion by introducing a “slow time” $\tilde{t} = \epsilon t$, use a frequency domain decomposition.

Post-adiabatic orbit evolution



The evolution of the orbit is determined by the post-adiabatic/self-force equations of motion:

$$\frac{d\Omega}{dt} = \epsilon\left[F_0^{\Omega}(\Omega) + \epsilon F_1^{\Omega}(\Omega)\right] \qquad \frac{d\phi_p}{dt} = \Omega$$

Adiabatic order

0PA: Adiabatic dissipation-driven rate of change, determined by first-order dissipative gravitational self-force/energy flux (dissipative part of $h^{(1)}$)

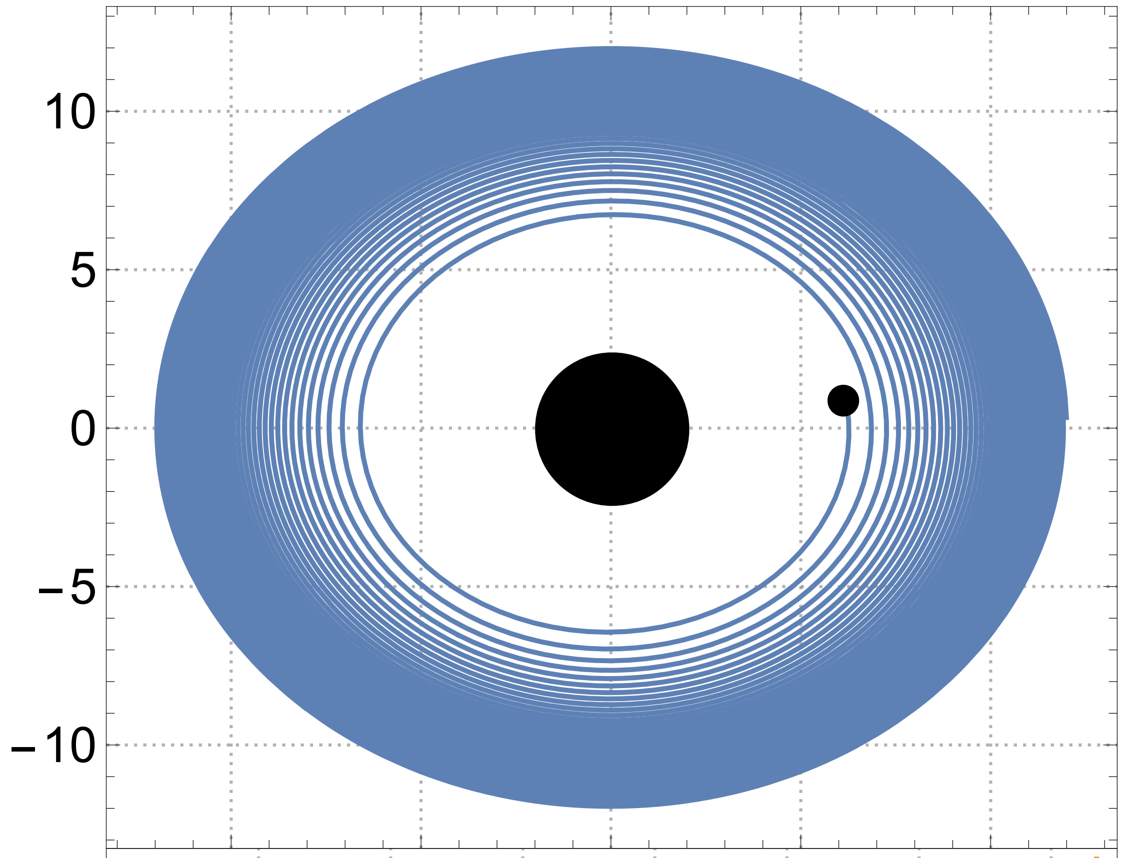
Post-Adiabatic order

1PA: first-post-adiabatic term determined by the full (conservative and dissipative) first-order gravitational self-force (full $h^{(1)}$) and second-order dissipation (dissipative part of $h^{(2)}$).

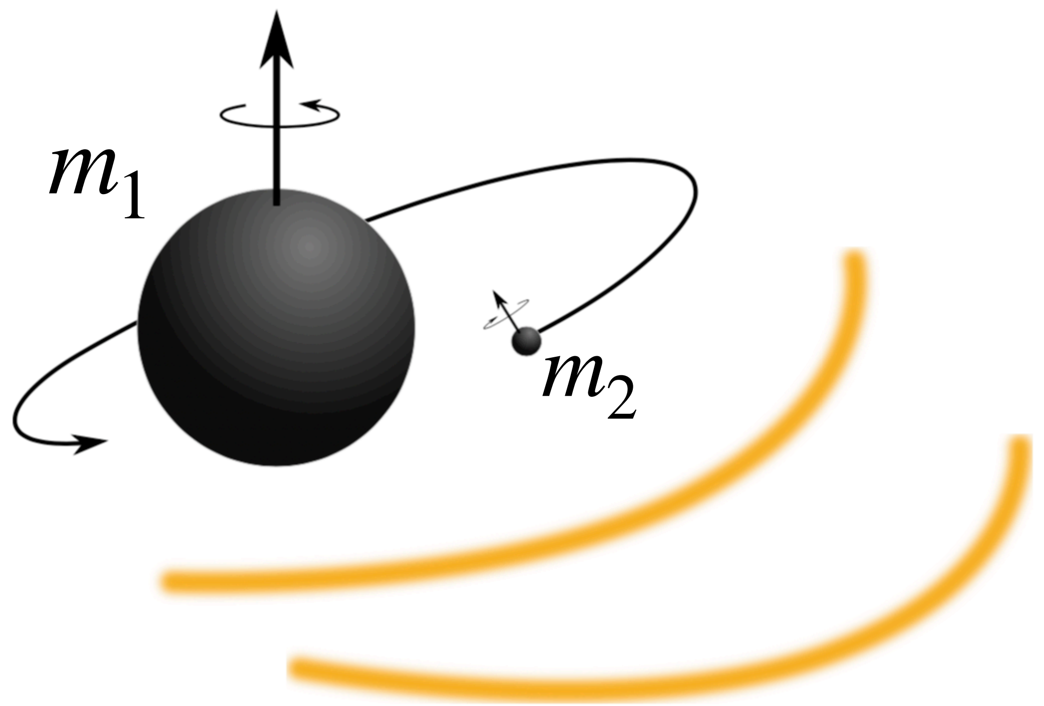
At 1PA order have to account for evolution of the mass and spin of the primary:

$$\frac{d\delta m_1}{dt} = \epsilon \mathscr{F}^{(1)}_{\mathscr{H}}(\Omega) \qquad \frac{d\delta s_1}{dt} = \epsilon\, \Omega^{-1}\, \mathscr{F}^{(1)}_{\mathscr{H}}(\Omega)$$

Specialise (for now) to quasi-circular orbits with orbital frequency Ω and spins aligned with orbital angular momentum.



EMRI waveforms



Split the waveform into an amplitude and orbital phase:

$$h_{\ell m}(t) = \left[\epsilon h^{(1)}_{\ell m}\left(\Omega(t)\right) + \epsilon^2 h^{(2)}_{\ell m}\left(\Omega(t)\right)\right] e^{-im\phi_p(t)}$$

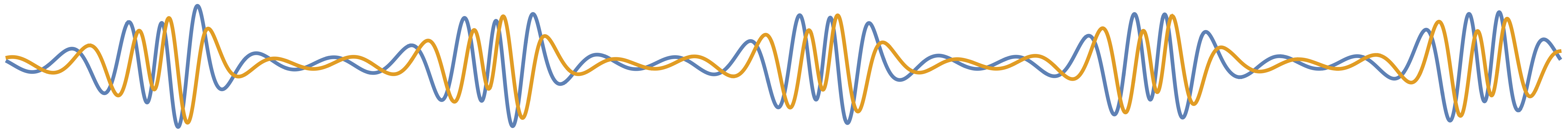
Amplitude is given by solving the linearised Einstein Equations.

Frequency evolution is given by solving the post-adiabatic equations of motion.

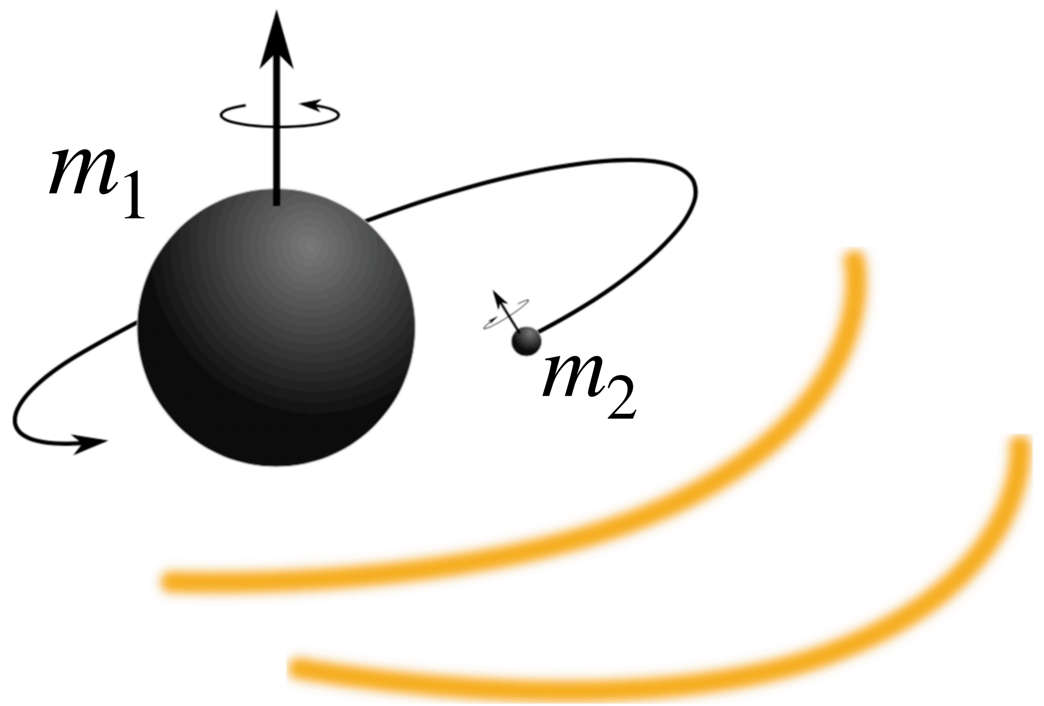
Orbital phase is given by integrating over the orbital frequencies.

Algorithm:

1. Precompute $h^{(1)}$ and $h^{(2)}$ on a grid of Ω values by solving Einstein's equations (hard)
2. Waveforms can be generated in **milliseconds** by solving ODEs (easy).



EMRI waveforms



Split the waveform into an amplitude and orbital phase:

$$h_{\ell m}(t) = \left[\epsilon h^{(1)}_{\ell m}\big(\Omega(t)\big) + \epsilon^2 h^{(2)}_{\ell m}\big(\Omega(t)\big)\right] e^{-im\phi_p(t)}$$

Amplitude is given by solving the linearised Einstein Equations.

Frequency evolution is given by solving the post-adiabatic equations of motion.

Orbital phase is given by integrating over the orbital frequencies.

For LISA: to $\mathcal{O}\left(\epsilon^0\right)$ the phase has contributions at adiabatic and post-adiabatic orders

$$\phi_p(t) = \epsilon^{-1}\phi_0\left[\langle h^1_{\text{diss}}\rangle\right] + \phi_1\left[h^1_{\text{diss,osc}} + h^1_{\text{cons}} + \langle h^2_{\text{diss}}\rangle\right] + \mathcal{O}(\epsilon)$$

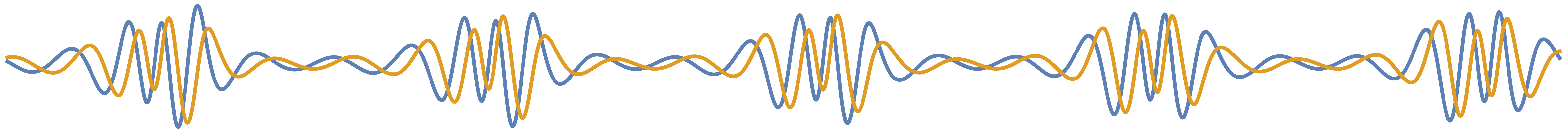
Adiabatic order

Can be obtained from asymptotic **fluxes**, avoiding a local calculation of the self-force

Post-Adiabatic order

Several contributions:

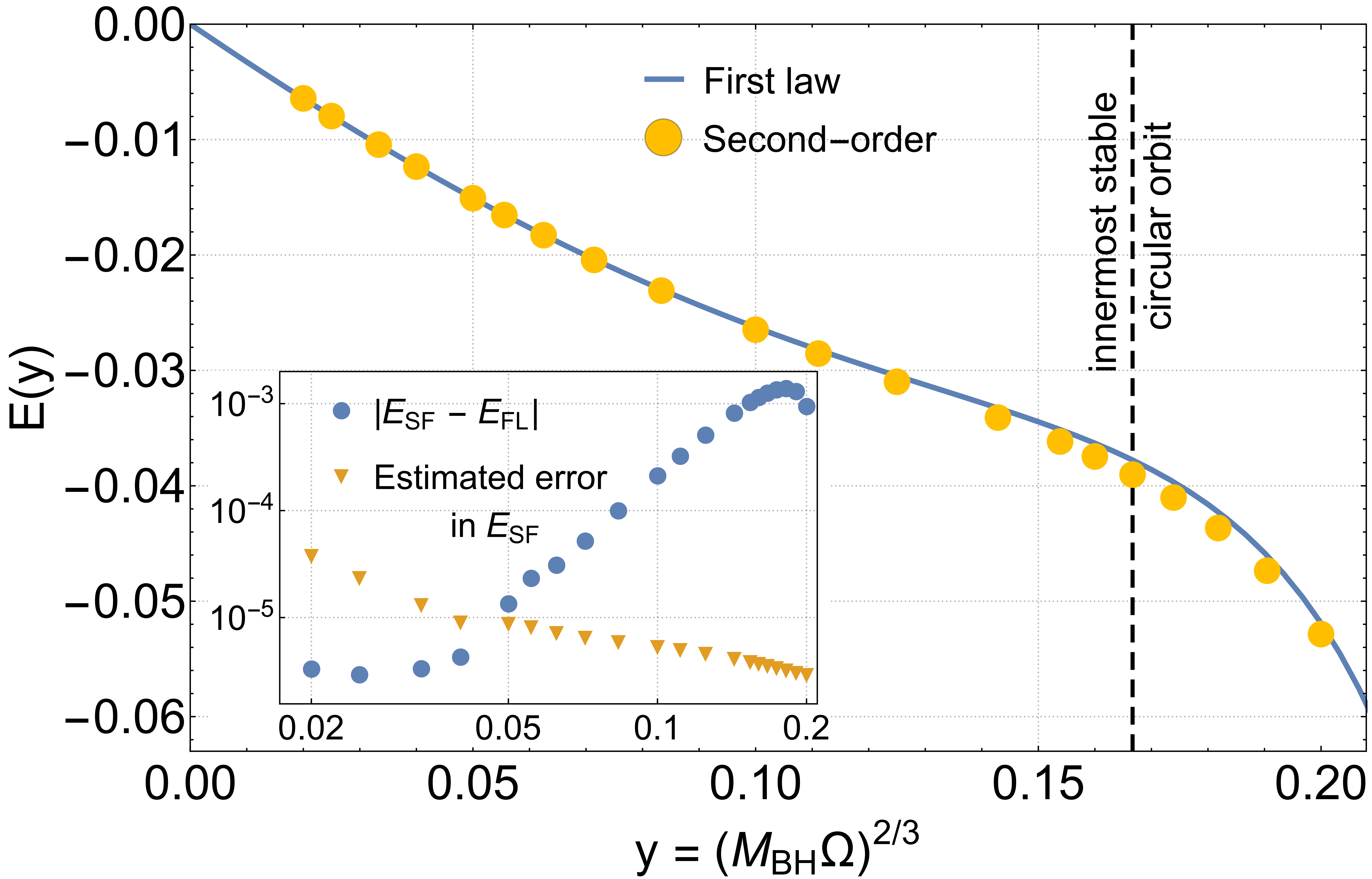
- Oscillatory first order self-force
- Spin of secondary
- Second-order averaged self-force



Results

Results: 1. Second order metric perturbation

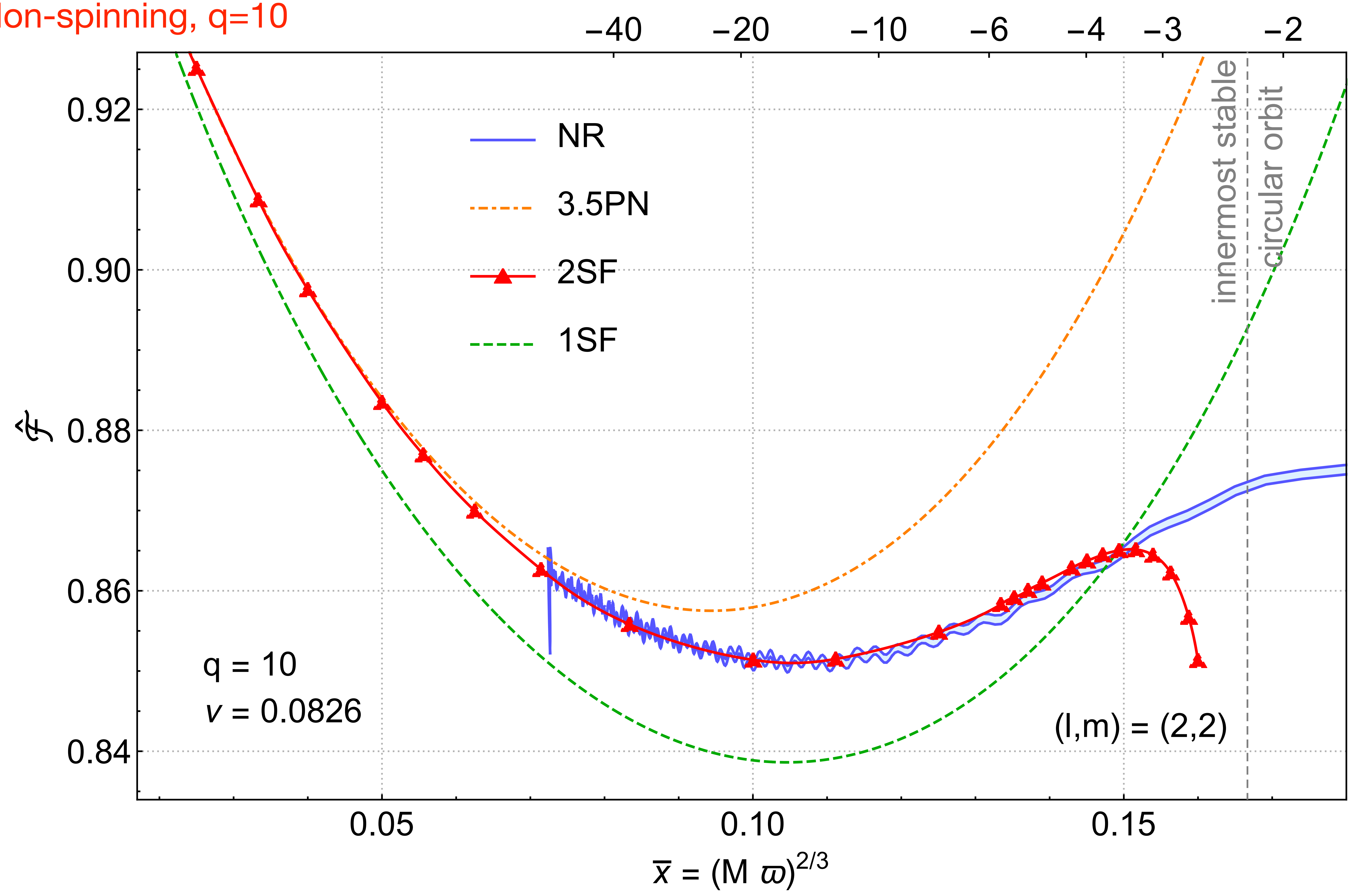
Binding energy ($l = 0 = m$)



Adam Pound, Barry Wardell, Niels Warburton and Jeremy Miller [Phys. Rev. Lett. 124, 021101]

Gravitational wave fluxes through $\mathscr{I}$ $(l \geq 2, m \neq 0)$

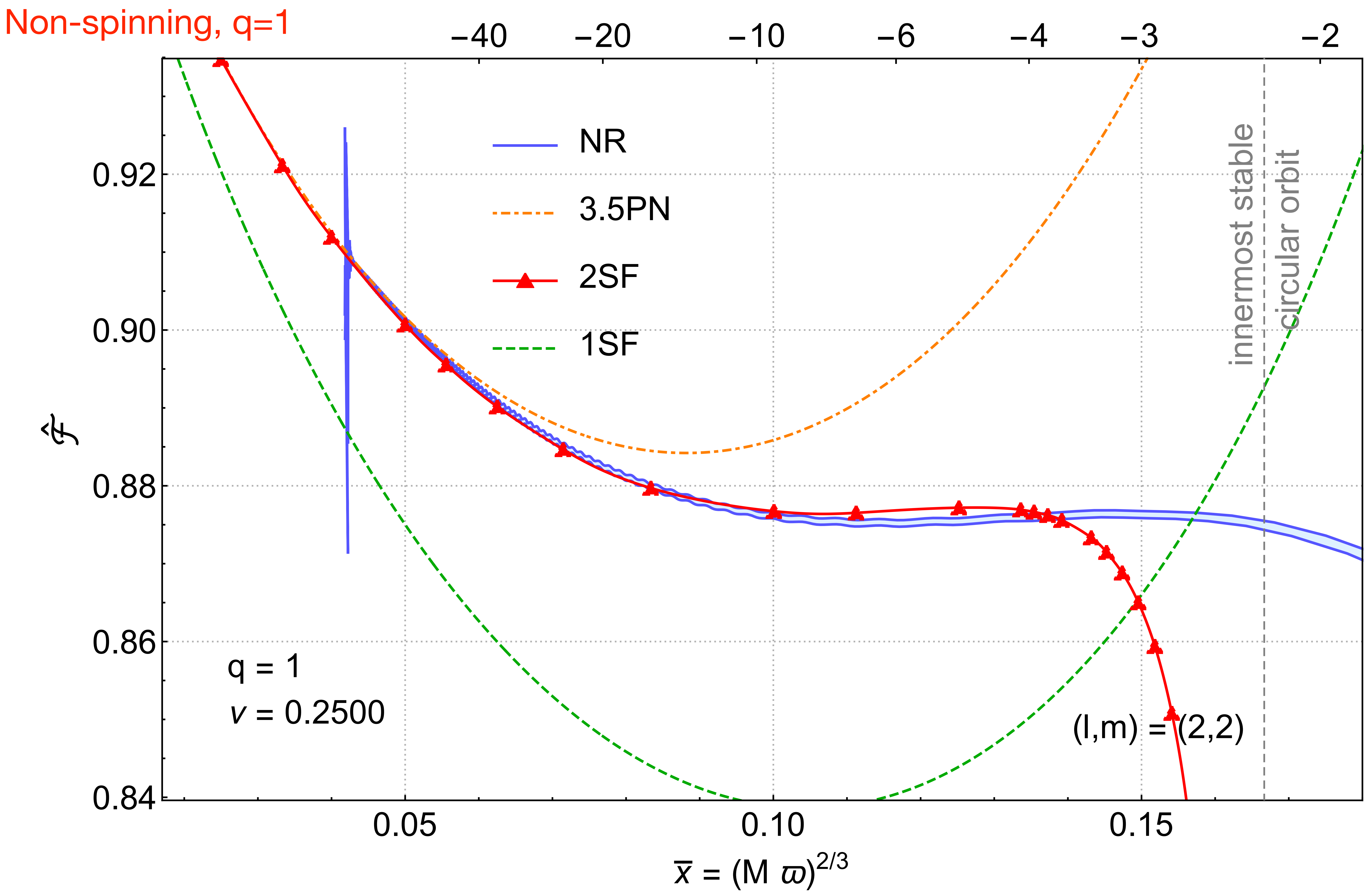
Non-spinning, q=10



$$\mathscr{F}_{lm}(t) = \frac{1}{16\pi} \left|\dot{A}_{lm}(t)\right|^2, \quad \varpi(t) = \dot{\Phi}_{22}(t)/2$$

NR waveform: SXS:BBH:1107

GW flux for **equal** mass binaries



$$\mathcal{F}_{lm}(t) = \frac{1}{16\pi}\left|\dot{A}_{lm}(t)\right|^2, \quad \varpi(t) = \dot{\Phi}_{22}(t)/2$$

NR waveform: SXS:BBH:1132

EMRI Waveforms

Factor waveform into amplitudes and **orbital** phase

$$h_{\ell m}(t) = \left[\epsilon h^{(1)}_{\ell m}\big(\Omega(t)\big) + \epsilon^2 h^{(2)}_{\ell m}\big(\Omega(t)\big)\right] e^{-im\phi_p(t)}$$

The **amplitude** is given by solving the first and second order Einstein Equations.

The **frequency evolution** is given by solving the post-adiabatic equations of motion.

The **orbital phase** is given by integrating over the orbital frequencies.

$$\frac{d\Omega}{dt} = \epsilon\left[F^{\Omega}_0(\Omega) + \epsilon F^{\Omega}_1(\Omega)\right] \qquad \frac{d\phi_p}{dt} = \Omega$$

Approximate 1PA equations of motion

Start from Bondi-Sachs mass-loss formula

$$-\mathcal{F}_\infty = \frac{dM_B}{dt}$$

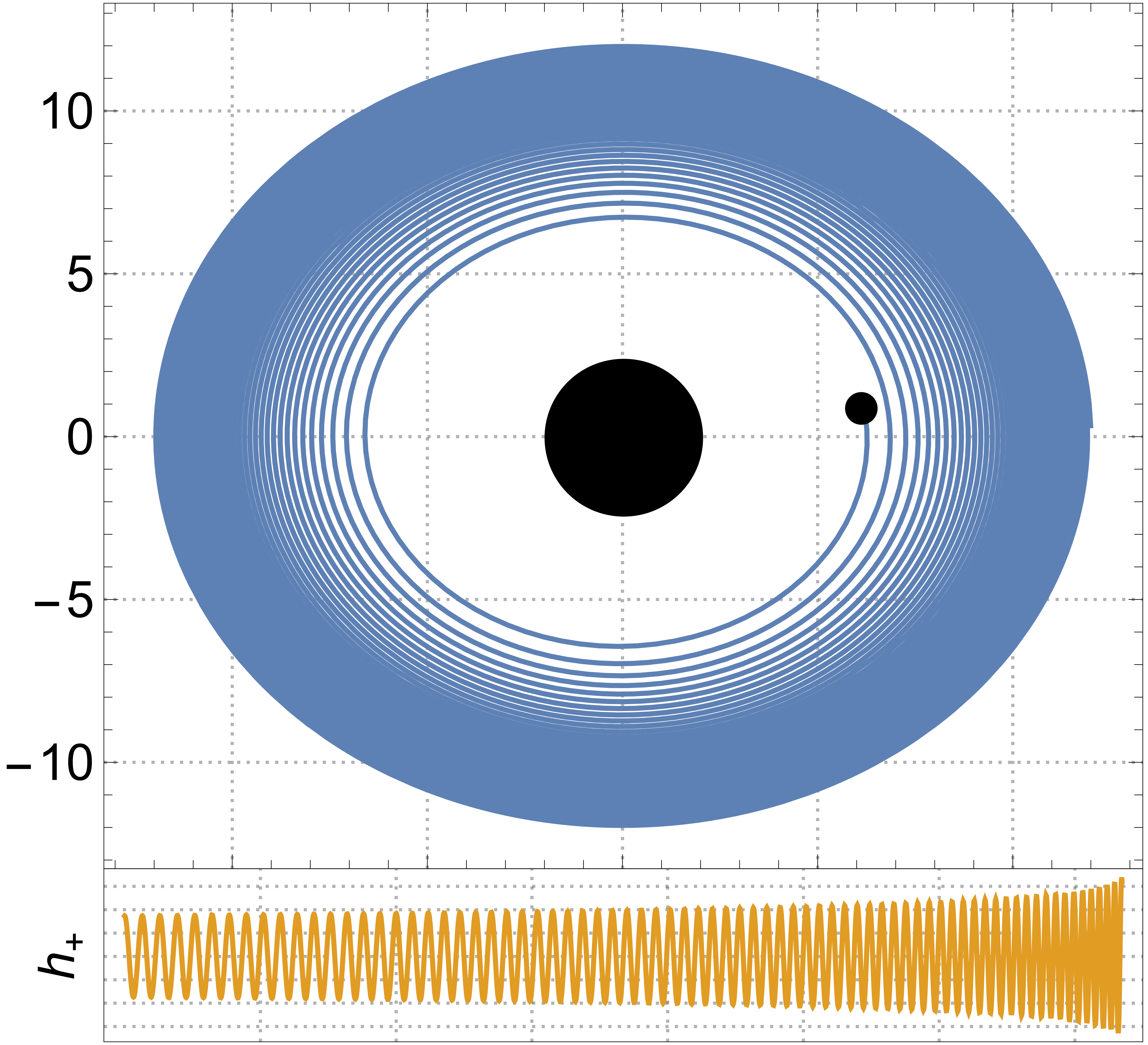
$$= \frac{d}{dt}\left(E_{\rm bind} + m_1 + \cancel{m_2}\right)$$

$$\approx \frac{dE_{\rm bind}}{d\Omega}\frac{d\Omega}{dt} + \frac{dE_{\rm bind}}{dm_1}\cancel{\frac{dm_1}{dt}} + \frac{dE_{\rm bind}}{dJ_1}\cancel{\frac{dJ_1}{dt}} + \frac{dm_1}{dt}$$

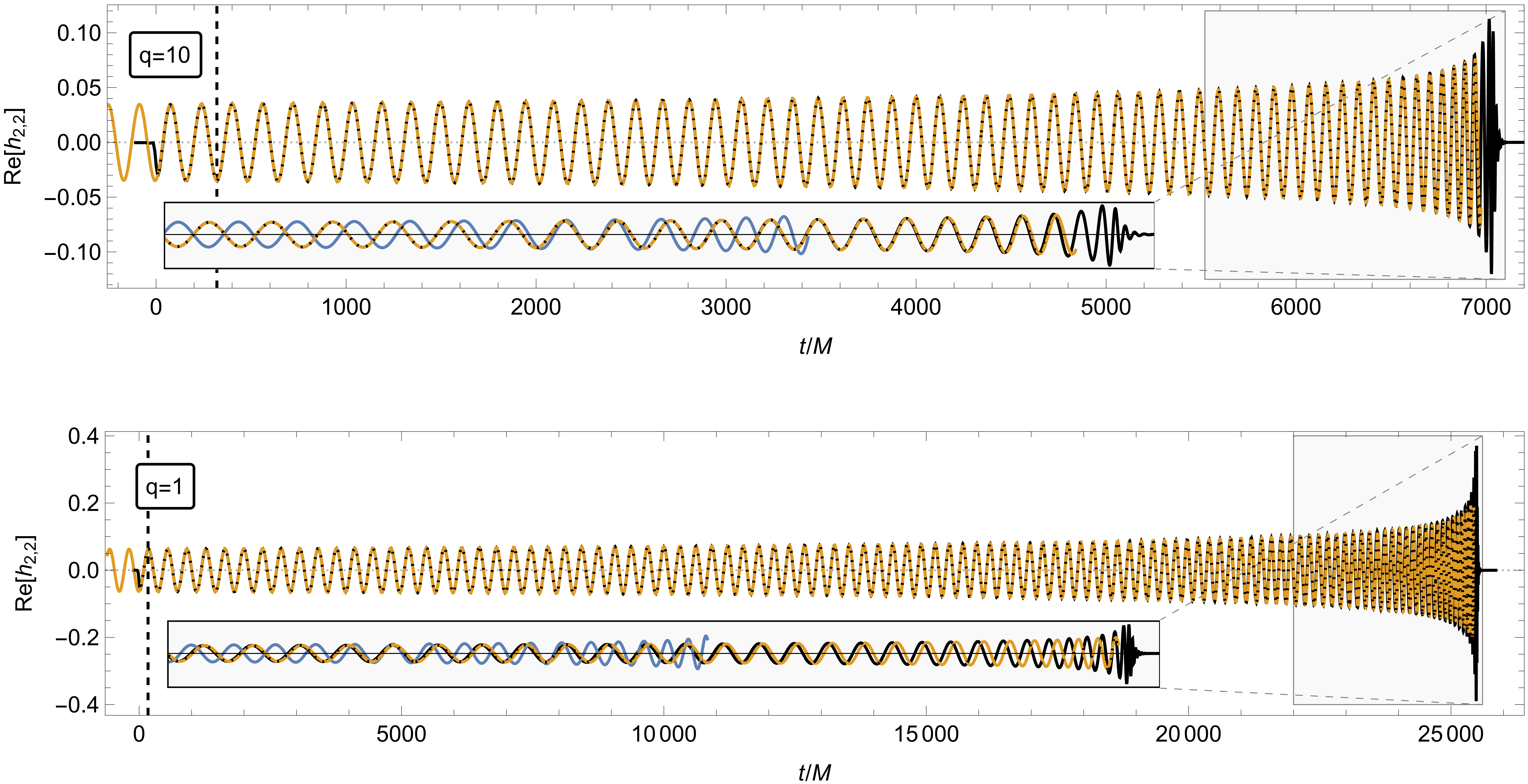
$$\boxed{\frac{d\Omega}{dt} = -\mathcal{F}\Big/\frac{dE_{\rm bind}}{d\Omega}}$$

$h^{(1)}_{\alpha\beta}, h^{(2)}_{\alpha\beta}$

Mass ratio $q = 10$ quasi-circular inspiral

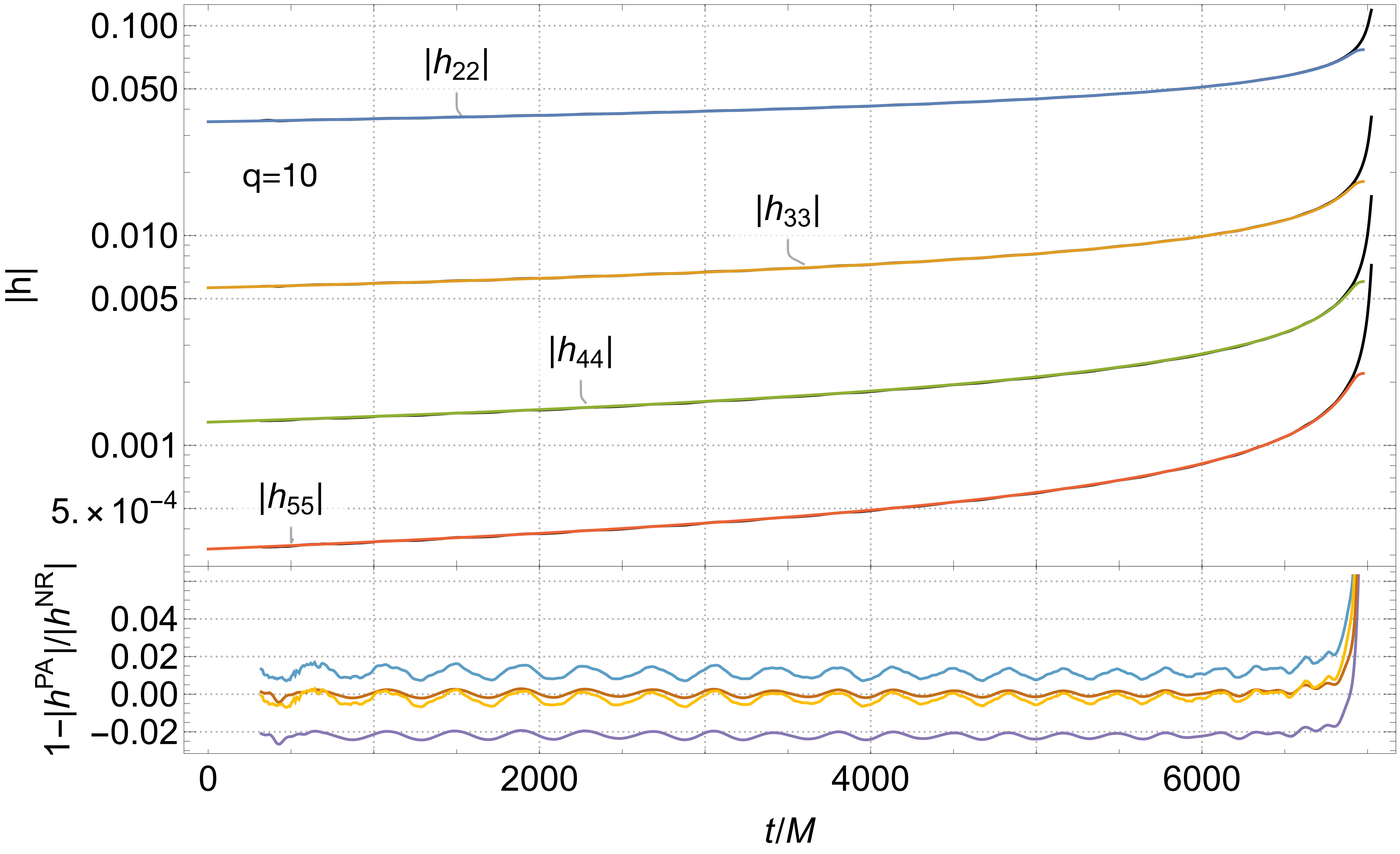


Waveform comparison with Numerical Relativity



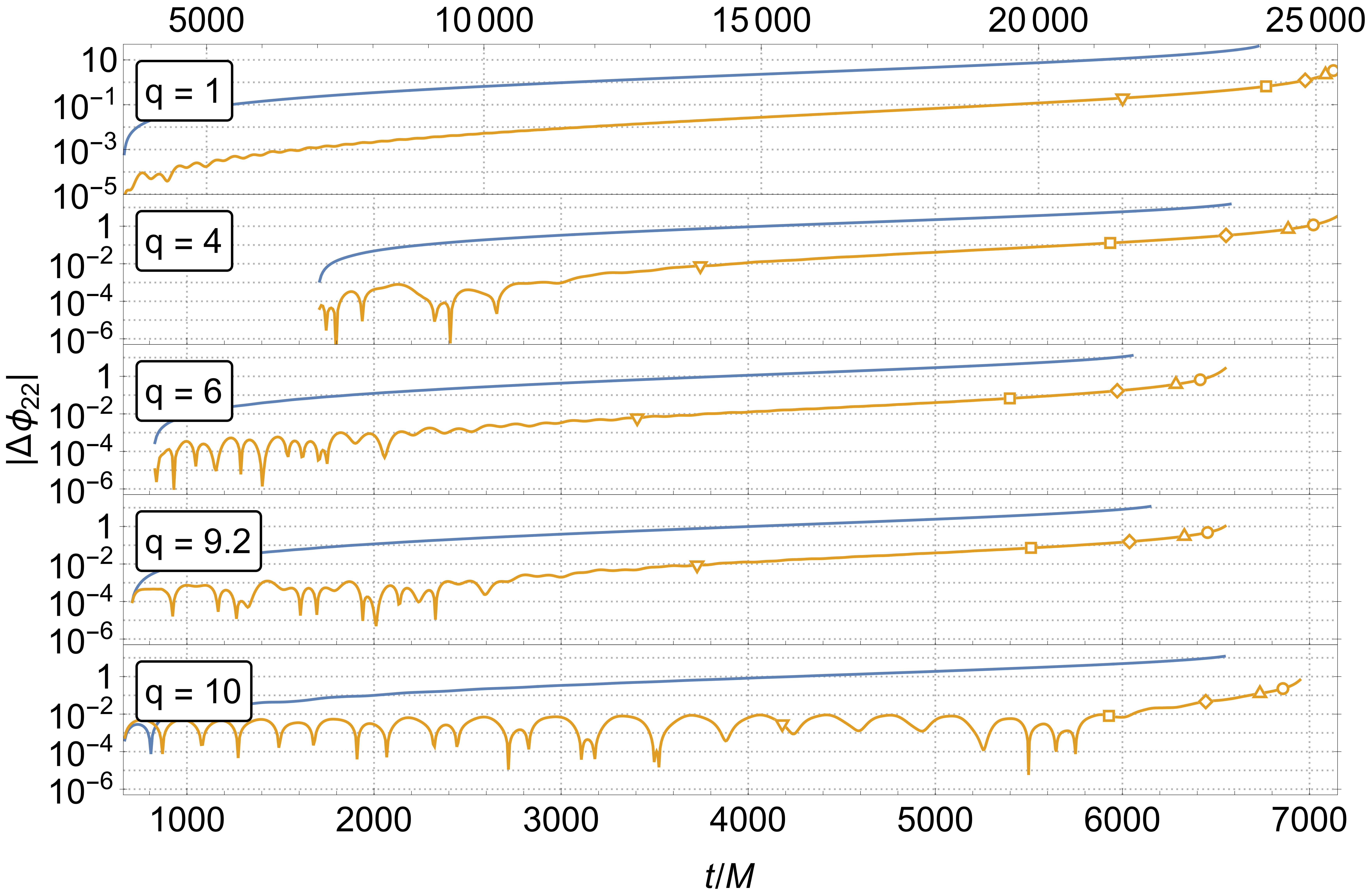
Barry Wardell, Adam Pound, Niels Warburton, Jeremy Miller, Leanne Durkan and Alexandre Le Tiec [Phys. Rev. Lett. 130, 241402]

Waveform comparison: amplitude, higher modes



Results: 2. Gravitational waveforms

Waveform comparison: phase



Including spin in waveforms

$$\chi \equiv S/\mu^2$$

$$S^2 \equiv S^\alpha S_\alpha = \frac{1}{2} S_{\alpha\beta} S^{\alpha\beta}$$

$$\mu^2 \chi_\parallel^2 \equiv S^\theta S_\theta$$

$$\chi_\perp^2 \equiv \chi^2 - \chi_\parallel^2$$

Josh Mathews, Adam Pound, Barry Wardell [Phys. Rev. D 105, 084031], Josh Mathews, Jonathan Thompson, et al. [in preparation]

Aligned secondary spin

$$q = 6, \quad \chi_2 = -0.8$$

$$(\chi_1 = 0)$$

Results: 3. Gravitational waveforms with spin

Precessing secondary spin

$$q = 4, \quad \chi \;\; \approx 0.01, \quad \chi_\perp \approx 0.8$$

SXS:BBH:1909

$$(\chi_1 = 0)$$

Josh Mathews, Adam Pound, Barry Wardell [Phys. Rev. D 105, 084031], Josh Mathews, Jonathan Thompson, et al. [in preparation]

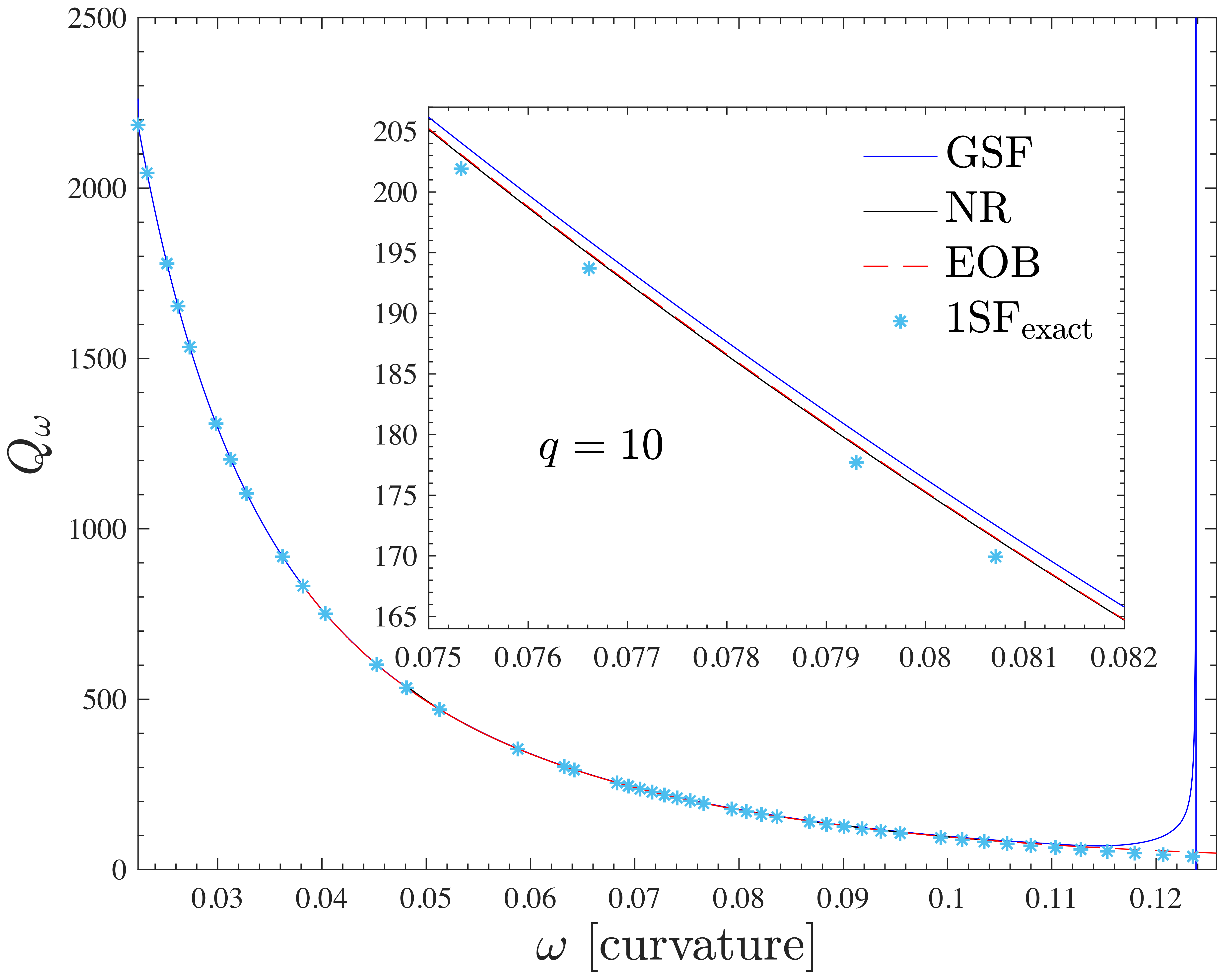
Comparison with TEOBResumS

Detailed comparison of 1PA GSF waveforms with those from the TEOBResumS effective one body model and with numerical relativity.

1. Effects of transition to plunge significant over a large frequency interval, restricting domain of validity to orbital frequencies much smaller than ISCO frequency.

2. 1PA GSF models yield satisfactory phase errors for mass ratios $\epsilon \lesssim 1/25$.

3. Identified key areas for improvement in TEOBResumS, particularly for small mass ratios.

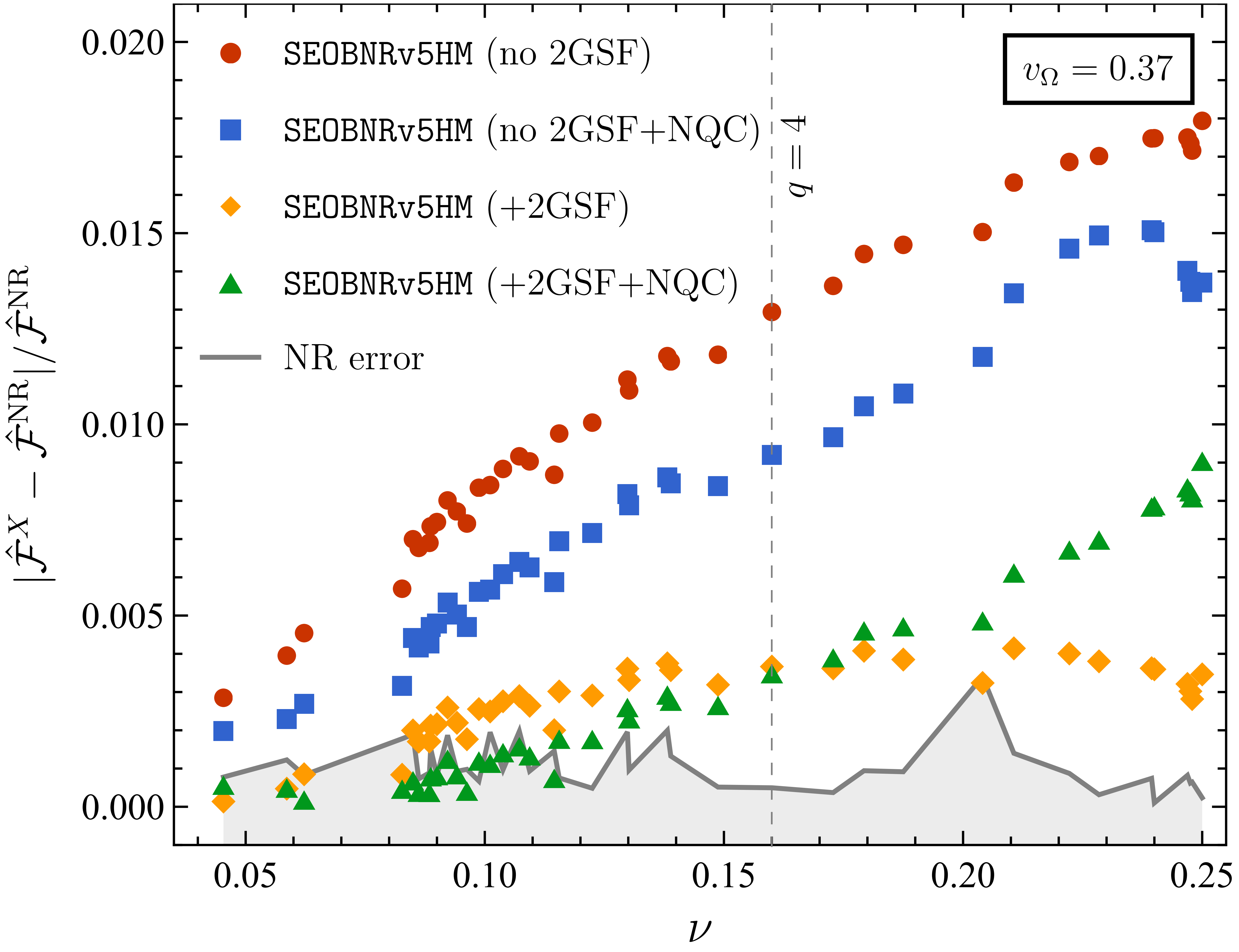


Calibration of SEOBNRv5

Incorporated 2SF flux information into latest SEOBNR models prepared for LIGO O4 data analysis.

1. Significant improvement in agreement with reference results provided by NR.

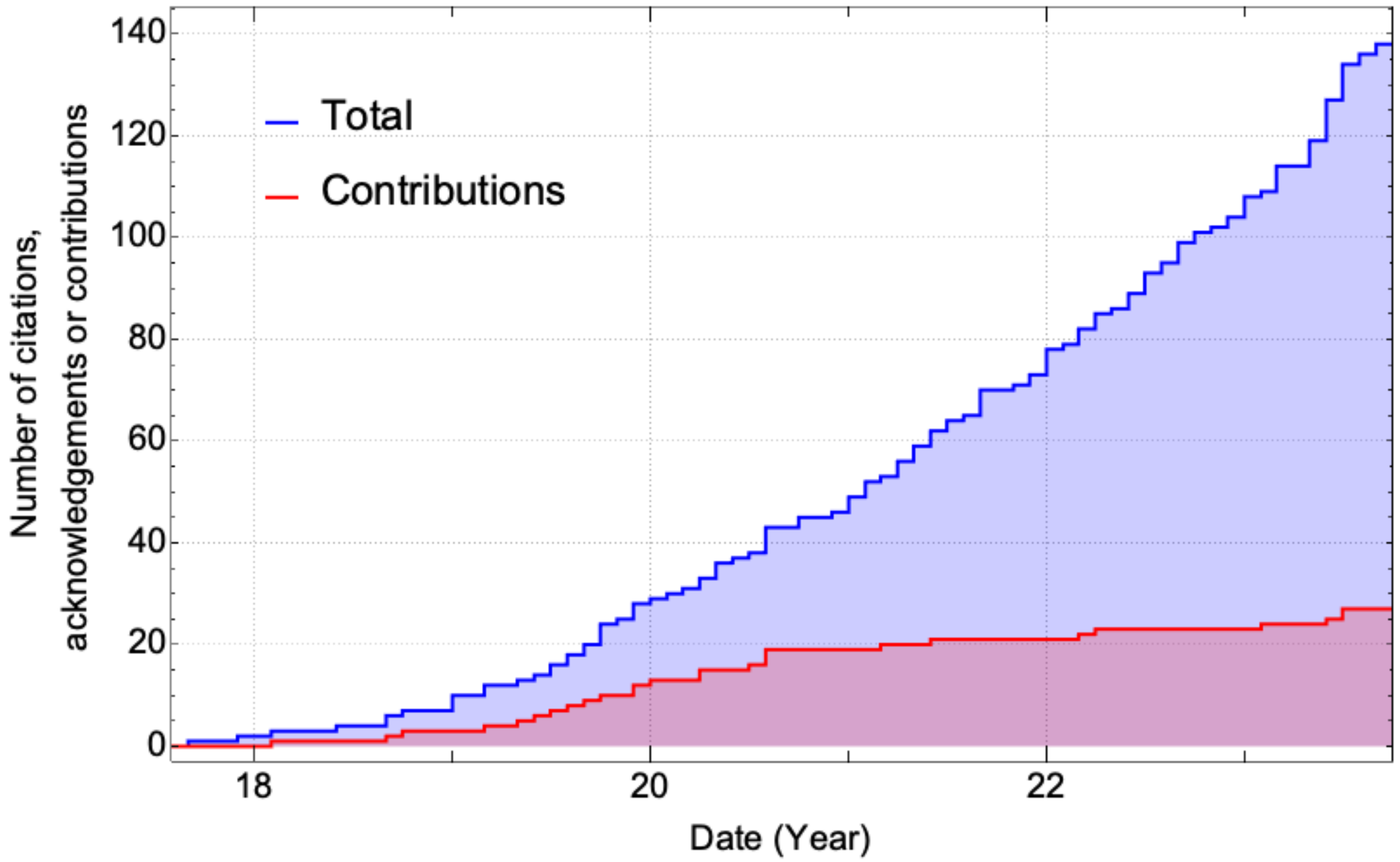
2. Reduces the need to rely on "NQC" corrections.



Black Hole Perturbation Toolkit

"Our goal is for less researcher time to be spent writing code and more time spent doing physics. Currently there exist multiple scattered black hole perturbation theory codes developed by a wide array of individuals or groups over a number of decades. This project aims to bring together some of the core elements of these codes into a Toolkit that can be used by all.

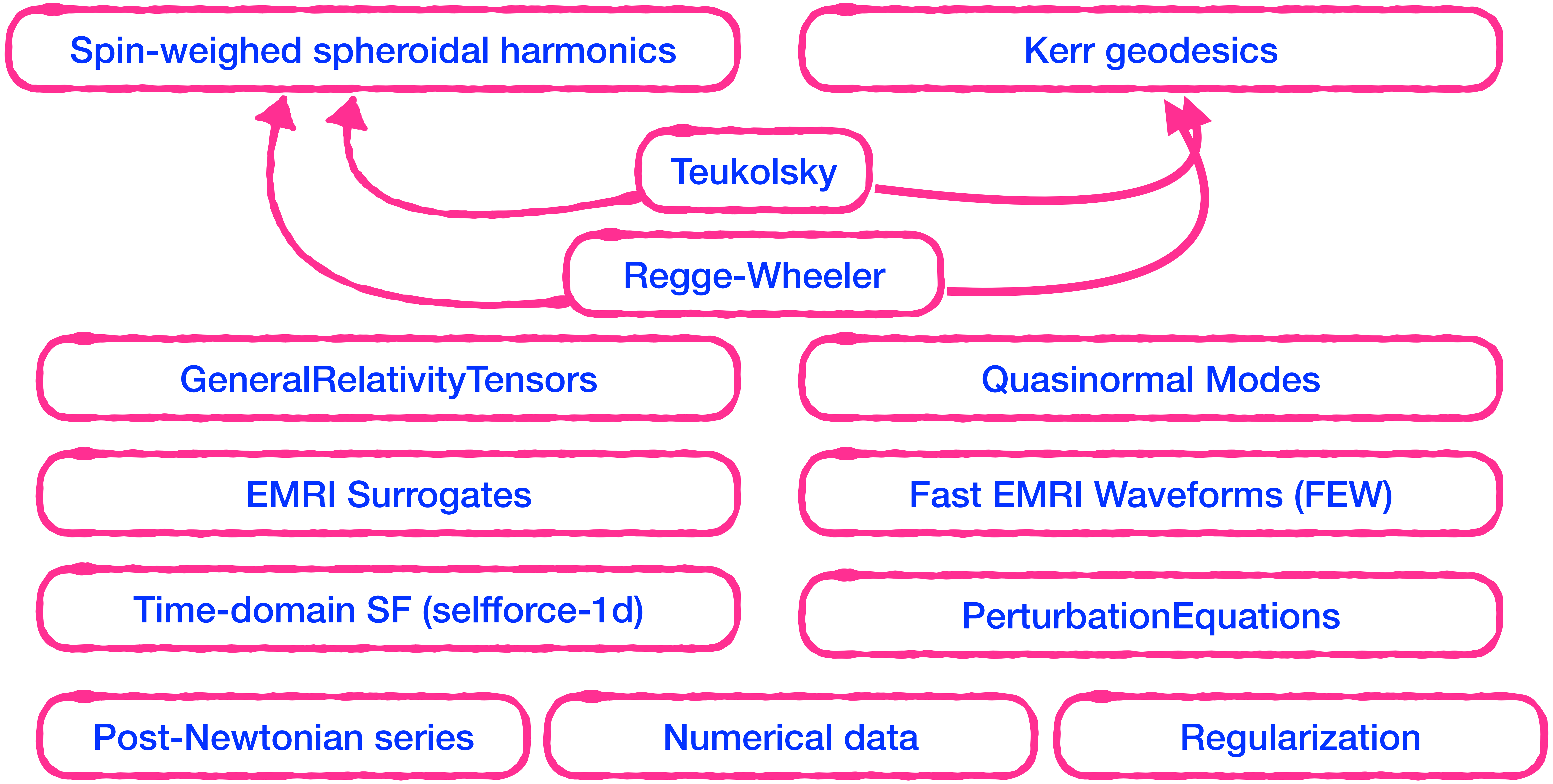
Additionally, we want to provide easy, open access to data from black hole perturbation codes and calculations."



Results: 5. Black Hole Perturbation Toolkit

Currently available toolkit components

The black hole perturbation toolkit has several packages for doing calculations in black hole perturbation theory, including post-adiabatic (1PA) waveforms.



Second order Einstein equations: PERTURBATIONEQUATIONS package

Untitled-1

```
In[1]:= Block[{Print}, << xAct`PerturbationEquations`]

In[2]:= SchwarzschildQuadraticOperator["d2G", "IngoingRadiationGauge", "Kinnersley", "Kinnersley"]["ll"]
```

Out[2]=

$$\frac{1}{f[r]^2 r^3} 2\Big(-C^{lm0}_{l_2 m_2 0 l_1 m_1 0}\Big(r\big(-r^2 \partial_0 h_{\kappa 34}{}^{0 l_1 m_1} \partial_0 h_{\kappa 34}{}^{0 l_2 m_2} + h_{\kappa 34}{}^{0 l_2 m_2}(4 M \partial_0 h_{\kappa 34}{}^{0 l_1 m_1} - 2 r^2 \partial_0\partial_0 h_{\kappa 34}{}^{0 l_1 m_1})\big) - f[r] r^2 \big(4 h_{\kappa 34}{}^{0l_2m_2} \partial_0 h_{\kappa 34}{}^{0l_1m_1} + r \partial_0 h_{\kappa 34}{}^{0l_2m_2} \partial_1 h_{\kappa 34}{}^{0l_1m_1} + r\partial_0 h_{\kappa 34}{}^{0l_1m_1}\partial_1 h_{\kappa 34}{}^{0l_2m_2} + 4 h_{\kappa 34}{}^{0l_2m_2} r \partial_1\partial_0 h_{\kappa 34}{}^{0l_1m_1}\big) - f[r]^2 r^2 \big(4 h_{\kappa 34}{}^{0l_2m_2}\partial_1 h_{\kappa 34}{}^{0l_1m_1} + r\partial_1 h_{\kappa 34}{}^{0l_1m_1}\partial_1 h_{\kappa 34}{}^{0l_2m_2} + 2 h_{\kappa 34}{}^{0l_2m_2} r \partial_1\partial_1 h_{\kappa 34}{}^{0l_1m_1}\big)\Big) + C^{lm0}_{l_2m_2-2l_1m_12}\Big(-2M h_{\kappa 44}{}^{-2l_2m_2} r\partial_0 h_{\kappa 33}{}^{2l_1m_1} - 2M h_{\kappa 33}{}^{2l_1m_1} r \partial_0 h_{\kappa 44}{}^{-2l_2m_2} + r^3\partial_0 h_{\kappa 33}{}^{2l_1m_1}\partial_0 h_{\kappa 44}{}^{-2l_2m_2} + h_{\kappa 44}{}^{-2l_2m_2} r^3 \partial_0\partial_0 h_{\kappa 33}{}^{2l_1m_1} + h_{\kappa 33}{}^{2l_1m_1} r^3\partial_0\partial_0 h_{\kappa 44}{}^{-2l_2m_2} + f[r] r^2 \big(r(\partial_0 h_{\kappa 44}{}^{-2l_2m_2}\partial_1 h_{\kappa 33}{}^{2l_1m_1} + \partial_0 h_{\kappa 33}{}^{2l_1m_1}\partial_1 h_{\kappa 44}{}^{-2l_2m_2}) + 2 h_{\kappa 44}{}^{-2l_2m_2}(\partial_0 h_{\kappa 33}{}^{2l_1m_1} + r\partial_1\partial_0 h_{\kappa 33}{}^{2l_1m_1}) + 2 h_{\kappa 33}{}^{2l_1m_1}(\partial_0 h_{\kappa 44}{}^{-2l_2m_2} + r\partial_1\partial_0 h_{\kappa 44}{}^{-2l_2m_2})\big) + f[r]^2 r^2 \big(r\partial_1 h_{\kappa 33}{}^{2l_1m_1}\partial_1 h_{\kappa 44}{}^{-2l_2m_2} + h_{\kappa 44}{}^{-2l_2m_2}(2\partial_1 h_{\kappa 33}{}^{2l_1m_1} + r\partial_1\partial_1 h_{\kappa 33}{}^{2l_1m_1}) + h_{\kappa 33}{}^{2l_1m_1}(2\partial_1 h_{\kappa 44}{}^{-2l_2m_2} + r\partial_1\partial_1 h_{\kappa 44}{}^{-2l_2m_2})\big)\Big)\Big)$$

```
In[3]:= lmReplacerule[%, 2, 2, 2, 0, 2, 2]
```

Out[3]=

$$\frac{1}{f[r]^2 r^3} 2\Bigg(\frac{1}{7}\sqrt{\frac{5}{\pi}}\Big(r\big(-r^2 \partial_0 h_{\kappa 34}{}^{020} \partial_0 h_{\kappa 34}{}^{022} + h_{\kappa 34}{}^{022}(4 M \partial_0 h_{\kappa 34}{}^{020} - 2 r^2 \partial_0\partial_0 h_{\kappa 34}{}^{020})\big) - f[r] r^2 \big(4 h_{\kappa 34}{}^{022} \partial_0 h_{\kappa 34}{}^{020} + r \partial_0 h_{\kappa 34}{}^{022} \partial_1 h_{\kappa 34}{}^{020} + r\partial_0 h_{\kappa 34}{}^{020}\partial_1 h_{\kappa 34}{}^{022} + 4 h_{\kappa 34}{}^{022} r \partial_1\partial_0 h_{\kappa 34}{}^{020}\big) - f[r]^2 r^2 \big(4 h_{\kappa 34}{}^{022}\partial_1 h_{\kappa 34}{}^{020} + r\partial_1 h_{\kappa 34}{}^{020}\partial_1 h_{\kappa 34}{}^{022} + 2 h_{\kappa 34}{}^{022} r \partial_1\partial_1 h_{\kappa 34}{}^{020}\big)\Big) + \frac{1}{7}\sqrt{\frac{5}{\pi}}\Big(-2M h_{\kappa 44}{}^{-222} r\partial_0 h_{\kappa 33}{}^{220} - 2M h_{\kappa 33}{}^{220} r \partial_0 h_{\kappa 44}{}^{-222} + r^3\partial_0 h_{\kappa 33}{}^{220}\partial_0 h_{\kappa 44}{}^{-222} + h_{\kappa 44}{}^{-222} r^3 \partial_0\partial_0 h_{\kappa 33}{}^{220} + h_{\kappa 33}{}^{220} r^3\partial_0\partial_0 h_{\kappa 44}{}^{-222} + f[r] r^2 \big(r(\partial_0 h_{\kappa 44}{}^{-222}\partial_1 h_{\kappa 33}{}^{220} + \partial_0 h_{\kappa 33}{}^{220}\partial_1 h_{\kappa 44}{}^{-222}) + 2 h_{\kappa 44}{}^{-222}(\partial_0 h_{\kappa 33}{}^{220} + r\partial_1\partial_0 h_{\kappa 33}{}^{220}) + 2 h_{\kappa 33}{}^{220}(\partial_0 h_{\kappa 44}{}^{-222} + r\partial_1\partial_0 h_{\kappa 44}{}^{-222})\big) + f[r]^2 r^2 \big(r\partial_1 h_{\kappa 33}{}^{220}\partial_1 h_{\kappa 44}{}^{-222} + h_{\kappa 44}{}^{-222}(2\partial_1 h_{\kappa 33}{}^{220} + r\partial_1\partial_1 h_{\kappa 33}{}^{220}) + h_{\kappa 33}{}^{220}(2\partial_1 h_{\kappa 44}{}^{-222} + r\partial_1\partial_1 h_{\kappa 44}{}^{-222})\big)\Big)\Bigg)$$

Parameter estimation

Incorporated 1PA waveform into FEW. Fast enough to be used in LISA MCMC parameter estimation studies (~6 hours on a GPU per configuration).

Focus on three configurations:

Config.	ϵ	M $[M_\odot]$	r_0/M	D_S [Gpc]	T_{obs} [yrs]	ρ_{AET}
(1)	10^{-5}	10^6	10.6025	1.0	2.0	70
(2)	10^{-4}	10^6	15.7905	2.0	1.5	65
(3)	10^{-3}	$5 \cdot 10^6$	16.8123	1.0	1.0	340

(1)

Extreme Mass Ratio Inspiral: $q = 10^{-5}$

Marginalized posteriors with shaded 68% credible intervals generated by injecting a true reference model cir1PA and recovering using different models

(2)

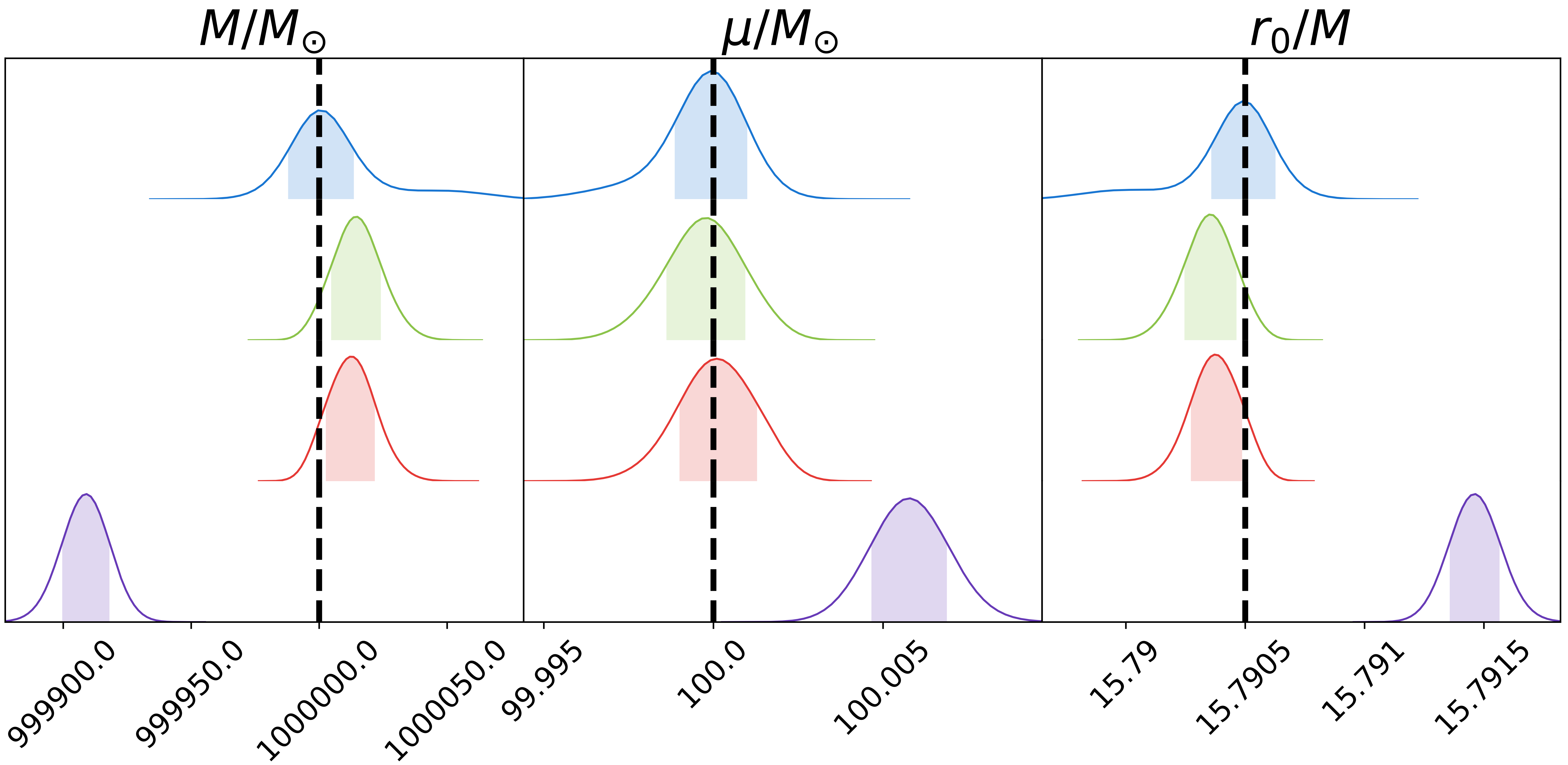
Extreme Mass Ratio Inspiral: $q = 10^{-4}$

cir1PA w/ spin

cir1PA w/o spin

cir0PA + 1PA-3PN w/o spin

cir0PA w/o spin



Marginalized posteriors with shaded 68% credible intervals generated by injecting a true reference model cir1PA and recovering using different models

(3)

Intermediate Mass Ratio Inspiral: $q = 10^{-3}$

Marginalized posteriors with shaded 68% credible intervals generated by injecting a true reference model cir1PA and recovering using different models

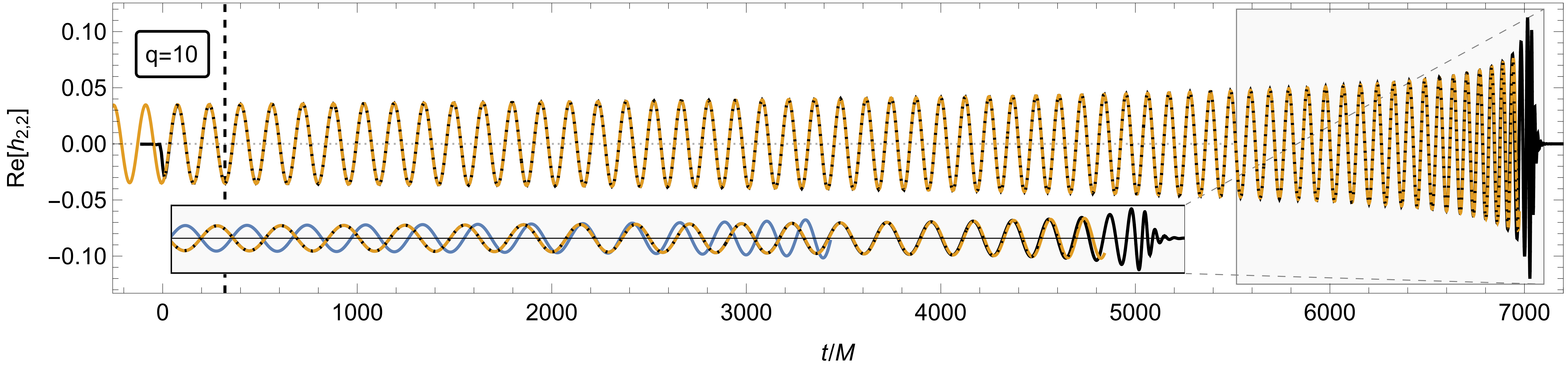
Dephasing, mismatches and degeneracy

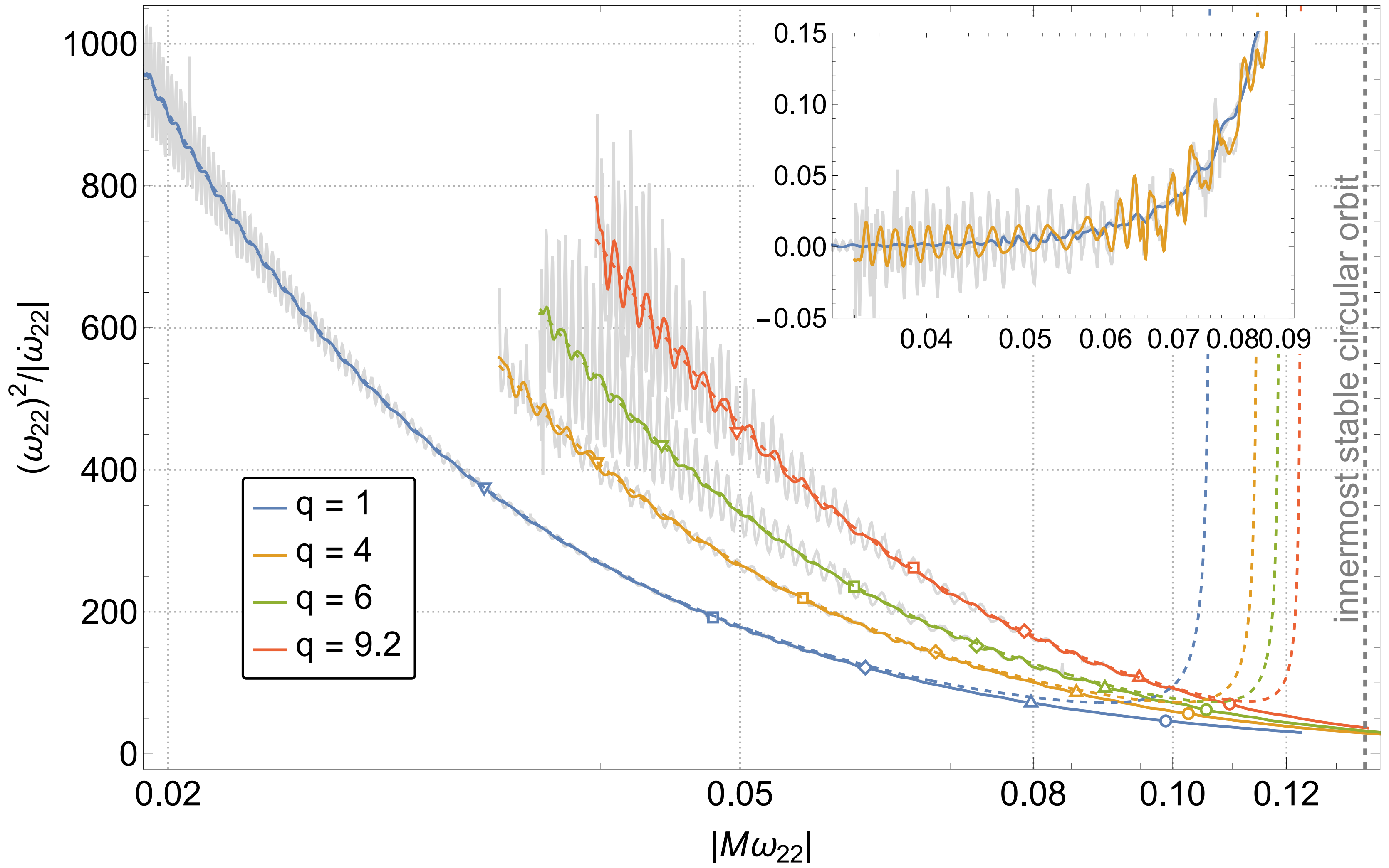
Bias in the parameters is degenerate with mis-modelling errors

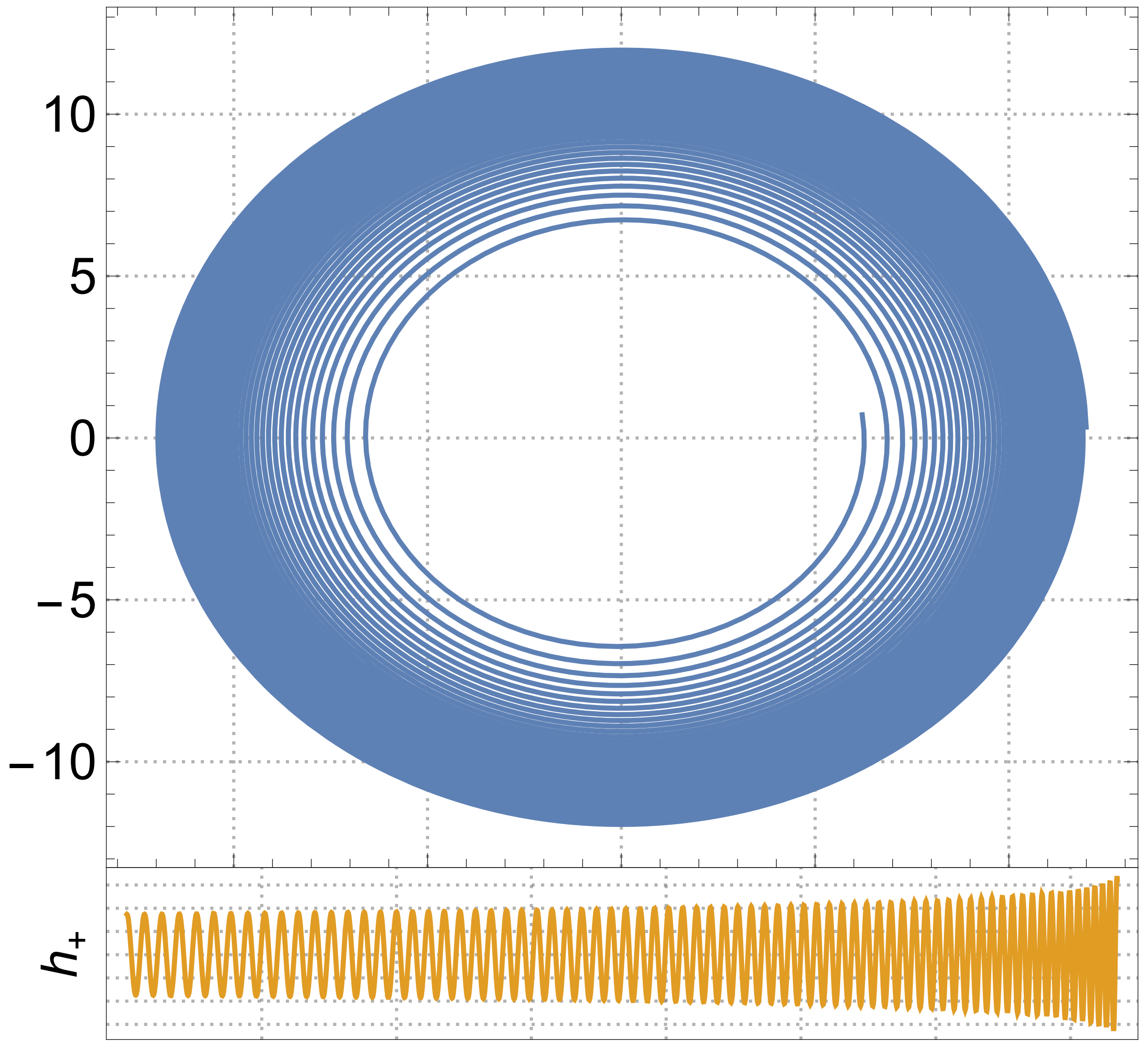
ϵ	**Model Waveform**	$\Delta\Phi^{(\mathrm{inj})}$	$\Delta\Phi^{(\mathrm{bf})}$	$\mathcal{M}^{(\mathrm{inj})}$	$\mathcal{M}^{(\mathrm{bf})}$	$\rho^{(\mathrm{inj})}/\rho^{(\mathrm{opt})}$	$\rho^{(\mathrm{bf})}/\rho^{(\mathrm{opt})}$	$\log\mathcal{L}^{(\mathrm{inj})}$	$\log\mathcal{L}^{(\mathrm{bf})}$
10^{-5}	Cir1PA w/o spin	0.779	0.0165	0.143	4.497×10^{-5}	83.4%	99.9%	−846	-0.250
	Cir0PA 1PA-3PN w/o spin	0.786	0.00179	0.163	4.293×10^{-6}	81.5%	99.8%	−943	-0.0324
	Cir0PA w/o spin	3.002	0.00532	0.889	2.412×10^{-6}	6.4%	99.8%	−4800	-0.0234
10^{-4}	Cir1PA w/o spin	3.994	0.00702	0.511	8.601×10^{-6}	30.3%	99.9%	-5019	-0.336
	Cir0PA 1PA-3PN w/o spin	4.310	0.0179	0.486	1.26×10^{-4}	34.2%	99.9%	-4799	-0.441
	Cir0PA w/o spin	13.093	0.0354	0.653	2.573×10^{-5}	19.0%	99.9%	-5506	-0.122
10^{-3}	Cir1PA w/o spin	4.518	0.00559	0.922	3.643×10^{-6}	3.3%	99.9%	-112938	-0.226
	Cir0PA 1PA-3PN w/o spin	4.882	0.0218	0.949	3.443×10^{-5}	3.4%	99.9%	-112827	-2.132
	Cir0PA w/o spin	14.958	0.153	0.938	6.854×10^{-3}	4.9%	99.1%	-122173	-524.798

Future improvements: Transition and Plunge

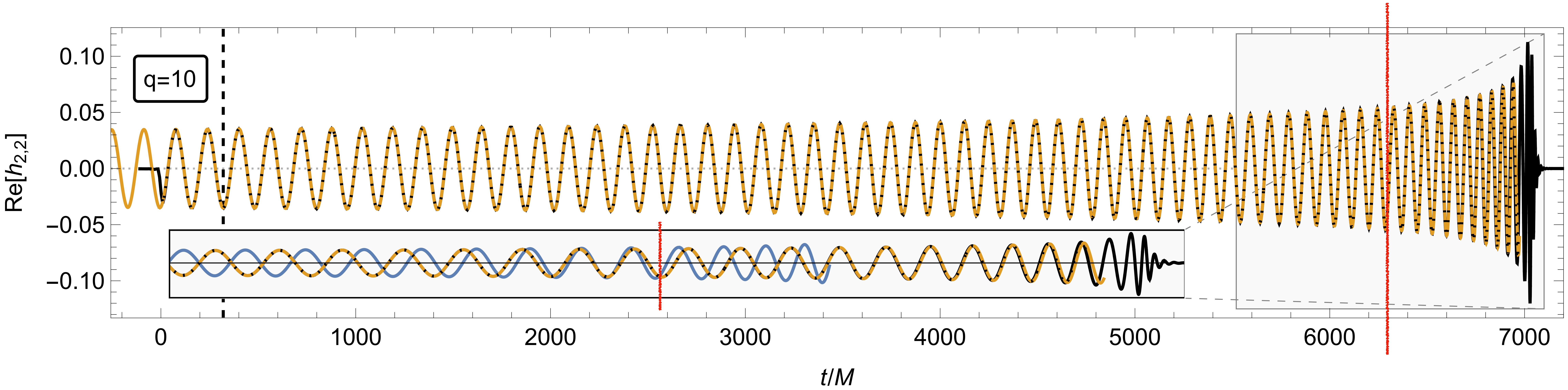
Transition and plunge







Transition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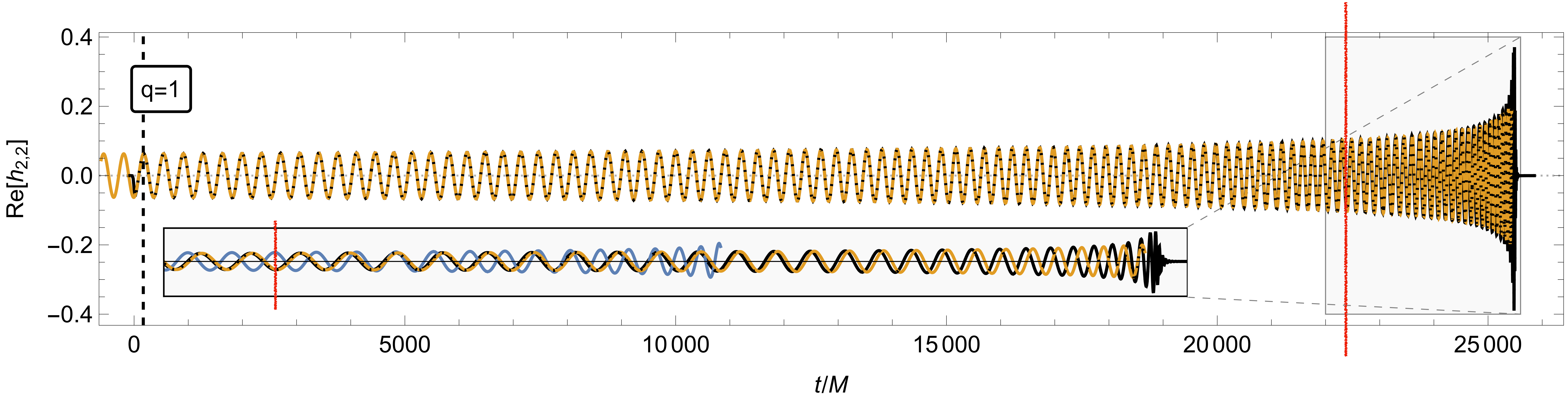
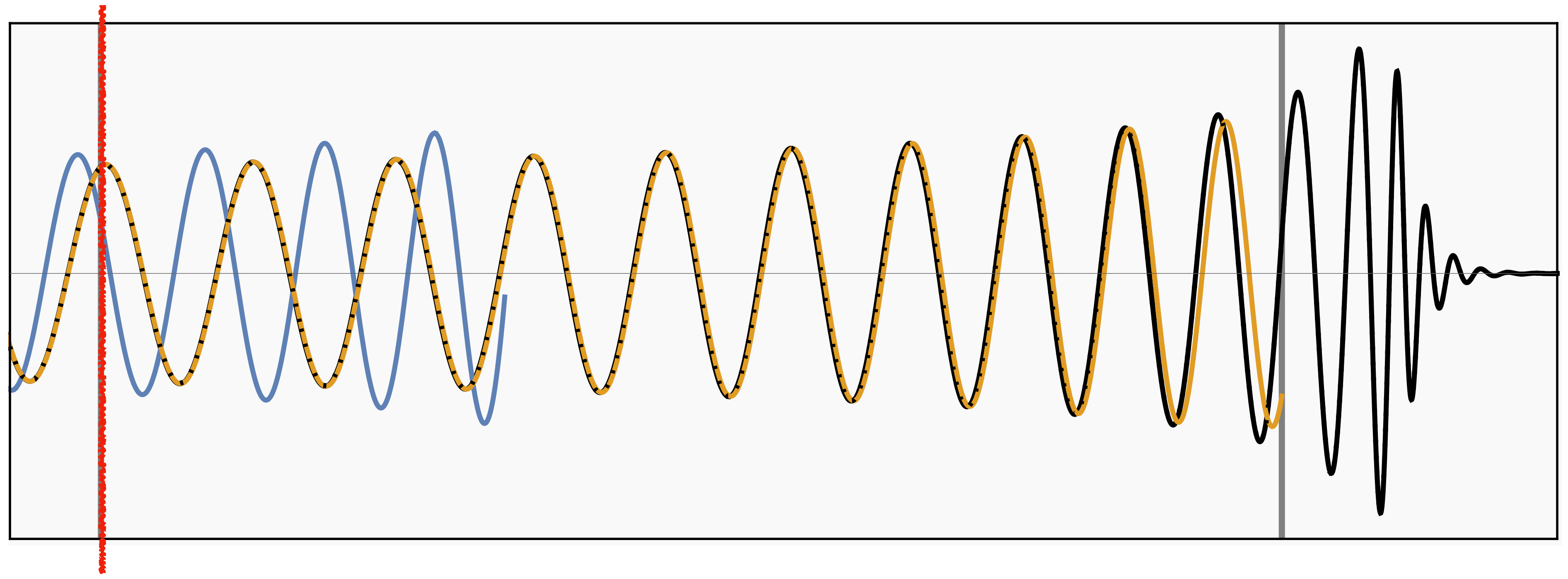
Re[$h_{2,2}$]
0.10
0.05
0.00
−0.05
−0.10
0
1000
2000
3000
4000
5000
6000
7000
t/M

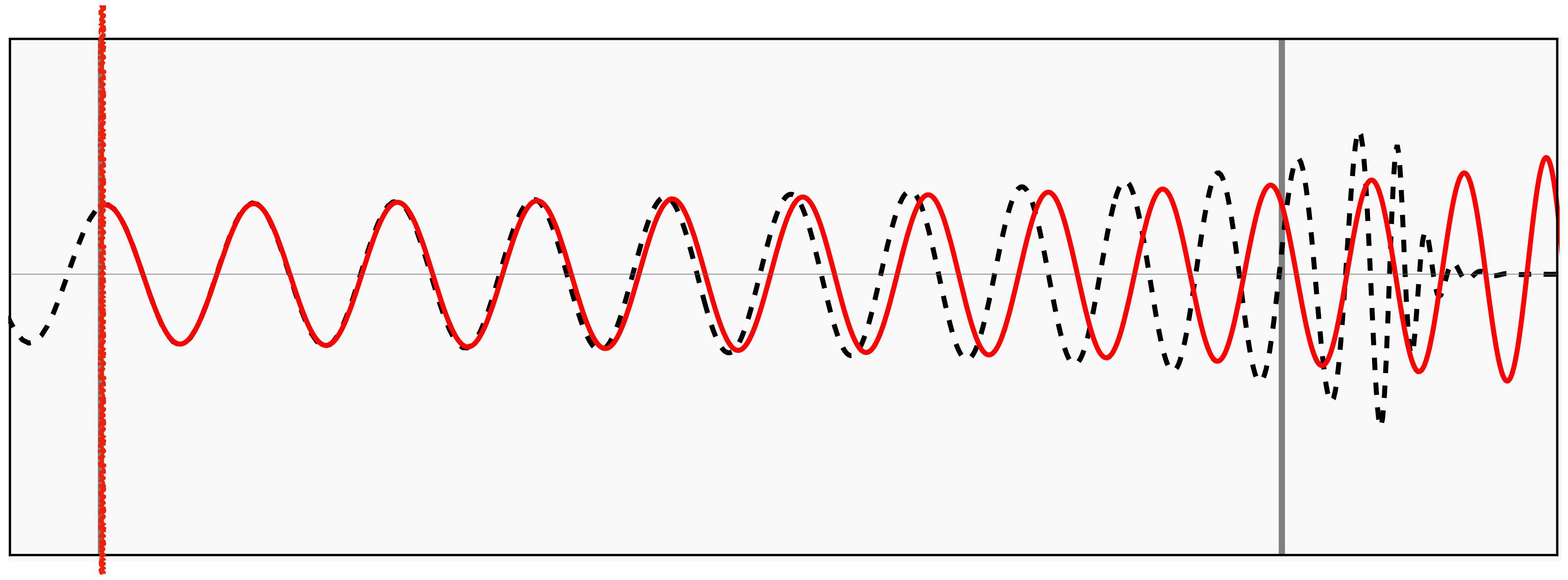

q=1
Re[$h_{2,2}$]
0.4
0.2
0.0
−0.2
−0.4
0
5000
10 000
15 000
20 000
25 000
t/M

Transition

Adiabatic and post-adiabatic waveform (q=10)



Leading order transition



Leading + first subleading order transition

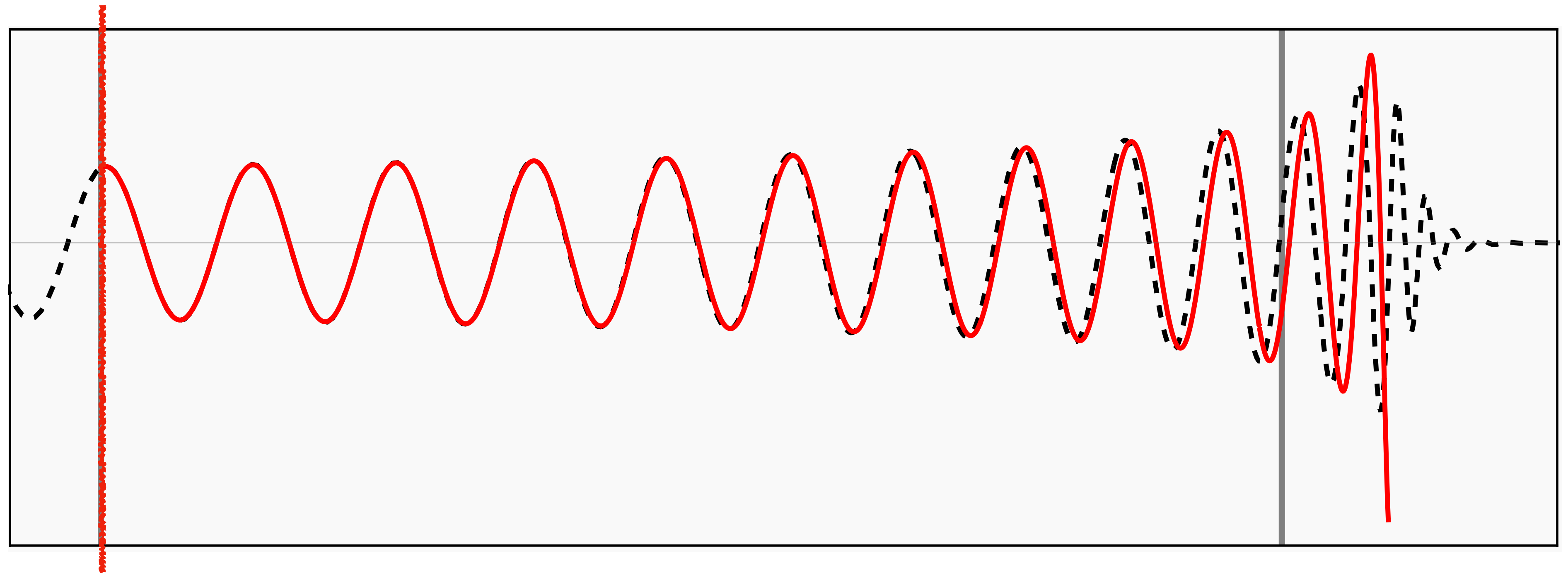


Figure credit:
Leanne Durkan
Lorenzo Küchler
Geoffrey Compère

Outlook

Conclusions

- We can now produce (quasi-circular) waveforms for **arbitrary mass ratios** in the time it takes to evaluate an interpolating function (milli-seconds).
- Can be used for **LISA data analysis**.
- For a complete waveform, we will need to attach a **transition to plunge** and **ringdown** at the point where our adiabatic approximation breaks down.
- Detailed comparisons with existing NR, PN and EOB show excellent agreement.
- Could be useful in the future as a test case for new EOB and PN results.
- Could be suitable for modelling IMRIs for **LIGO** once we have attached a model for the transition, plunge and ringdown.
- Used to calibrating other models (TEOBResumS and SEOBNRv5)
- It is relatively easy to add non-aligned spin on the secondary (precession), small spin on the primary, small eccentricity.

Outlook

- We are near the end of the beginning, but there are many more important things to get EMRI waveforms ready for LISA and IMRI waveforms for LIGO:
  - Improved formulations: **Teukolsky**, **Regge-Wheeler** gauges are much easier to work with as they only require us to solve a single scalar equation, but some foundational issues still to be worked out.
  - Check that certain components of the calculation can be left out without significant effects on waveform. For example, how well justified are we to **ignore** the slow evolution of the mass and angular momentum of the big black hole?
  - Everything described here extends **in principle** to generic orbits, but significant human effort required **in practice**.
  - Need a practical method for doing things in **Kerr** spacetime.
  - Incorporate finite-size (e.g. spin effects from smaller body) into waveform.
  - Can second order be done analytically (using MST-PN expansions)?

Thank you!