Non-linear stability of black holes: a mathematical overview

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Nonlinear aspects of General Relativity, PCTS, October 2023

Nonlinear stability of a black hole family

Theorem

General initial data sufficiently close to a member of a black hole family evolve according to the Einstein equation to a spacetime which

- *possesses a complete future null infinity* $\mathscr{I}^+$ *whose past is bounded by a future complete event horizon* $\mathcal{H}^+$*,*
- *asymptotes back to a a member of the same black hole family as (an appropriate notion of) time goes to infinity.*

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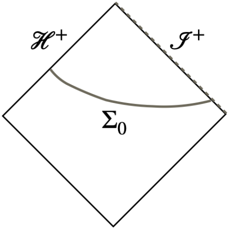

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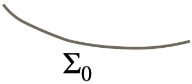

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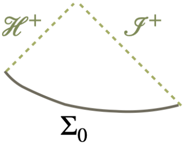

Known results in nonlinear stability

- $\Lambda = 0$:
  - Minkowski [Christodoulou-Klainerman 1993, Lindblad-Rodnianski 2003]
  - Schwarzschild
    - for axially symmetric polarized perturbations [Klainerman-Szeftel 2017],
    - for data which lie on a codimension-3 "submanifold" of moduli space [Dafermos-Holzegel-Rodnianski-Taylor 2021]
  - Kerr for $|a| \ll M$ [Klainerman-Szeftel 2019, Klainerman-Szeftel 2021, Shen 2022, G.-Klainerman-Szeftel 2022]

- $\Lambda > 0$:
  - de Sitter [Friedrich 1986]
  - Kerr-de Sitter for $|a| \ll M$ [Hintz-Vasy 2016]
  - Kerr-Newman-de Sitter for $|a| \ll M$ [Hintz 2018]

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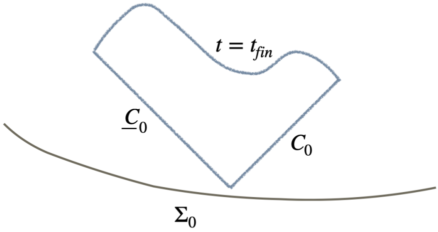

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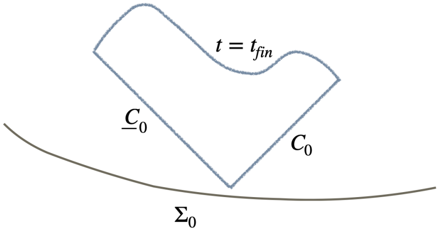

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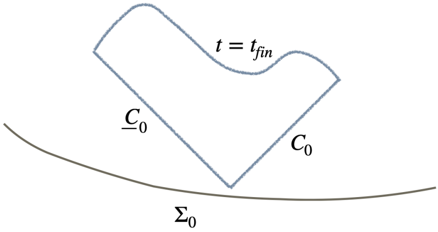

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$$\mathcal{B} = \{t_{fin} \in [0, \infty) : \textbf{bootstrap} \,\&\, \textbf{gauge} \text{ assumptions hold in } \mathcal{D}_{fin}\}.$$

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$$\sup_{\mathcal{D}_{fin}} |g - g_{M_f, a_f}|, |\Gamma - \Gamma_{M_f, a_f}|, |R - R_{M_f, a_f}| \leq \epsilon.$$

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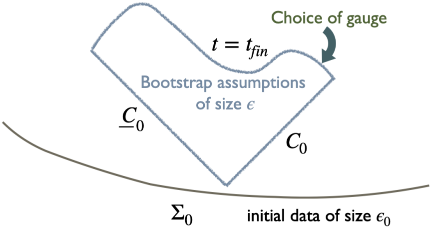

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    $$\sup_{\mathcal{D}_{fin}} |g - g_{M_f, a_f}|, |\Gamma - \Gamma_{M_f, a_f}|, |R - R_{M_f, a_f}| \leq C\epsilon_0 < \epsilon.$$

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Then $\mathcal{B} = [0, \infty)$ and therefore the solution is global.

Improvement of the bootstrap assumptions

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Aside: Second order perturbations of Kerr (Ripley-Loutrel-G.-Pretorius 2020)

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g &= g_{M,a} + \epsilon \cdot g^{(1)} + O(\epsilon^2) \\
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- Linear perturbations are described by the Teukolsky equations for $\Psi_4^{(1)}$:

$$\mathcal{T}^{[-2]}[\Psi_4^{(1)}] = 0$$

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  The source term depends on all first order geometric quantities in addition to $\Psi_4^{(1)}$: need *reconstruction of the metric perturbation.*

In the outgoing radiation gauge

$$g_{\mu}^{(1)\,\mu} = g^{\mu\nu} g_{\mu\nu}^{(1)} = 0$$

all first order quantities can be derived from the solution of the Teukolsky equation for $\Psi_4^{(1)}$ through null transport equations:

$$\begin{aligned}
&\mathcal{T}^{[-2]}[\Psi_4^{(1)}] = 0 \\
&(\Delta + \mu + \overline{\mu} + 3\gamma - \overline{\gamma})^{(0)}\, \lambda^{(1)} = -\Psi_4^{(1)} \\
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This allows to compute the source term $\mathcal{S}^{(2)}[\Gamma^{(1)}, R^{(1)}]$ and solve the Teukolsky equation for $\Psi_4^{(2)}$.

Back to the improvement of the bootstrap ass. for $\mathcal{T}^{[\pm 2]}(\alpha^{\pm 2}) = O(\epsilon^2)$

The scalar version of the linear **Teukolsky equation** for $s = \pm 2$ is

$$\Box_{g_{M,a}} \alpha^{[s]} + \frac{2s}{\rho^2}(r - M)\partial_r \alpha^{[s]} + \frac{2s}{\rho^2}\left(\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\phi \alpha^{[s]}$$
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We actually consider tensorial equations for

$$\alpha_{ab} = W(e_a, e_4, e_b, e_4), \qquad \underline{\alpha}_{ab} = W(e_a, e_3, e_b, e_3),$$

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It turns out that the Teukolsky equation is difficult to analyze directly in view of its first order terms. One needs to consider instead higher order quantities[Dafermos-Holzegel-Rodnianski 2016-2017, Ma 2017] that satisfy more treatable equations, called **(generalized) Regge-Wheeler equations**, which can be analyzed in much the same way as the scalar wave equation

$$\Box_g \psi = 0.$$

A little bit of history...

Historically, there are two versions of linearizing the Einstein equation:

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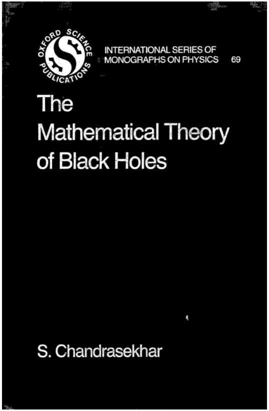

Consider metric perturbations of the form

$$g = e^{2\nu}dt^2 - e^{2\psi}\left(d\phi - \omega dt - q_2 dx^2 - q_3 dx^3\right)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2$$

of the Schwarzschild metric, with

$$e^{2\nu} = e^{-2\mu_2} = 1 - \frac{2M}{r}, \qquad e^{\mu_3} = r, \qquad e^{\psi} = r\sin\theta, \qquad \omega = q_2 = q_3 = 0.$$

The decoupled equation for axial perturbations (i.e. those modifying ω, q_2, q_3) is given by the so-called **Regge-Wheeler equation**:

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In general, we call **Regge-Wheeler equation** an equation of the form

$$\Box_g\psi - V\psi = 0, \qquad \text{for a positive real potential } V$$

In order to obtain a decoupled equation for perturbations of Kerr, one needs to use curvature perturbations. The relevant Weyl scalars are the spin ± 2 quantities

$$\begin{aligned}\alpha^{[+2]} &= \Psi_0 = -W(l, m, l, m), \\ \alpha^{[-2]} &= (r - ia\cos\theta)^4 \Psi_4 = -(r - ia\cos\theta)^4 W(n, \overline{m}, n, \overline{m})\end{aligned}$$

that satisfy the so-called **Teukolsky equation** of spin s

$$\Box_{g_{M,a}} \alpha^{[s]} + \frac{2s}{\rho^2}(r - M)\partial_r \alpha^{[s]} + \frac{2s}{\rho^2}\left(\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right)\partial_\phi \alpha^{[s]}$$
$$+\frac{2s}{\rho^2}\left(\frac{M(r^2 - a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t \alpha^{[s]} + \frac{1}{\rho^2}(s - s^2\cot^2\theta)\alpha^{[s]} = 0$$

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$$\Box_g \alpha - V\alpha = c_1 \partial_r \alpha + c_2 \partial_\phi \alpha + c_3 \partial_t \alpha$$

with c_1, c_2, c_3 complex functions, and V a real function.

Chandrasekhar describes a **transformation theory** from mode-decomposed curvature perturbations (solution to the Teukolsky equation) to mode-decomposed metric perturbations (solution to the Regge-Wheeler equation):

$$\begin{array}{rcl} \text{Teukolsky equation} & \rightarrow & \text{Regge-Wheeler equation} \\ \Box_g \alpha - V\alpha = c_1\partial_r \alpha + c_2 \partial_\phi \alpha + c_3 \partial_t \alpha & \rightarrow & \Box_g \psi - V\psi = 0 \end{array}$$

30. The transformation theory

The problem is to express the solution of an equation of the form

$$\Lambda^2 Y + P\Lambda_- Y - QY = 0$$

in terms of the solution of a one-dimensional wave-equation

$$\Lambda^2 Z = VZ,$$

Since Y and Z both satisfy equations of the second order, there is no restriction to assuming that Y is a linear combination of Z and its derivative. But instead of making this assumption simply as we did in §26 when considering a similar problem, we shall now assume that

$$Y = f\Lambda_+\Lambda_+ Z + W\Lambda_+ Z, \tag{287}$$

where (cf. equation (29))

$$\Lambda_\pm = \frac{d}{dr_*} \pm i\sigma \quad \text{and} \quad \frac{d}{dr_*} = \frac{\Delta}{r^2}\frac{d}{dr}.$$

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Shaking up the past: a new Chandrasekhar transformation

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so that

$$\Box_2\psi_{ab} - \frac{4}{r^2}\left(1-\frac{2M}{r}\right)\psi_{ab} = 0$$

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Energy estimates for the Teukolsky variables can be obtained from the Regge-Wheeler equation through the Chandrasekhar transformation[DHR 2016].

In Kerr, a similar transformation holds[Ma 2017, DHR 2017], and the Teukolsky variables get transformed into a **generalized version of the Regge-Wheeler equation**:

$$\square_2 \psi - V\psi - i\frac{4a\cos\theta}{\rho^2}\nabla_{\partial_t}\psi = a \cdot L_\psi[\alpha], \tag{1}$$

where

- V is a positive real potential and $\rho^2 = r^2 + a^2\cos^2\theta$
- $L_\psi[\alpha]$ denotes linear terms in up to two derivatives of α. Schematically,

$$L_\psi[\alpha] = c_1(r,\theta)\nabla_{\partial_t}\nabla_{e_3}\alpha + c_2(r,\theta)\nabla_{\partial_\phi}\nabla_{e_3}\alpha + c_3(r,\theta)\nabla_{e_3}\alpha + c_4(r,\theta)\alpha$$

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- Even though equation (1) has a first order term in ∂_t, it satisfies good divergence properties.
- Due to the presence of $L_\psi[\alpha]$ on the right hand side of (1), one has to view the wave equation in (1) as coupled with the defining equations for ψ.

In the non-linear picture, we define [G.-Klainerman-Szeftel 2020-2022] the Chandrasekhar transformation as

$$\psi_{ab} = f(r,\theta)\Big(\nabla_{e_3}\nabla_{e_3}A_{ab} + C_1\nabla_{e_3}A_{ab} + C_2A_{ab}\Big), \qquad A = \alpha + i\,{}^\star\alpha.$$

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Error terms which, according to the bootstrap assumptions, would decay "too slowly" to close the argument do not appear in $\mathrm{Err}[\check{\Gamma},\check{R}]$: this is a manifestation of the null condition:

$$\square\psi = m(d\psi, d\psi), \qquad \text{with} \quad m(\xi,\xi) = 0 \qquad \text{if } g(\xi,\xi) = 0.$$

The bootstrap assumptions for α and $\underline{\alpha}$ are finally improved by obtaining **energy estimates** for the gRW equation for ψ and $\underline{\psi}$.

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Conservation of $E(t) = \int_{\Sigma_t} |\partial_t \psi|^2 + |\nabla \psi|^2$ is obtained through

$$0 = \Box\psi \cdot \partial_t \psi = \big(- \partial_t^2 \psi + \triangle \psi \big) \cdot \partial_t \psi = -\frac{1}{2} \partial_t \big(|\partial_t \psi|^2 + |\nabla \psi|^2 \big) + \nabla \cdot (\partial_t \psi \nabla \psi)$$

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Using the bootstrap assumptions on $\check{\Gamma}$, $\check{R}$, one obtains energy estimates for ψ and α, which are "almost" gauge invariant.

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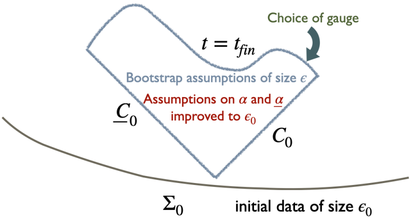
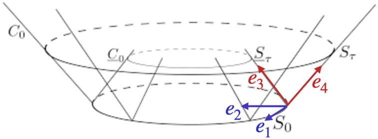


Preamble to gauge: non-integrable frames

To each point of $\mathcal{D}_{fin}$ we can associate a null frame $\{e_3, e_4, e_a\}$, with e_3, e_4 null vectorfields and $\{e_a\}_{a=1,2}$ orthogonal to e_3 and e_4.



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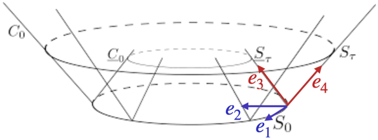

In order to use the Teukolsky equation, we consider null frames which are small perturbations of the **principal null frame** in Kerr, such as

$$e_{3,4} = \frac{(r^2+a^2)}{\rho\sqrt{\Delta}}\partial_t + \frac{a}{\rho\sqrt{\Delta}}\partial_\phi \pm \frac{\sqrt{\Delta}}{\rho}\partial_r, \quad e_1 = \frac{1}{\rho}\partial_\theta, \quad e_2 = \frac{1}{\rho\sin\theta}\partial_\phi + \frac{a\sin\theta}{\rho}\partial_t.$$

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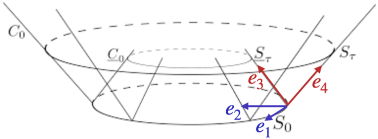

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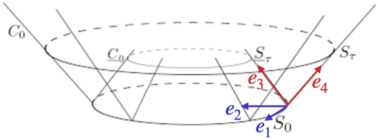

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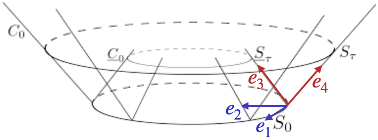

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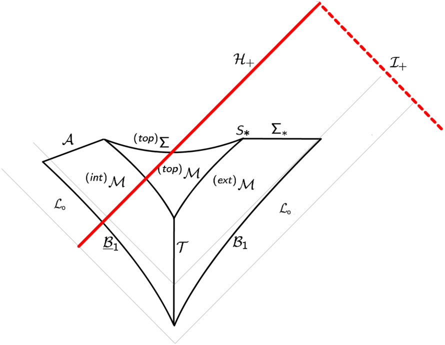
[One could instead use the double null coordinates[Pretorius-Israel 1998], but then the Teukolsky variables would not be small.]

Gauge assumptions at final time

The gauge assumptions on $\mathcal{D}_{fin}$ are imposed at the "final" sphere S_* and hypersurface Σ_* [Klainerman-Szeftel 2019].



The sphere S_* is a codimension 2 compact surface, unrelated to the initial conditions, on which some geometric quantities have the same value as in Schwarzschild, and which are equipped with effective coordinates (θ, ϕ).

Gauge assumptions on $\mathcal{D}_{fin}$

In order to finally improve the bootstrap assumptions for all the $\check{\Gamma}$ and $\check{R}$, we need gauge assumptions on $\mathcal{D}_{fin}$ as well.

Two gauges are introduced[Klainerman-Szeftel 2021]:

- Geodesic gauge, which is a generalization of the geodesic foliation in the non-integrable case: good for decay estimates, bad for loss of derivatives

$$\nabla_{e_4}\check{\Gamma} = \nabla\check{\Gamma} + \check{R}$$

- Temporal gauge, which favors transport equations along a null direction: no loss of derivatives, bad for decay estimates

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(The control of gauge-dependent quantities holds for $|a| < M$, if you have control of the almost-gauge invariant quantities!)

Mass, angular momentum and center of mass frame

On the final sphere S_* one defines the mass M_f, the angular momentum a_f and the virtual axis of rotation of $\mathcal{D}_{fin}$, which converge in the limit to the final parameters M_∞, a_∞ [Klainerman-Szeftel 2019]:

$$\frac{2M_f}{r} := 1 + \frac{1}{16\pi}\int_{S_*} \mathrm{tr}\chi \mathrm{tr}\underline{\chi}, \qquad a_f := \frac{r^3}{8\pi M}\int_{S_*} J^{(0)}\mathrm{curl}\beta.$$

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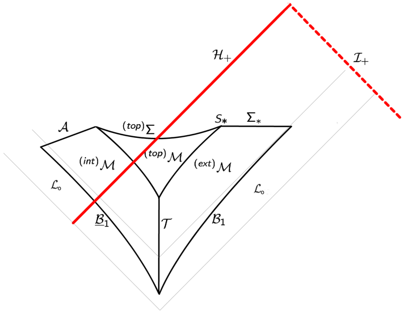
Since the initialization of S_* does not make a direct reference to the initial conditions, when it is transported along Σ_* to a sphere on the initial data this induces a new foliation on the initial data which differs substantially from the original one, due to a shift to the center of mass frame of the final black holes (gravitational wave recoil).



Thank you for your attention!