High frequency spacetimes in general relativity

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Nonlinear aspects of general relativity, October 11th 2023

Joint works with Cécile Huneau (Ecole Polytechnique),
Igor Rodnianski (Princeton)

Einstein equations

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- Goal: describe "high-frequency limits" of vacuum solutions.
- Are the limits vacuum? Can we describe the effective matter field of the limiting spacetime metric?
    - Describe the effective stress-energy-momentum tensor $= Ric(g) - \frac{1}{2}R(g)g$ of the limits.

Plane wave example

- For $\lambda \in (0, \lambda_0]$, consider plane waves:

$$g_\lambda = -2du\,dv + H_\lambda(v)^2(e^{G_\lambda(v)}(dy)^2 + e^{-G_\lambda(v)}(dz)^2).$$

  - The only non-trivial component of Ricci is $\mathrm{Ric}_\lambda(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = -\frac{1}{2}(G_\lambda'(v))^2 - \frac{2H_\lambda''(v)}{H_\lambda(v)}$
  - Prescribe G_λ and solve H_λ (locally) to get a vacuum solution.

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  - Prescribe G_λ and solve H_λ (locally) to get a vacuum solution.
- For $k \in C^\infty$ non-zero, define

$$G_\lambda(v) = \lambda k(v) \sin(\frac{v}{\lambda}).$$

- $H_\lambda(v)$ solves

$$\frac{2H_\lambda''(v)}{H_\lambda(v)} = -\frac{1}{2}(G_\lambda'(v))^2, \quad H_\lambda(0) = 1, \quad H_\lambda'(0) = 0.$$

Plane wave example

- $G_0(v) = \lim_{\lambda\to 0} G_\lambda(v) = 0$.
- H_λ admits a limit H_0 which satisfies

$$-\frac{2H_0''(v)}{H_0(v)} - \frac{1}{2}(G_0'(v))^2 = \frac{1}{4}(k(v))^2 \neq 0.$$

- $$g = -2du\,dv + H_0(v)^2((dy)^2 + (dz)^2)$$

  is the **non-vacuum(!)** limit and in fact solves

$$\mathrm{Ric}(g) = \frac{1}{4}(k(v))^2\,dv \otimes dv, \quad g^{-1}(dv, dk) = 0.$$

- This is a solution to the Einstein–null dust system.

High frequency limits in general relativity

- Isaacson (1968), Choquet-Bruhat (1969), MacCallum–Taub (1973), Burnett (1989), Ali–Hunter (1999), Green–Wald (2011), etc.
- Suppose there is a sequence of smooth metric g_n with $Ric(g_n) = 0$ and a smooth limit metric g_∞ such that for $\lambda_n \to 0$,

$$\|\partial^k (g_n - g_\infty)\|_{L^\infty} \lesssim \lambda_n^{1-k}, \quad k = 0, 1, 2, \cdots .$$

  What can we say about g_∞?
- $g_n - g_\infty$ "has" amplitude $\sim \lambda_n$ and frequency $\sim \lambda_n^{-1}$.
- Is g_∞ vacuum? If not, what is the "effective matter"?

Burnett's conjecture

The high-frequency limit in general relativity

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Conjecture (Burnett (1989))

Any solution to the Einstein–massless Vlasov system can locally be achieved as a high-frequency limit of vacuum solutions.

Burnett's conjecture

Conjecture: For any tensor field $T_{ab} = \mu_{[b|[am]|n]}{}^{mn}$ obtained from a one-parameter family of metrics satisfying conditions (i)–(iv), there exists a scalar field $a^2(x,k)$ defined on the null cotangent bundle, such that

$$T_{ab}(x) = \int a^2(x,k) k_a k_b \, dV_k \,, \tag{27}$$

$$k^m \nabla_m a^2(x,k) = 0 \,, \tag{28}$$

where $x \in M$, (x,k) is a point of the null cotangent bundle, and the integral is performed over the null cone.

It would also be interesting to know if the converse of the above conjecture is true: Given any metric g_{ab} on a fixed manifold M, and a scalar field $a^2(x,k)$ defined on the null cotangent bundle constructed from M, such that $k^m \nabla_m a^2(x,k) = 0$ and

$$G_{ab}[g] = \int a^2(x,k) k_a k_b \, dV_k \,,$$

then there exists a one-parameter family of metrics $g_{ab}(\lambda)$ satisfying conditions (i)–(iv) with $g_{ab}(0) = g_{ab}$. If this is true, we do not need to impose any restrictions on what fields $a^2(x,k)$ and g_{ab} we use, other than (27) and (28) above.

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then there exists a one-parameter family of metrics $g_{ab}(\lambda)$ satisfying conditions (i)–(iv) with $g_{ab}(0) = g_{ab}$. If this is true, we do not need to impose any restrictions on what fields $a^2(x,k)$ and g_{ab} we use, other than (27) and (28) above.

- The null dust equation above can be viewed as the massless Vlasov equation when the integral is a finite sum.

Physical interpretation of the Burnett conjecture

Think of $a^2(x,k)$ as a "particle" distribution function on the null cotangent bundle. Then, if this conjecture is true, we see that these particles do not interact directly, but that they do affect one another by their effect on the background space-time. Further, we then have a complete system for describing the effect of high-frequency waves on the background space-time. That is, on a fixed smooth manifold M, we have a metric g_{ab}, and on the null cotangent bundle constructed from M, we have a scalar field $a^2(x,k)$. The fields g_{ab} and $a^2(x,k)$ then evolve together via $G_{ab}[g] = T_{ab}$, with T_{ab} given in Eq. (27), and via Eq. (28).

Vlasov field as "poor man's gravitational waves"

Burnett's conjecture suggests studying the Einstein–massless Vlasov (or Einstein–null dust) equations to gain insights into the vacuum problem.

- Martin's talk yesterday: Israel–Poisson (1989), Moschidis (2018), Weissenbacher (2023)

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- Martin's talk yesterday: Israel–Poisson (1989), Moschidis (2018), Weissenbacher (2023)
- The Einstein–massless Vlasov (or Einstein–null dust) equations could be easier.
- This is useful when the phenomenon is dominated by high-frequency gravitational waves.
- Could provide insights on the physical phenomenon and even suggest a mathematical approach.

(In)stability of Cauchy horizons

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Inner-Horizon Instability and Mass Inflation in Black Holes

E. Poisson and W. Israel
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(Received 6 July 1989)

these questions.

The problem becomes tractable if one considers a simple model: A charged, spherical (Reissner-Nordström) black hole perturbed by crossflowing radial streams of infalling and outgoing lightlike particles. This model is very idealized, but there are good reasons for believing that it captures the essential physics. In the first place, the causal and horizon structures of the Reissner-Nordström and Kerr black holes are known to be very similar.[1] Secondly, the large blueshift of infalling gravitational waves means that high-frequency components will dominate near the Cauchy horizon, so that Isaacson's "effective stress-energy" description[4] for the waves (in effect, the "optical," graviton approximation) should be an adequate approximation.

We begin by setting up the basic equations of the

Remarks on Burnett's conjecture

- If Burnett's conjecture is true, then the effective stress-energy-momentum is trace-free, satisfies the dominant energy condition, and is positive semi-definite.

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- If Burnett's conjecture is true, then the effective stress-energy-momentum is trace-free, satisfies the dominant energy condition, and is positive semi-definite.
- In practice, one is also interested in describing how the limiting Vlasov field related to the sequence of solutions.

Green–Wald theorem

Theorem (Green–Wald (2011))

Given the setting as in Burnett's conjecture, the limiting stress-energy momentum tensor $T_{\mu\nu}$ satisfies

$$T_{\mu}{}^{\mu} = 0$$

and

$$T_{\mu\nu}X^{\mu}X^{\nu} \geq 0 \text{ for all causal vector } X.$$

Further remarks on Burnett's conjecture

The first conjecture is a question of compactness.

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- Weak convergence: $\psi_n \rightharpoonup \psi$ weakly if $\int \varphi\psi_n \to \int \varphi\psi$ for all nice enough φ.
  - $\cos(nx)$ has no pointwise limit as $n \to 0$, but $\cos(nx) \rightharpoonup 0$.
  - $\psi_n^{(1)} \rightharpoonup \psi^{(1)}$ and $\psi_n^{(2)} \rightharpoonup \psi^{(2)}$ do **not** imply $\psi_n^{(1)}\psi_n^{(2)} \rightharpoonup \psi^{(1)}\psi^{(2)}$
  - $\cos^2(nx) = \frac{1}{2}(1+\cos(2nx)) \rightharpoonup \frac{1}{2} \neq 0$.

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  - $\cos^2(nx) = \frac{1}{2}(1+\cos(2nx)) \rightharpoonup \frac{1}{2} \neq 0$.
- Model problems on $\mathbb{R}^2$:

$$\Delta\phi = (\partial_x\phi)^2, \quad \Box\phi = (\partial_t\phi)^2, \quad \Box\phi = (\partial_t\phi)^2 - (\partial_x\phi)^2,$$

  where $\Delta = \partial_x^2 + \partial_y^2$, $\Box = -\partial_t^2 + \partial_x^2$.
- Suppose ϕ_n are solutions, $\phi_n \to \phi_\infty$ uniformly and $\sup_n \|\partial\phi_n\|_{L^\infty} < +\infty$. What can we say about the limits?

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- Compare: incompressible Euler equations

$$\partial_t v + \mathrm{div}(v \otimes v) = -\nabla P, \quad \mathrm{div}\, v = 0. \tag{1}$$

Any smooth solution to the Euler–Reynolds system

$$\partial_t v + \mathrm{div}(v \otimes v) = -\nabla P + \mathrm{div} \mathring{R}, \quad \mathrm{div}\, v = 0 \tag{2}$$

can be achieved as limits of solutions to (1) (De Lellis–Székelyhidi, Isett).

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- Fact: If a sequence of vacuum metrics g_n have up to second order derivatives uniformly bounded in L^2, then any subsequential limit must also be vacuum.

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- The best known general result requires the curvature to be in L^2 (Klainerman–Rodnianski–Szeftel (2015)).
- Fact: If a sequence of vacuum metrics g_n have up to second order derivatives uniformly bounded in L^2, then any subsequential limit must also be vacuum.
- Thus we must go below the lowest known general threshold.

$\mathbb{U}(1)$ symmetry and gauge condition

We will study the Burnett's conjecture under symmetry.

- Take $(t = x^0, x^1, x^2, x^3) \in I \times \mathbb{R}^2 \times \mathbb{S}^1$ as coordinates. We impose a $\mathbb{U}(1)$ (translational) symmetry (i.e. no x^3 dependence)

$$^{(4)}g = e^{-2\psi} g + e^{2\psi}(dx^3 + \mathfrak{A}_\mu dx^\mu)^2.$$

- Impose that g is put into the form:

$$g = -N^2\,(dt)^2 + e^{2\gamma}\delta_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$

- Impose also that $\{t = \mathrm{const.}\}$ hypersurfaces have zero mean curvature.

Reduced equation in $\mathbb{U}(1)$ symmetry

Under $\mathbb{U}(1)$ symmetry, the Einstein equations for ${}^{(4)}g$ become an Einstein–wave map system (with target $\mathbb{H}^2$) on the $(2+1)$-dimensional space

$$\begin{cases} \Box_g \psi = -\frac{1}{2} e^{-4\psi} (g^{-1})^{\mu\nu} \partial_\mu \omega \partial_\nu \omega, \\ \Box_g \omega = (g^{-1})^{\mu\nu} \partial_\mu \omega \partial_\nu \psi, \\ Ric_{\mu\nu}(g) = 2\partial_\mu \psi \partial_\nu \psi + \frac{1}{2} e^{-4\psi} \partial_\mu \omega \partial_\nu \omega, \end{cases}$$

where $d\mathfrak{A} = \frac{1}{2} e^{-4\psi} ({}^* d\omega)$.

$\mathbb{U}(1)$ symmetry and gauge condition

- Together with the gauge conditions, equations become schematically (for $\Phi = (\psi, \omega)$ and $\mathfrak{g}$ components of g)

$$\begin{cases} \Box_g \Phi = \Gamma(\Phi) Q(\partial\Phi, \partial\Phi), \\ \Delta \mathfrak{g}_{\mu\nu} = (\partial\mathfrak{g})^2_{\mu\nu} + (\partial_\mu \Phi)(\partial_\nu \Phi). \end{cases}$$

  where $\Box_g$ is wave operator, Δ the Euclidean Laplacian on $\{t = \mathrm{const.}\}$, Q a g-null form.

- We now ask the PDE problem: If $(\Phi_n, \mathfrak{g}_n)$ are solutions and

$$\|\partial^k(\Phi_n - \Phi_\infty)\|_{L^\infty} \lesssim \lambda_n^{1-k}, \quad \|\partial^k(\mathfrak{g}_n - \mathfrak{g}_\infty)\|_{L^\infty} \lesssim \lambda_n^{1-k},$$

  what can we say about $(\Phi_\infty, \mathfrak{g}_\infty)$?

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Theorem (Huneau–L. (2019))

Under the symmetry assumption and gauge condition, the first Burnett conjecture is true.

- Moreover, the massless Vlasov field is given by a microlocal defect measure.

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- Moreover, the massless Vlasov field is given by a microlocal defect measure.
- Guerra–Teixeira da Costa (2021) gave a simplified treatment of one of the frequency regimes.

Simplified model

Recall the schematic equation

$$\begin{cases} \Box_{g_n}\Phi_n = \Gamma(\Phi_n)Q(\partial\Phi_n, \partial\Phi_n), \\ \Delta(g_n)_{\mu\nu} = (\partial\mathfrak{g}_n)^2_{\mu\nu} + (\partial_\mu\Phi_n)(\partial_\nu\Phi_n). \end{cases}$$

- Q is a null form with respect to g_n.
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- Q is a null form with respect to g_n.
- $\mathfrak{g}$ obeys elliptic equations.

The first (easy) step is to show that the limit satisfies

$$\begin{cases}\Box_{g_\infty}\Phi_\infty = \Gamma(\Phi_\infty)Q(\partial\Phi_\infty,\partial\Phi_\infty),\\ \Delta(g_\infty)_{\mu\nu} = (\partial\mathfrak{g}_\infty)^2_{\mu\nu} + \text{w-lim}_{k\to\infty}(\partial_\mu\Phi_{n_k})(\partial_\nu\Phi_{n_k}).\end{cases}$$

- A weaker form of compensated compactness.

Microlocal defect measures

Suppose $\psi_n \to \psi_\infty$ uniformly and $|\partial\psi_n| \lesssim 1$. Then

$$\int_{\mathbb{R}^{d+1}} (\partial_\alpha(\psi_{n_k} - \psi_\infty)) b(x) (m(\frac{1}{i}\nabla)(\partial_\beta(\psi_{n_k} - \psi_\infty)))\, dx$$

$$\to \int_{S^*\mathbb{R}^{d+1}} b(x) m(\xi) \xi_\alpha \xi_\beta \, d\nu.$$

for every smooth $b(x)$ and smooth $m(\xi)$ homogeneous of order 0, where $m(\frac{1}{i}\nabla)$ is the corresponding Fourier multiplier, i.e. $[\widehat{m(\frac{1}{i}\nabla)} f](\xi) = m(\xi)\widehat{f}(\xi)$.

- Gérard (1991), Tartar (1990).
- Measures the location and direction of failure of strong limit.

Microlocal defect measures and linear wave equation

- The microlocal defect measure associated with solutions to the linear wave equation satisfies the massless Vlasov equation.

Microlocal defect measures and linear wave equation

- The microlocal defect measure associated with solutions to the linear wave equation satisfies the massless Vlasov equation.
- Our setting is quasilinear: need special structure and cancellations.

Construction of high-frequency spacetimes

Theorem (Huneau–L. (2017))

Given a generic, small data, smooth, local-in-time, $\mathbb{U}(1)$*-symmetric solution to the Einstein–null dust system with a finite number of families of null dust, there exists a sequence of vacuum solutions whose limit is the given solution.*

Geometric optics construction

- The proof is based on an ansatz

$$\Phi_\lambda = \Phi_0 + \lambda \sum_{\mathbf{A}} F_{\mathbf{A}} \sin(\tfrac{u_{\mathbf{A}}}{\lambda}) + \text{error}.$$

- A precise understanding of interaction of waves and (non)-generation of higher harmonics.
- Null dust is generated in the limit: $\partial_\mu \Phi_\lambda \partial_\nu \Phi_\lambda \rightharpoonup \partial_\mu \Phi_0 \partial_\nu \Phi_0 + \frac{1}{2} \sum_{\mathbf{A}} F_{\mathbf{A}}^2 \partial_\mu u_{\mathbf{A}} \partial_\nu u_{\mathbf{A}}$.

From vacuum to dust to Vlasov

- Finite sums of delta measures are weak-* dense in the set of all Radon measures.
- Work in progress: use this theorem as a building block to construct more general limiting solutions to the Einstein–massless Vlasov system.

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- Work in progress: use this theorem as a building block to construct more general limiting solutions to the Einstein–massless Vlasov system.
- Key difficulty: understand (and sharpen) the dependence on the number of families of dust.
- Need to use more precise eikonal functions.

Geometric optics construction

- One may wish to carry out the whole program in 3+1 dimensions, e.g., in generalized wave coordinates.

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Theorem (Touati (2022))

There exists one parameter families of vacuum solutions representing high-frequency geometric optics constructions, whose limits are solutions to the Einstein–null dust system.

- This upgrades the approximate solutions of Choquet-Bruhat (1969) to actual solutions.

Angularly regular spacetimes

If one considers “angularly regular” spacetimes, there is still a good low-regularity well-posedness result.

Theorem (L.–Rodnianski (2013))

Consider metric of the form

$$g = -4\Omega^2\,\mathrm{d}u\mathrm{d}\underline{u} + \gamma_{AB}(\mathrm{d}\theta^A - b^A\mathrm{d}u)(\mathrm{d}\theta^B - b^B\mathrm{d}u).$$

Local existence and uniqueness holds with the class where for $\mathfrak{g} \in \{\log\Omega, \gamma, b\}$,

$$\partial_u\mathfrak{g} \in L^2_u, \quad \partial_{\underline{u}}\mathfrak{g} \in L^2_{\underline{u}}$$

as long as there are extra regularity in the θ*-directions.*

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as long as there are extra regularity in the θ*-directions.*

- The profile of $\mathfrak{g}$ can be very general, not necessarily oscillatory with a particular profile.

Burnett's conjectures

Theorem (L.–Rodnianski (2020))

Both conjectures of Burnett are true in the setting of angularly regular spacetimes and the limits correspond to solutions to the Einstein–null dust system.

Null dust shell solutions

PROCEEDINGS

OF

THE ROYAL IRISH ACADEMY

PAPERS READ BEFORE THE ACADEMY.

1

A MODEL IN GENERAL RELATIVITY FOR THE INSTANTANEOUS TRANSFORMATION OF A MASSIVE PARTICLE INTO RADIATION

By J. L. SYNGE

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[Read 27 May, 1957. Published 23 December, 1957.]

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- Many other explicit solutions known, including interaction of null dust shells. (Dray, Gibbons, Penrose, Redmount, t' Hooft, ...)
- Question: Can one prove a general local existence result for null dust shells?

Plane wave solutions with a null dust shell

- Plane wave ansatz

$$g = -2du\,dv + H(v)^2(e^{G(v)}(dy)^2 + e^{-G(v)}(dz)^2).$$

- Let $G(v) = 0$ and $H(v) = \begin{cases} 1 & \text{if } v \leq 0 \\ 1 - v & \text{if } v > 0 \end{cases}$
- We compute

$$Ric(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}) = -\frac{2H''}{H} - \frac{1}{2}(G')^2 = -\frac{2H''}{H} = 2\delta_0.$$

- Solution to the Einstein–null dust system where the null dust is a delta measure.

Null dust shell solution as a vacuum limit

- Plane wave ansatz

$$g = -2du\,dv + H(v)^2(e^{G(v)}\,(dy)^2 + e^{-G(v)}\,(dz)^2),$$

- Let $\mathfrak{G}(v)$ be smooth and compactly supported, with $\int \mathfrak{G}'^2(v)\,dv = 4$.
- Define $G_\epsilon(v) = \epsilon^{\frac{1}{2}}\mathfrak{G}(\epsilon^{-1}v)$.
- Can solve for H_ϵ in a uniform region to obtain vacuum solutions.
- $\lim_{\epsilon\to 0} G_\epsilon(v) = 0$,
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- The limit is the null dust shell solution we saw earlier. This involves concentration and is slightly different from "high-frequency limit".

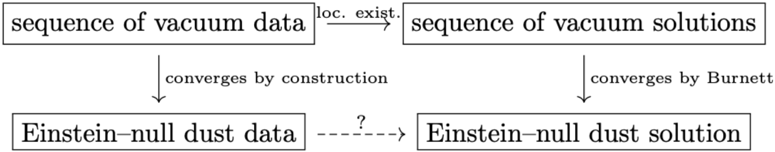
Local existence for general null dust shells

Using the general low-regularity local existence result in **vacuum**, we prove

Theorem (L.–Rodnianski (2020))

Given any suitable null dust shell data, there exists a unique local solution to the Einstein–null dust system which features the interactions of two null shells.

Local existence for general null dust shells



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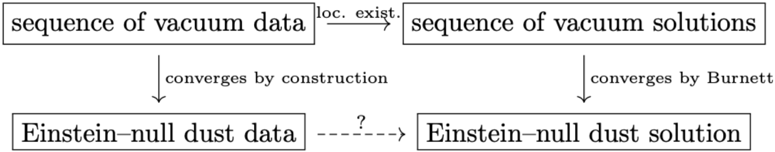

- The local existence result holds **without any symmetry assumptions**.
- The theorem applies more generally to construct solutions to the Einstein–null dust system where the null dust is **measure-valued**.

From Synge to Christodoulou

- Synge's solution is not just a local propagating null shell, but is also an example of gravitational collapse.
  - A trapped surface forms in dynamical evolution!
- A much harder problem is whether a trapped surface can form in evolution in vacuum.

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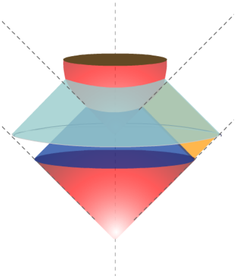
Theorem (Christodoulou (2008))

A trapped surface can form dynamically in vacuum by the focusing of gravitational waves.

- An, Athanasiou, Jaffe, Klainerman, Le, Lesourd, Li, Liu, L., Mei, Reiterer, Rodnianski, Trubowitz, Yu, ...

From Synge to Christodoulou and back

- Christodoulou's construction is based on what he called the **short pulse method**.



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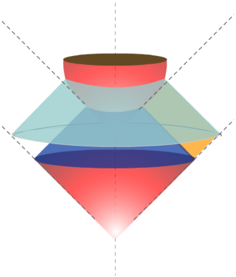

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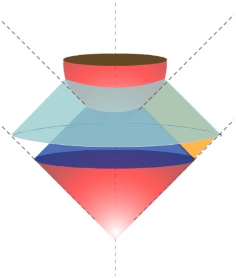

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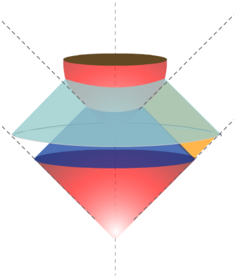

- If we take the length scale $\to 0$, the spacetime converges to the Synge solution.
- Thus Christodoulou's construction can be viewed as an "approximation" of the Synge solution.

Thank you!