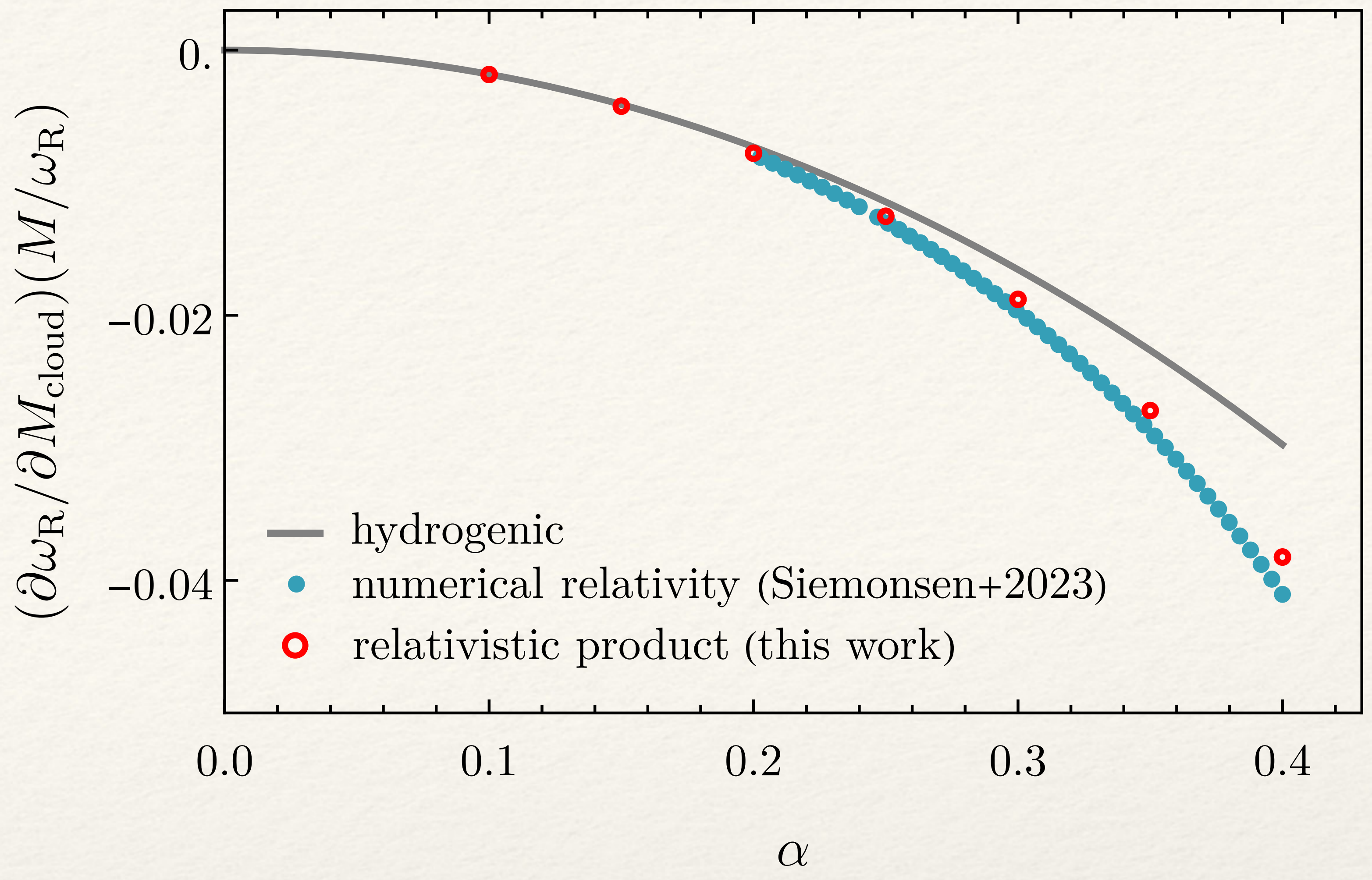
$$\langle\langle \psi_{\ell_1 m_1 \omega_1}, \psi_{\ell_2 m_2 \omega_2} \rangle\rangle = 8\pi M^{4/3} \delta_{m_1 m_2} e^{-i(\omega_2-\omega_1)t} \int_{C_*} dr_* \int_0^\pi d\theta \, \frac{(r^2+a^2)\sin\theta}{\Delta} S_1(\theta) S_2(\theta) R_1(r) R_2(r) \left( -\frac{i\Lambda}{\Delta}(\omega_1+\omega_2) + \frac{2iMra}{\Delta}(m_1+m_2) + 2\left[ -r - ia\cos\theta + \frac{M}{\Delta}(r^2-a^2) \right] \right)$$




*PCTS/PGI Workshop on Nonlinear Aspects of General Relativity*

# Orthogonality of quasinormal modes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Based on 2210.15935, 2309.10021

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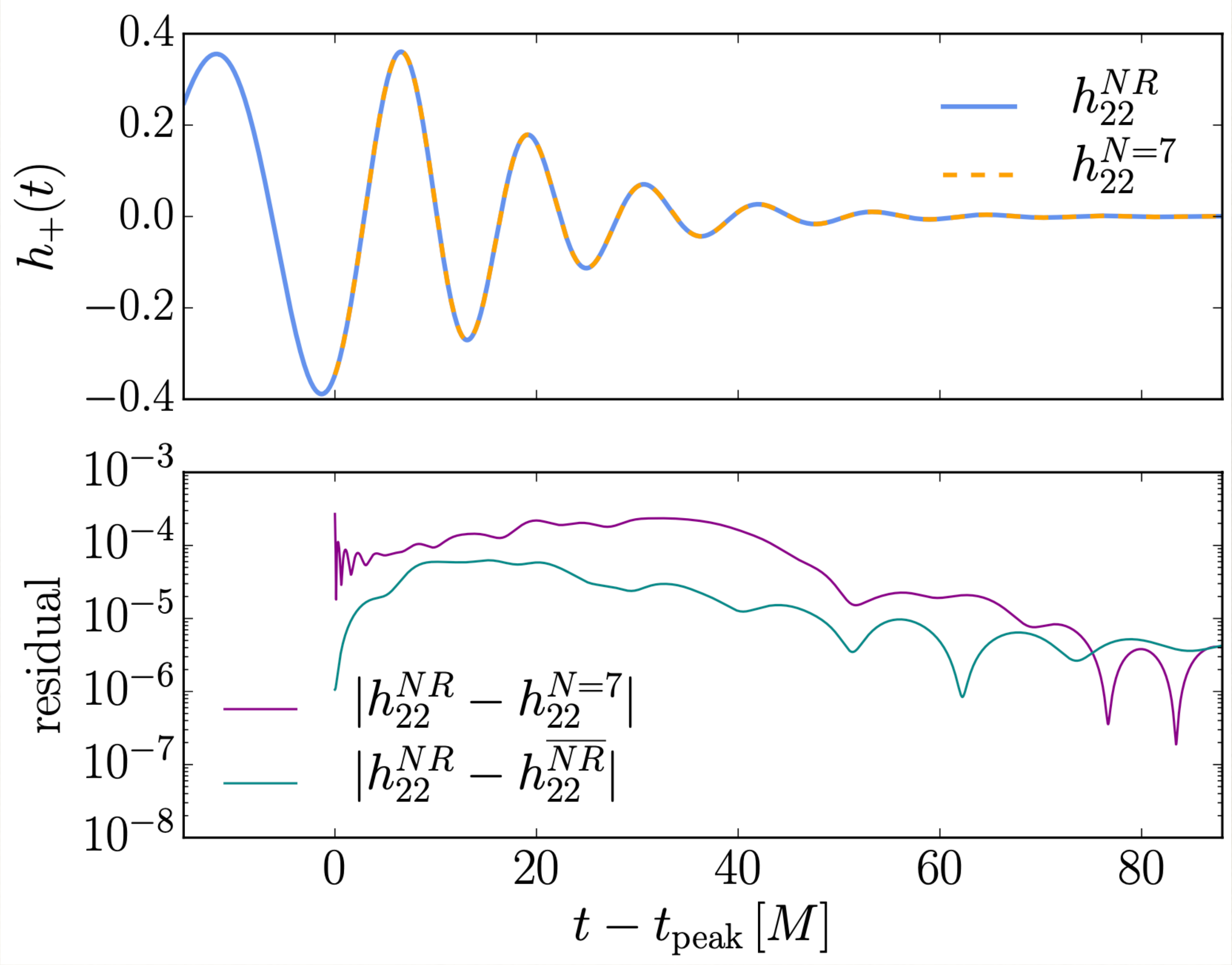
# Outline

- Motivation
- Conserved currents for Kerr $\longrightarrow$ relativistic product
- Quasinormal modes and orthogonality
- Time-dependent perturbation theory for black holes
- Application to perturbative frequency shifts

# Motivation

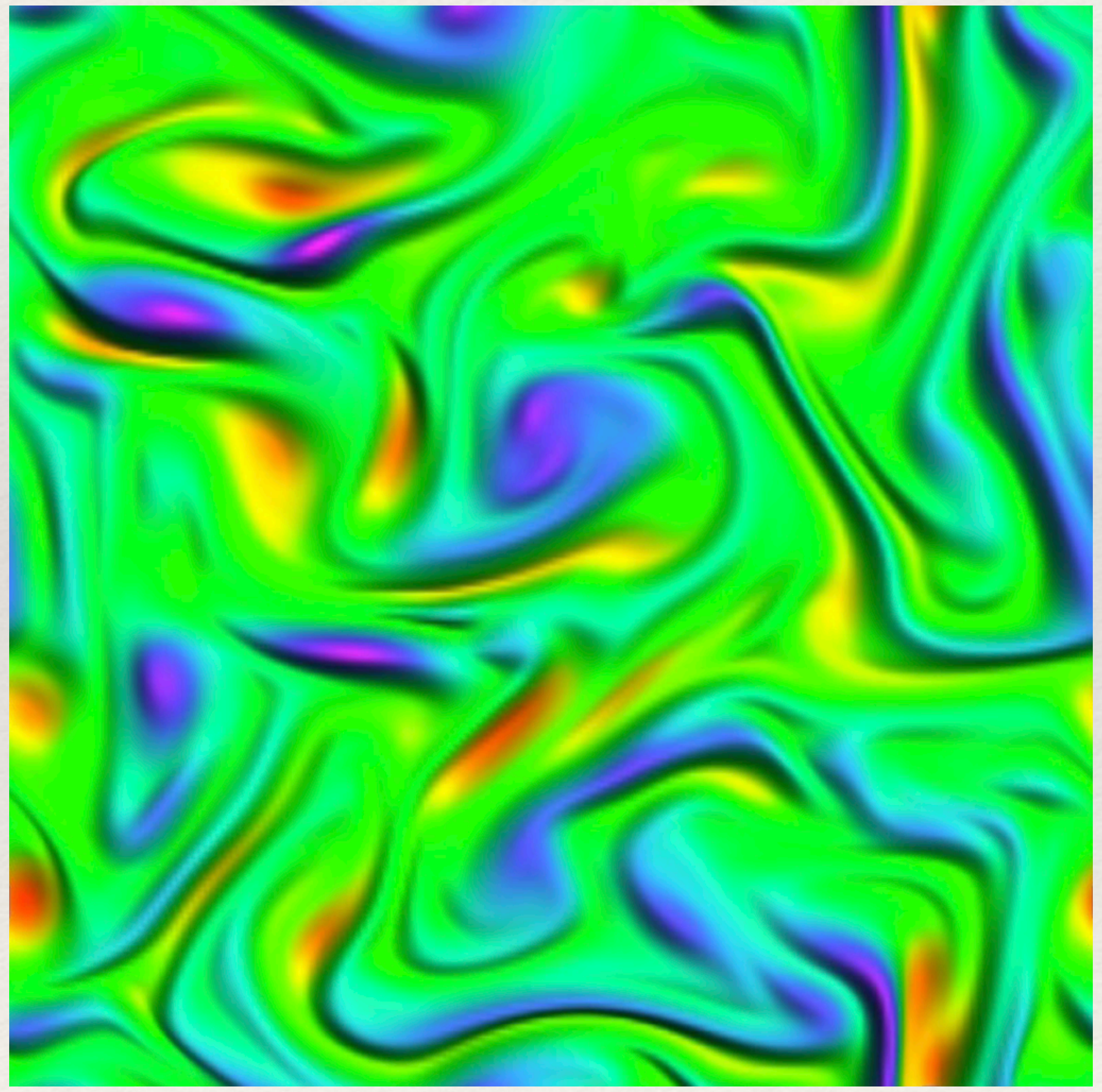
## How can we use quasinormal modes in perturbation theory beyond linear order?

- Post-merger ringdown



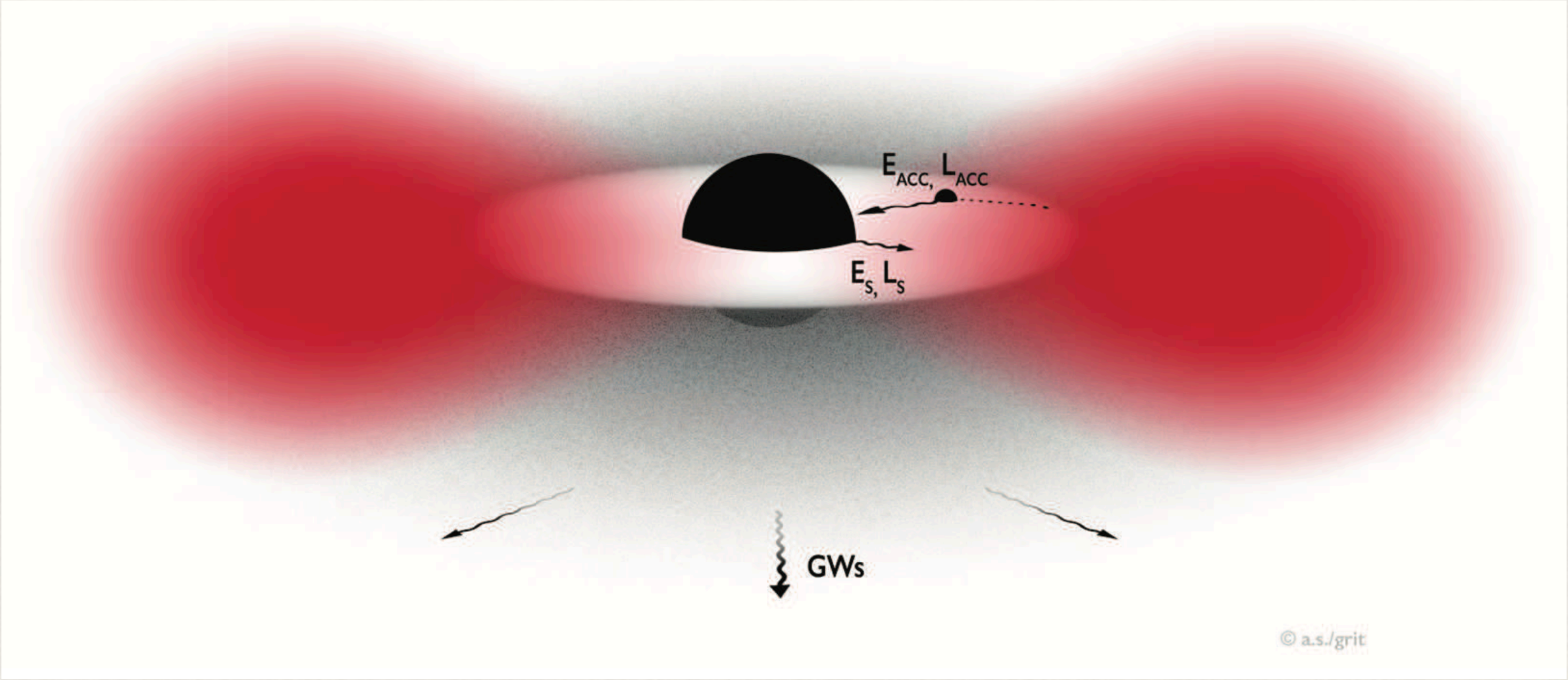

Giesler+ (2019)

- Gravitational turbulence



Green+ (2013)

- Black-hole boson clouds



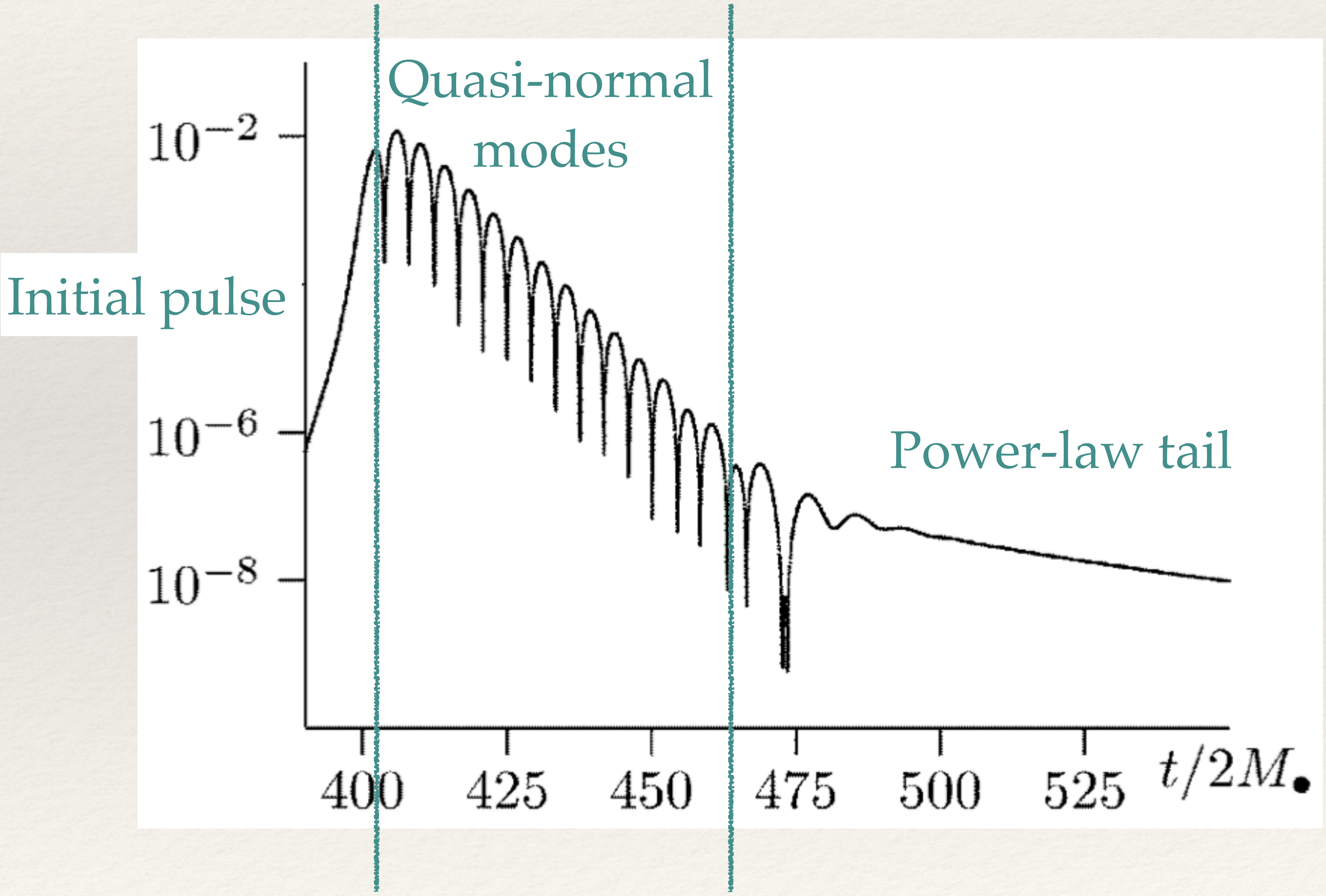

Brito+ (2014)

# Normal and quasinormal modes

| | Normal | Quasinormal |
|---|---|---|
| System | self-adjoint | dissipative |
| Frequencies | $\omega \in \mathbb{R}$ | $\omega \in \mathbb{C}$ |
| Orthogonal | ✅ (inner product) | ? |
| Complete | ✅ | ❌ |

spectral theorem

- Incomplete description of ringdown



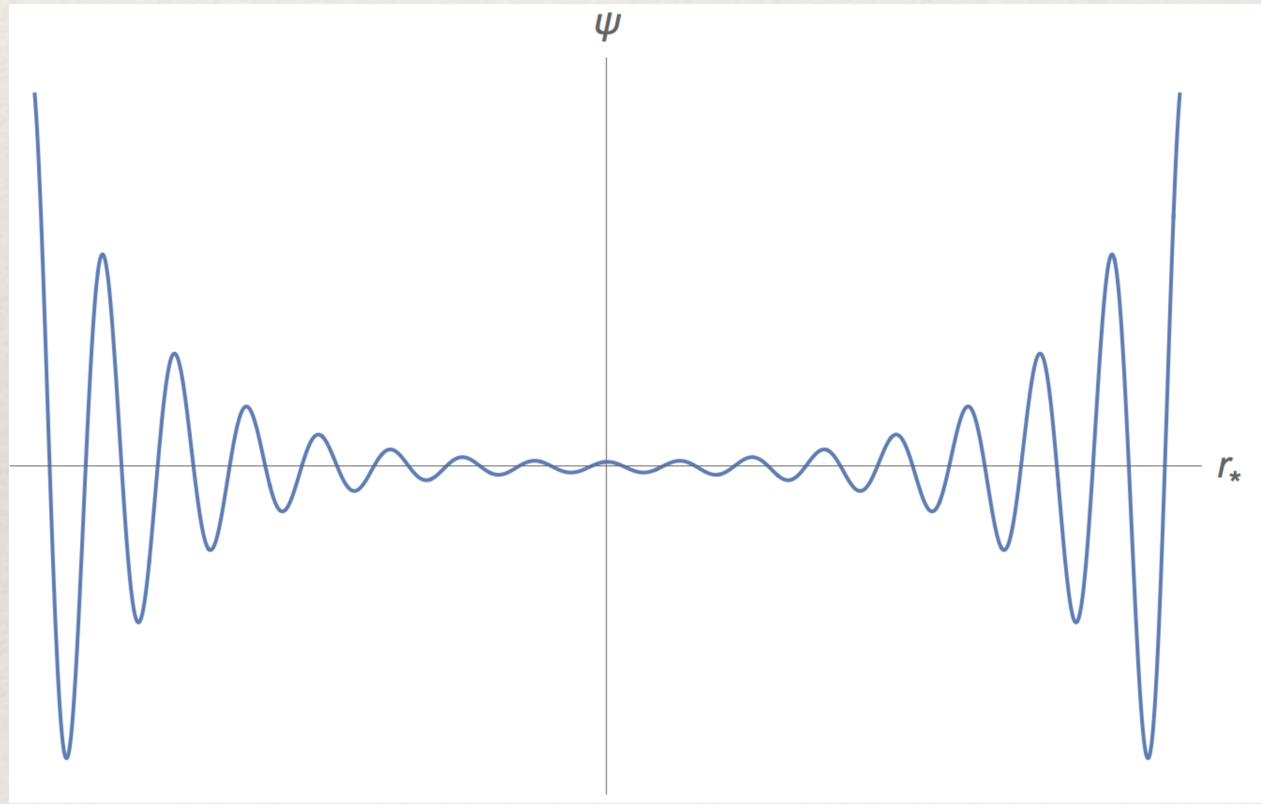
# Normal and quasinormal modes

| | Normal | Quasinormal |
|---|---|---|
| System | self-adjoint | dissipative |
| Frequencies | $\omega \in \mathbb{R}$ | $\omega \in \mathbb{C}$ |
| Orthogonal | ✅ (inner product) | ? |
| Complete | ✅ | ❌ |

spectral theorem

- Typically blow up at bifurcation surface and infinity

$$e^{-i\omega t} \xrightarrow{t\to\infty} 0 \quad \Longrightarrow \quad e^{-i\omega r_*} \xrightarrow{r_*\to-\infty} \infty \qquad e^{i\omega r_*} \xrightarrow{r_*\to\infty} \infty$$




- Not clear how to define a product between modes

# Normal and quasinormal modes

| | Normal | Quasinormal |
|---|---|---|
| System | self-adjoint | dissipative |
| Frequencies | $\omega \in \mathbb{R}$ | $\omega \in \mathbb{C}$ |
| Orthogonal | ✅ (inner product) | ✅ with suitable **bilinear form** |
| Complete | ✅ | ❌ |

spectral theorem

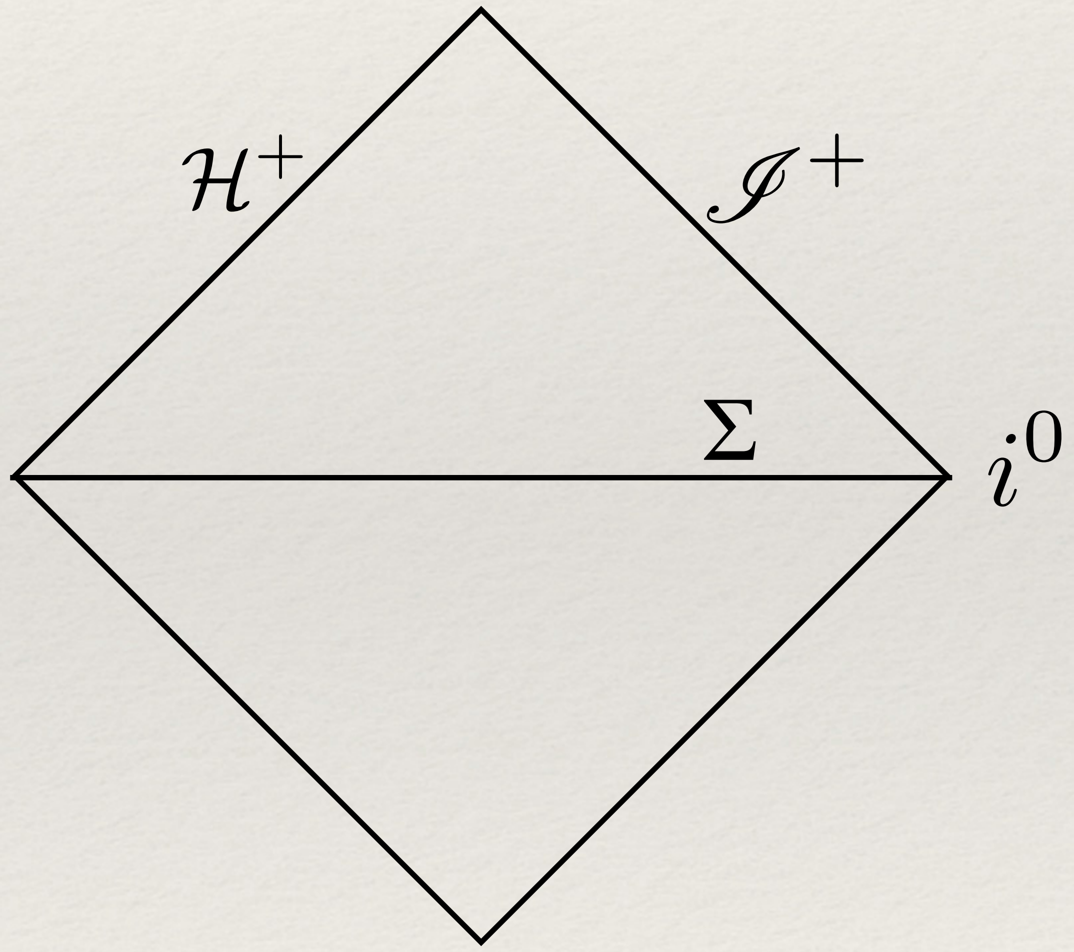
this work

Rest of talk:

- Develop the bilinear form (aka relativistic product)
- Time-dependent perturbation theory
- Practical example

# Conserved currents

- **Basic idea:** Consider the equation $\mathcal{O}\psi = 0$, where $\mathcal{O}$ is a differential operator.
- Adjoint $\mathcal{O}^\dagger$ is defined by $$(\mathcal{O}^\dagger\tilde{\psi})\psi - \tilde{\psi}\mathcal{O}\psi = \nabla_a \underbrace{\pi^a[\tilde{\psi},\psi]}_{\text{current}}$$ (integration by parts)
- For solutions $\mathcal{O}^\dagger\tilde{\psi} = 0 = \mathcal{O}\psi$, the current is conserved, $\nabla_a\pi^a[\tilde{\psi},\psi] = 0$.
  - Hence if $\tilde{\psi},\psi$ decay sufficiently rapidly at infinity, we obtain a conserved (base) bilinear form $$\Pi^a[\tilde{\psi},\psi] = \int_\Sigma \pi^a[\tilde{\psi},\psi]\, d\Sigma_a$$




- E.g., if $\mathcal{O} = \nabla^a\nabla_a - m^2$ is the Klein-Gordon operator, then $\pi^a[\tilde{\psi},\psi] = -\tilde{\psi}\nabla^a\psi + \psi\nabla^a\tilde{\psi}$ is the Klein-Gordon current.

# Conserved currents

$$(\mathcal{O}^\dagger\tilde{\psi})\psi - \tilde{\psi}\mathcal{O}\psi = \nabla_a \pi^a[\tilde{\psi}, \psi]$$

- For perturbations of Kerr, apply this to the Teukolsky operator, which can be written in the useful form

$$\Theta_a = \nabla_a - \frac{p+q}{2} n^b \nabla_a l_b + \frac{p-q}{2} \bar{m}^b \nabla_a m_b$$

$$\mathcal{O} = g^{ab}(\Theta_a + 4B_a)(\Theta_b + 4B_b) - 16\Psi_2$$

$$B_a = -\rho n_a + \tau \bar{m}_a$$

  - This acts on GHP scalars $\tilde{\psi}$ of weight $(p,q) = (4{,}0)$
  - The adjoint Teukolsky operator $\mathcal{O}^\dagger$ acts on GHP scalars $\psi$ of weight $(p,q) = (-4{,}0)$
- Gives rise to the conserved quantity

$$\boxed{\Pi[\tilde{\psi}, \psi] = \int_\Sigma \left[\tilde{\psi}(\Theta^a - 4B^a)\psi - \psi(\Theta^a + 4B^a)\tilde{\psi}\right] \, d\Sigma_a}$$

# Tower of conservation laws

$$\boxed{\Pi[\tilde{\psi}, \psi] = \int_\Sigma \left[\tilde{\psi}(\Theta^a - 4B^a)\psi - \psi(\Theta^a + 4B^a)\tilde{\psi}\right] \, d\Sigma_a}$$

- From this "base" bilinear form, construct an infinite number of conserved quantities combining with symmetries:
  - A differential operator $\mathscr{C}$ is a symmetry operator if it takes solutions into solutions, i.e., $\mathscr{O}\mathscr{C}\psi = \mathscr{D}\mathscr{O}\psi$, for some operator $\mathscr{D}$.
    - $\Pi[\tilde{\psi}, \mathscr{C}\psi]$ is also conserved.
  - For Kerr, we have symmetry operators $[\mathscr{C}, \mathscr{O}] = 0 = [\mathscr{C}, \mathscr{O}^\dagger]$ arising from the Killing vectors and Killing tensor,

$$\mathcal{C} \to \begin{cases} Ł_t & \text{time translation} \\ Ł_\phi & \text{rotation} \\ \mathcal{K} & \text{Carter operator} \end{cases}$$

  - These symmetry operators moreover commute with each other. Compositions of symmetries are also symmetries, hence we obtain an infinite number of conserved quantities (see also Grant and Flanagan, 2020).

# $t$–$\phi$ reflection

$$\boxed{\Pi[\tilde{\psi}, \psi] = \int_\Sigma \left[\tilde{\psi}(\Theta^a - 4B^a)\psi - \psi(\Theta^a + 4B^a)\tilde{\psi}\right] \, d\Sigma_a}$$

- Our bilinear form will be constructed using the discrete reflection symmetry $J : (t, \phi) \to (-t, -\phi)$
  - Acts on null tetrad as $J_* l^a = -\Lambda n^a$, $J_* n^a = -\Lambda^{-1} l^a$, $J_* m^a = e^{i\Gamma} \bar{m}^a$
  - Define action on GHP scalars $\eta \circeq (p, q)$ as $\mathcal{J}\eta = i^{p+q} \lambda^{-p} \bar{\lambda}^{-q} \eta \circ J \circeq (-p, -q)$, where $\lambda^2 = \Lambda e^{i\Gamma}$
    - GHP prime + $t$-$\phi$ reflection
  - Satisfies important property, $\mathcal{O} \Psi_2^{4/3} \mathcal{J} = \Psi_2^{4/3} \mathcal{J} \mathcal{O}^\dagger$
    - $\Psi_2^{4/3} \mathcal{J}$ takes solutions of adjoint Teukolsky to solutions of Teukolsky

Exactly what we need to construct a product on two solutions in the **same** space!

# Bilinear form

- Let $\psi_1, \psi_2 \mathrel{\mathring{=}} (-4,0)$ be smooth with compact support in $\ker \mathscr{O}^\dagger$. Define **bilinear form**

$$\boxed{\langle\langle \psi_1, \psi_2 \rangle\rangle = \Pi\left[\Psi_2^{4/3} \mathscr{J} \psi_1, \psi_2\right]}$$

- **Properties**

1. $\mathbb{C}$-linear in both entries
2. conserved (independent of choice of Cauchy surface $\Sigma$)
3. $\langle\langle \psi_1, \psi_2 \rangle\rangle = \langle\langle \psi_2, \psi_1 \rangle\rangle$ (symmetric)
4. $\langle\langle \mathrm{L}_t \psi_1, \psi_2 \rangle\rangle = \langle\langle \psi_1, \mathrm{L}_t \psi_2 \rangle\rangle$ (time translation operator is symmetric)

# Bilinear form

- **Proof of (4),** $\langle\langle \mathrm{L}_t\psi_1, \psi_2\rangle\rangle = \langle\langle \psi_1, \mathrm{L}_t\psi_2\rangle\rangle$

closed on solutions

$$\mathscr{L}_t\pi = d(t\cdot\pi) + t\cdot \cancel{d\pi} \qquad \text{Cartan's magic formula}$$

$$\Big\downarrow \text{Compact support}$$

$$\int_\Sigma \mathscr{L}_t\pi = \int_{\partial\Sigma} t\cdot\pi = 0$$

$$\begin{aligned}
&\mathcal{L}_t\pi(\Psi_2^{4/3}\mathcal{J}\psi_1,\psi_2)\\
&\quad= \pi(\Psi_2^{4/3}\text{Ł}_t\mathcal{J}\psi_1,\psi_2) + \pi(\Psi_2^{4/3}\mathcal{J}\psi_1,\text{Ł}_t\psi_2)\\
&\quad= -\pi(\Psi_2^{4/3}\mathcal{J}\text{Ł}_t\psi_1,\psi_2) + \pi(\Psi_2^{4/3}\mathcal{J}\psi_1,\text{Ł}_t\psi_2)
\end{aligned}$$

$L_t\mathscr{J} = -\mathscr{J}L_t$

$\square$

- Note that in a Hamiltonian formulation, this corresponds to

$$\langle\langle \mathscr{H}\psi_1, \psi_2\rangle\rangle = \langle\langle \psi_1, \mathscr{H}\psi_2\rangle\rangle$$

# Bilinear form

In Boyer-Lindquist coordinates and Kinnersley frame,

$$\langle\langle \psi_1, \psi_2 \rangle\rangle = 4M^{4/3} \int_\Sigma dr\, d\theta d\phi\, \frac{\sin\theta}{\Delta^2} \left[ \psi_1\Big|_{\substack{t\to -t\\ \phi\to-\phi}} \left( \frac{\Lambda}{\Delta}\partial_t + \frac{2Mra}{\Delta}\partial_\phi + 2\left[ -r - ia\cos\theta + \frac{M}{\Delta}(r^2 - a^2) \right] \right) \psi_2 \right.$$
$$\left. + \psi_2 \left[ \left( \frac{\Lambda}{\Delta}\partial_t + \frac{2Mra}{\Delta}\partial_\phi + 2\left[ -r - ia\cos\theta + \frac{M}{\Delta}(r^2 - a^2) \right] \right) \psi_1 \right]_{\substack{t\to -t\\ \phi\to-\phi}} \right]$$

- In contrast to inner product, there is no complex conjugation on first argument — **bilinear**
- Not positive — **not an inner product**

# Quasinormal modes

$$ {}_s\psi_{\ell m\omega} = e^{-i\omega t + im\phi}\, {}_sR_{\ell m\omega}(r)\, {}_sS_{\ell m\omega}(\theta) $$

- Teukolsky equation separates into angular and radial parts

$$ \left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + \left(\boxed{K} - \frac{m^2+s^2+2ms\cos\theta}{\sin^2\theta} - a^2\omega^2\sin^2\theta - 2a\omega s\cos\theta\right)\right] {}_sS_{\ell m\omega}(\theta) = 0 $$

$$ \left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \left(\frac{H^2 - 2is(r-M)H}{\Delta} + 4is\omega r + 2am\omega - \boxed{K} + s(s+1)\right)\right] {}_sR_{\ell mn}(r) = 0 $$

separation constant

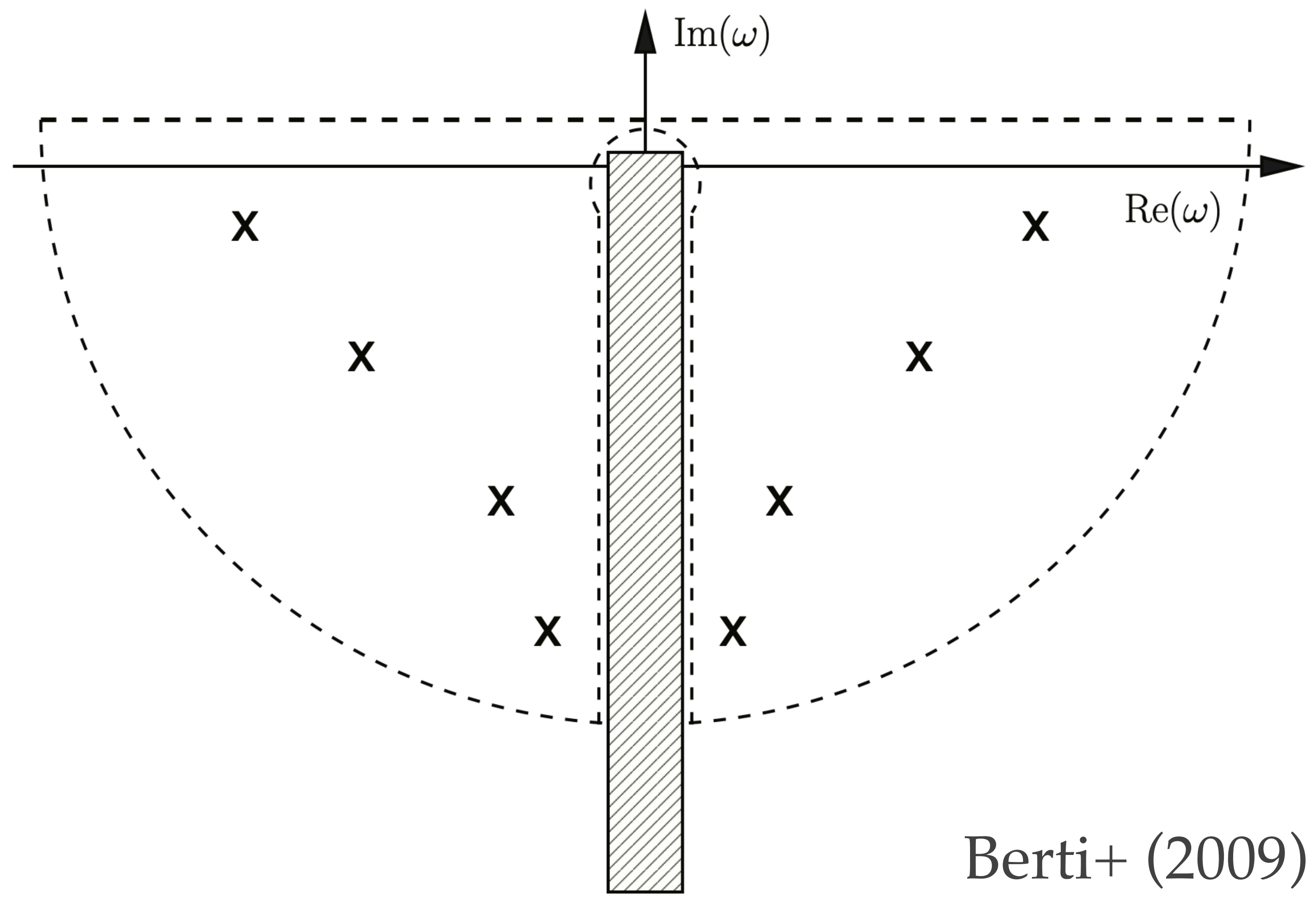
- **Angular part** gives spheroidal harmonics, indexed by integer $\ell \geq \max(|m|, |s|)$
- For fixed $s, m, \omega \in \mathbb{R}$, these are orthogonal, $\int_0^\pi d\theta\, \sin\theta\, {}_sS_{\ell m\omega}(\theta) {}_sS_{\ell' m\omega}(\theta) = \delta_{\ell\ell'}$

# Quasinormal modes

$$ {}_s\Psi_{\ell m\omega} = e^{-i\omega t + im\phi}\, {}_sR_{\ell m\omega}(r)\, {}_sS_{\ell m\omega}(\theta) $$

- **Radial part** $$\left[\Delta^{-s}\frac{d}{dr}\left(\Delta^{s+1}\frac{d}{dr}\right) + \left(\frac{H^2 - 2is(r-M)H}{\Delta} + 4is\omega r + 2am\omega - K + s(s+1)\right)\right] {}_sR_{\ell mn}(r) = 0$$
- Assume ingoing at horizon / outgoing at infinity $$\begin{cases} R^{\rm in} \sim \dfrac{e^{-ikr_*}}{\Delta^s}, & r_* \to -\infty, \\ R^{\rm up} \sim \dfrac{e^{i\omega r_*}}{r^{2s+1}}, & r_* \to \infty \end{cases}$$
- Obtain discrete spectrum $\omega_{\ell mn}$ with $\mathrm{Im}(\omega_{\ell mn}) < 0$

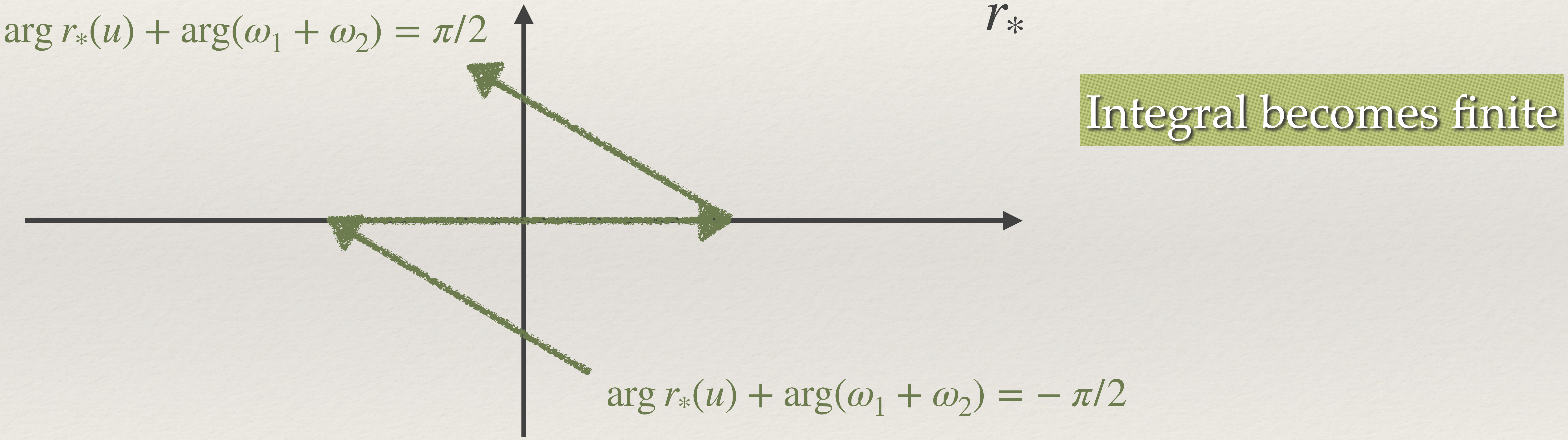
$\Longrightarrow$ Modes decay in time but **blow up** at $|r_*| \to \infty$
$\Longrightarrow$ $\langle\langle \cdot , \cdot \rangle\rangle$ **divergent** on quasinormal modes



Berti+ (2009)

# Bilinear form for quasinormal modes

- Extend $\langle\langle \cdot , \cdot \rangle\rangle$ from compact support $\longrightarrow$ quasinormal mode data (following Leung+, 1994)
- Deform radial integration into complex plane




- Properties (1)-(4) continue to hold.

# Bilinear form for quasinormal modes

**Main result**

Let $\psi_1, \psi_2$ be quasinormal modes. Then
$$\begin{aligned} 0 &= \langle\langle \psi_1, Ł_t \psi_2 \rangle\rangle - \langle\langle Ł_t \psi_1, \psi_2 \rangle\rangle \\ &= (\omega_2 - \omega_1) \langle\langle \psi_1, \psi_2 \rangle\rangle \end{aligned}$$

$\Longrightarrow$ either $\omega_1 = \omega_2$ or $\langle\langle \psi_1, \psi_2 \rangle\rangle = 0$.

- Explicitly on modes,

$$\langle\langle \psi_{\ell_1 m_1 \omega_1}, \psi_{\ell_2 m_2 \omega_2} \rangle\rangle = 8\pi M^{4/3} \delta_{m_1 m_2} e^{-i(\omega_2 - \omega_1)t} \int_{C_*} dr_* \int_0^\pi d\theta \, \frac{(r^2 + a^2)\sin\theta}{\Delta} S_1(\theta) S_2(\theta) R_1(r) R_2(r)$$
$$\left( -\frac{i\Lambda}{\Delta}(\omega_1 + \omega_2) + \frac{2iMra}{\Delta}(m_1 + m_2) + 2\left[ -r - ia\cos\theta + \frac{M}{\Delta}(r^2 - a^2) \right] \right)$$

- This is fundamentally a 2D integral!

# Excitation coefficients

- Given initial data, expand QNM part of solution as

$$\psi_s \sim \sum_{\ell mn} c_{\ell mn} \, {}_s\Psi_{\ell mn}$$

- Using bilinear form…

$$c_{\ell mn} = \frac{\langle\langle {}_s\Psi_{\ell mn}, \psi_s \rangle\rangle}{\langle\langle {}_s\Psi_{\ell mn}, {}_s\Psi_{\ell mn} \rangle\rangle}$$

- From Laplace transform…

initial data

$$c_{n\ell m} = -\frac{i}{d\mathscr{W}/d\omega\,\big|_{\omega_n}} \int_{r_+}^{\infty} {}_sI_{\ell mn}(r')\, {}_sR_{\ell mn}(r') \Delta^s(r')\, dr'$$

$$\mathscr{W}[R_1, R_2] = \Delta^{1+s}\left[R_1 \frac{dR_2}{dr} - R_2 \frac{dR_1}{dr}\right]$$

- Can prove equivalence (following Leung+, 1994). In particular, **norm** of a QNM is related to **derivative of the Wronskian**,

$$\frac{d}{d\omega}\mathscr{W}[R^{\text{in}}_{\omega}, R^{\text{up}}_{\omega}]\bigg|_{\omega=\omega_n} = \frac{-i}{8\pi M^{4/3}} \langle\langle \psi^{\text{in}}_{\omega_n}, \psi^{\text{up}}_{\omega_n} \rangle\rangle$$

# “Norm” of a QNM

$$\frac{d}{d\omega}\mathscr{W}[R^{\text{in}}_{\omega}, R^{\text{up}}_{\omega}]\Big|_{\omega=\omega_n} = \frac{-i}{8\pi M^{4/3}}\langle\langle \psi^{\text{in}}_{\omega_n}, \psi^{\text{up}}_{\omega_n}\rangle\rangle$$

**Sketch of proof**

1. If $S_1, S_2$ satisfy the angular equation, then by explicit calculation

$$8\pi M^{4/3}\mathscr{W}[R_1, R_2] = \int_{S^2(t,r)} t \cdot \pi(\Psi_2^{4/3}\mathcal{J}\psi_1, \psi_2)$$

Integral on sphere

Note that both sides vanish on QNMs. Derivative, however, will relate to QNM norm.

# "Norm of a QNM"

2. For $R_1, R_2$ solutions, ingoing at horizon, outgoing at infinity

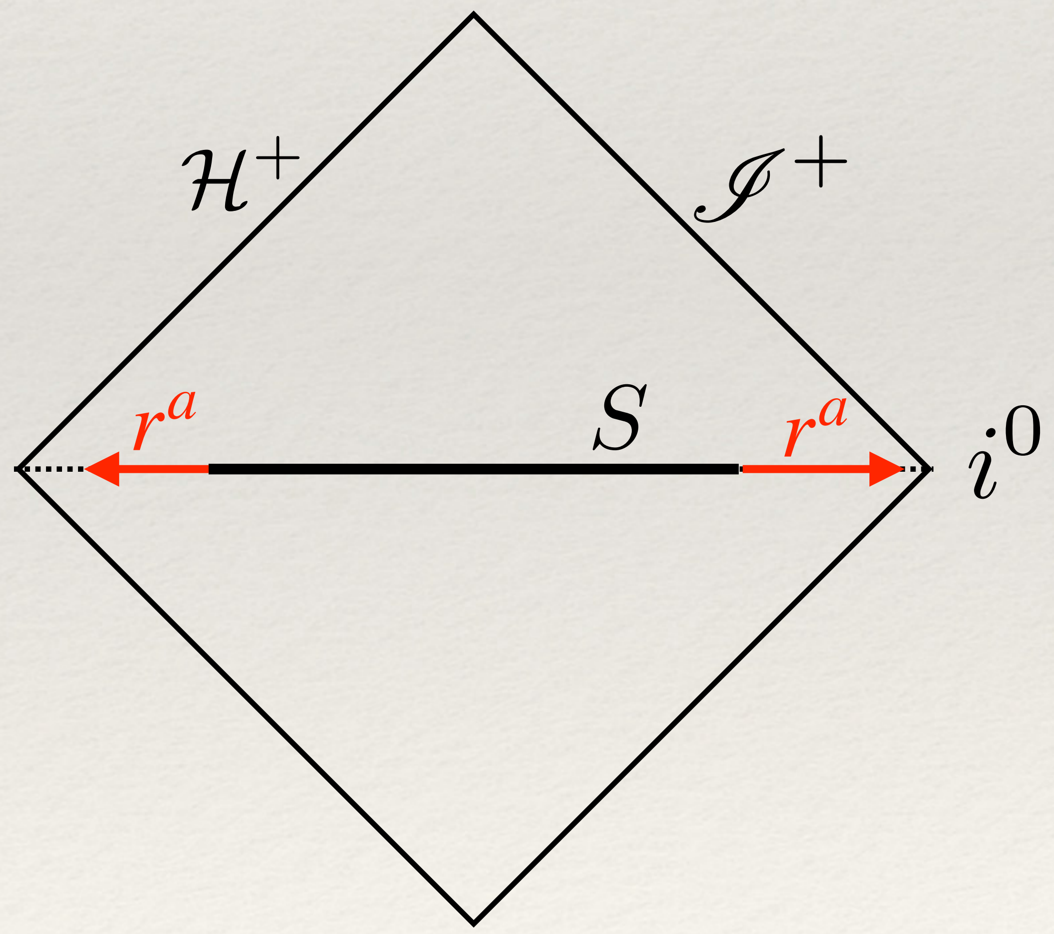
$$d\left(t\cdot\pi\left(\Psi_2^{4/3}\mathcal{J}\underbrace{\psi^{\text{in}}_{\omega_n}}_{\text{QNM}},\underbrace{\psi^{\text{up}}_{\omega}}_{\text{any }\omega}\right)\right) = \pounds_t\pi\left(\Psi_2^{4/3}\mathcal{J}\psi^{\text{in}}_{\omega_n},\psi^{\text{up}}_{\omega}\right)$$

$$= -i(\omega-\omega_n)\pi\left(\Psi_2^{4/3}\mathcal{J}\psi^{\text{in}}_{\omega_n},\psi^{\text{up}}_{\omega}\right)$$

Cartan's magic formula

- Integrate over $S$

$$\int_{\partial S} t\cdot\pi(\Psi_2^{4/3}\mathcal{J}\psi^{\text{in}}_{\omega_n},\psi^{\text{up}}_{\omega}) = -i(\omega-\omega_n)\int_S \pi(\Psi_2^{4/3}\mathcal{J}\psi^{\text{in}}_{\omega_n},\psi^{\text{up}}_{\omega})$$




- Differentiate wrt $\omega$ and take $\omega \to \omega_n$

# "Norm" of a QNM

$$\int_{\partial S} t \cdot \pi(\Psi_2^{4/3} \mathcal{J} \psi^{\text{in}}_{\omega_n}, \psi^{\text{up}}_{\omega}) = -i(\omega - \omega_n) \int_S \pi(\Psi_2^{4/3} \mathcal{J} \psi^{\text{in}}_{\omega_n}, \psi^{\text{up}}_{\omega})$$

Differentiate wrt $\omega$ and take $\omega \to \omega_n$

$$\int_{\partial S_+} t \cdot \pi\left(\Psi_2^{4/3} \mathcal{J} \psi^{\text{in}}_{\omega_n}, \left.\frac{d}{d\omega}\right|_{\omega=\omega_n} \psi^{\text{up}}_{\omega}\right)$$

$$- \left.\frac{d}{d\omega}\right|_{\omega=\omega_n} \int_{\partial S_-} t \cdot \pi\left(\Psi_2^{4/3} \mathcal{J} \psi^{\text{in}}_{\omega}, \psi^{\text{up}}_{\omega}\right)$$ Wronskian

$$+ \int_{\partial S_-} t \cdot \pi\left(\left.\frac{d}{d\omega}\right|_{\omega=\omega_n} \Psi_2^{4/3} \mathcal{J} \psi^{\text{in}}_{\omega}, \psi^{\text{up}}_{\omega_n}\right)$$ Vanish

$$-i \int_S \pi(\Psi_2^{4/3} \mathcal{J} \Upsilon^{\text{in}}_{\omega_n}, \psi^{\text{up}}_{\omega_n})$$

- Rearranging, obtain result $\left.\frac{d}{d\omega} \mathscr{W}[R^{\text{in}}_{\omega}, R^{\text{up}}_{\omega}]\right|_{\omega=\omega_n} = \frac{-i}{8\pi M^{4/3}} \langle\langle \psi^{\text{in}}_{\omega_n}, \psi^{\text{up}}_{\omega_n} \rangle\rangle$

# Summary (so far)

- Defined a **bilinear form (or relativistic product)** $\langle\langle \cdot , \cdot \rangle\rangle$ for Teukolsky solutions
  - Comes from combining $\pi^a[\cdot , \cdot]$ current with $t$-$\phi$ reflection operator $\mathcal{J}$
  - Complex radial integration makes $\langle\langle \cdot , \cdot \rangle\rangle$ finite on QNMs
  - QNMs with $\omega_1 \neq \omega_2$ are orthogonal with respect to the bilinear form
- Mode expansions $\psi_s \sim \sum_{\ell mn} c_{\ell mn}\, {}_s\psi_{\ell mn}$ with the bilinear form give rise to the well-known expression for the excitation coefficients.

# Application to black hole boson clouds

$$\boxed{(\Box - \mu^2)\Phi = 0}$$

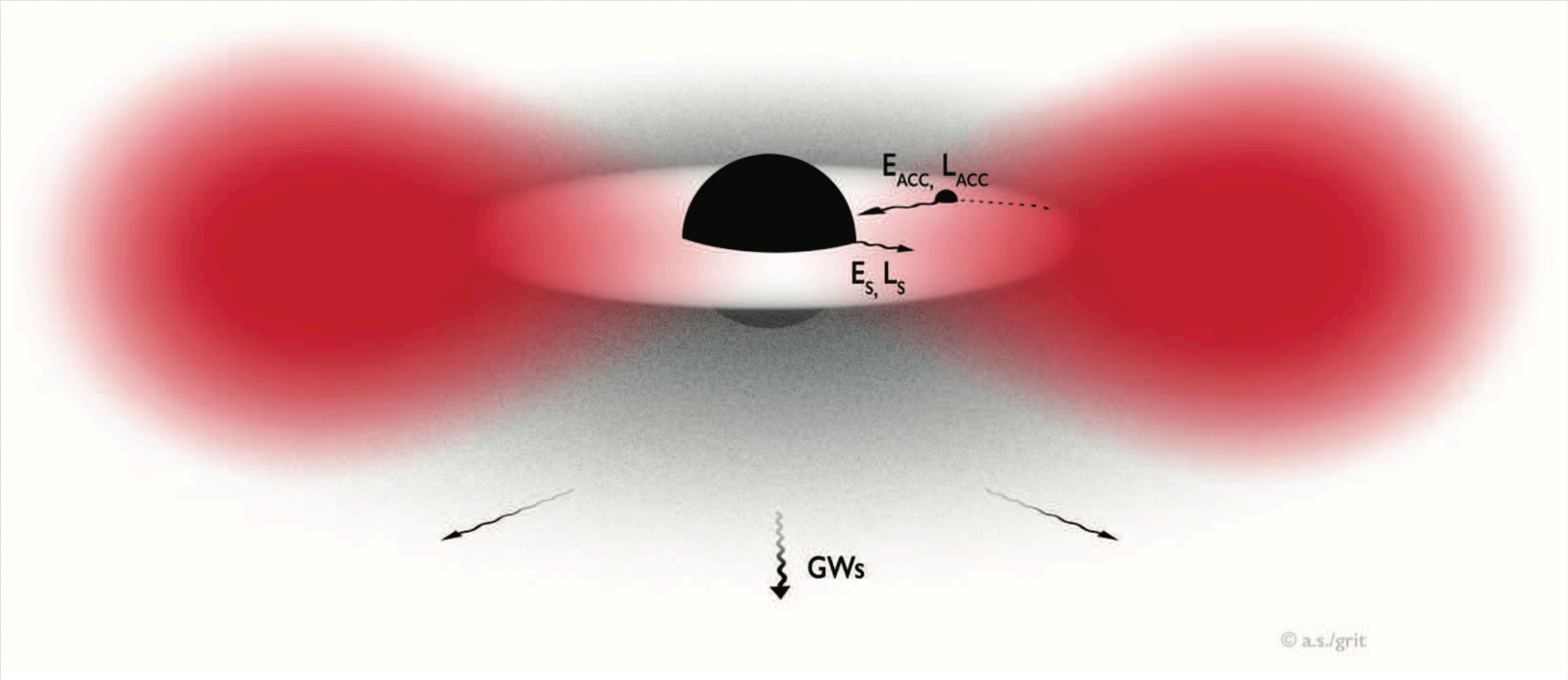
massive scalar field
in Kerr

- Massive fields give rise to **quasibound states (QBSs)** with $|\omega| < \mu$

$$\Phi \sim r^{-1} e^{ikr_*}, \qquad r_* \to \infty \quad \text{(QNMs)},$$
$$\Phi \sim r^{-1} e^{-ikr_*}, \qquad r_* \to \infty \quad \text{(QBSs)},$$

$$k = \sqrt{\omega^2 - \mu^2}$$

- Confined by Yukawa suppression $\Longrightarrow$ no radiation at $\mathscr{I}^+$.
- Superradiantly unstable if $m\Omega_{\mathrm{H}} > \omega_{\mathrm{R}}$.
- Astrophysically, leads to boson clouds for $\mu \approx 10^{-18} - 10^{-19}$ eV.




Brito+ (2014)

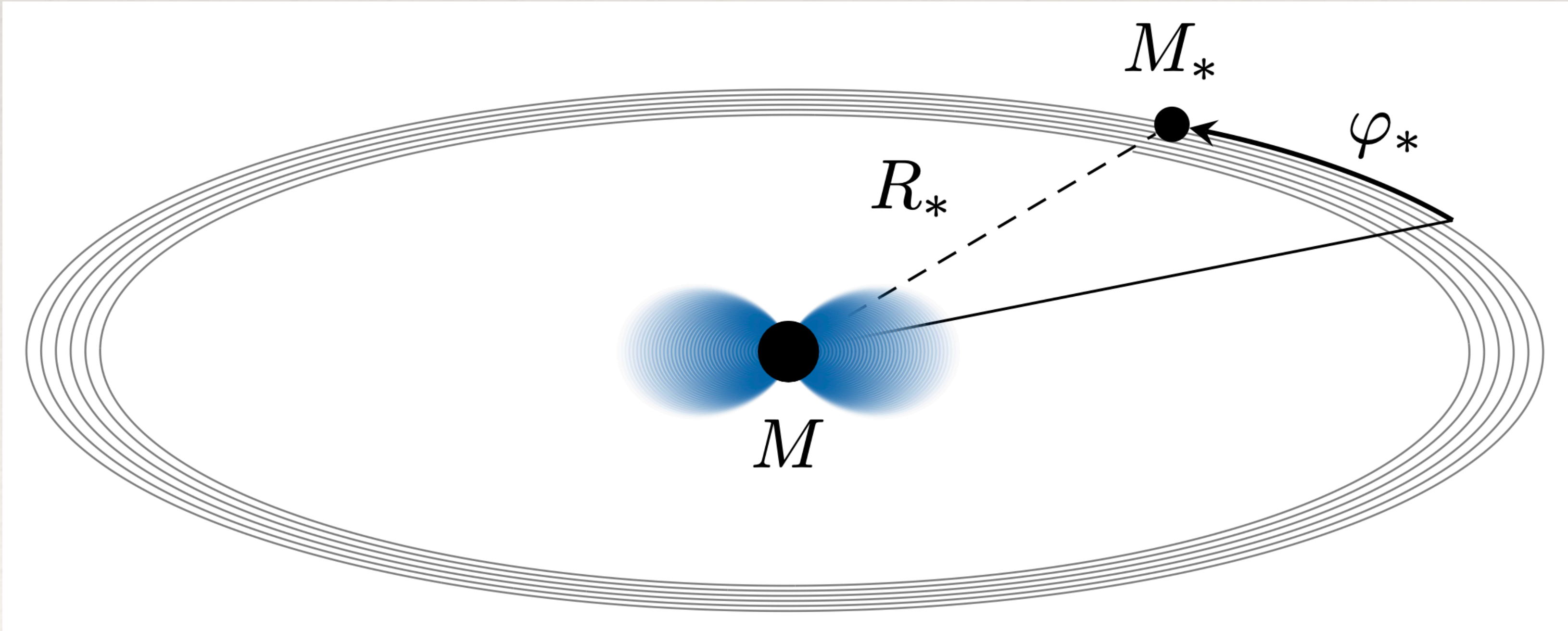
# Gravitational atom

**Quasibound states** $\xrightarrow[\text{leading order in } \alpha = \mu M \text{ and } r \gg \mu^{-1}]{\text{approximate}}$ **Hydrogen bound states**

- Regular at origin
- *Hydrogenic* inner product $\langle \cdot | \cdot \rangle_H$
- $\omega \in \mathbb{R}$
- Complete, orthonormal set of modes

Use bilinear form $\langle\langle \cdot , \cdot \rangle\rangle$ !

- Many applications:
  - Level mixing due to potentials $\langle n\ell m | \delta V | n'\ell' m' \rangle_H$, e.g., binary companion or self-interaction
  - Self-gravity leading to frequency shifts $\propto \langle n\ell m | \delta V | n\ell m \rangle_H$




Baumann+ (2022)

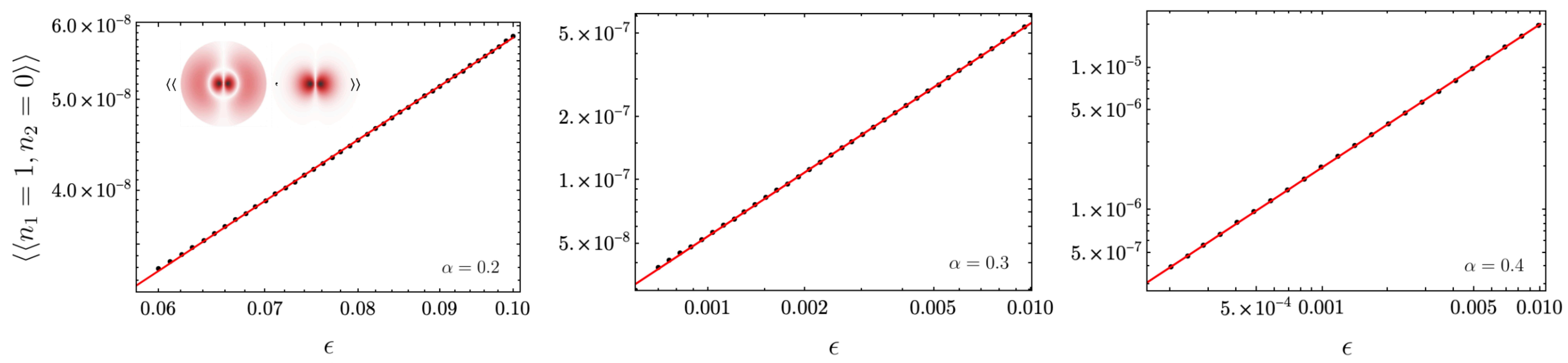
# Bilinear form for QBSs

- Straightforward extension to massive scalar fields and QBS states with $\Phi \sim r^{-1}e^{-ikr_*}$ as $r \to \infty$.
- For Schwarzschild QBS, can alternatively regularize using counter-term subtraction

$X(r) = rR(r)$

$$\langle\langle \Phi_1, \Phi_2 \rangle\rangle_{\text{Schwarzschild QBS}}$$
$$= i\delta_{m_1 m_2}\boxed{\delta_{l_1 l_2}}(\omega_1+\omega_2)\lim_{\bar{r}_*\to-\infty}\left[\int_{\bar{r}_*}^{\infty} dr_* X_1(r'_*)X_2(r'_*) + \boxed{\frac{i}{\omega_1+\omega_2}X_1(\bar{r}_*)X_2(\bar{r}_*)} + (\text{higher orders})\right]$$

**Numerical check of orthogonality**




Cutoff radius

# Relativistic perturbation theory

$$\mathcal{O}\Phi + \delta V\Phi = 0$$

Potential $\delta V$

Mode ansatz $\Phi = \sum_q c_q(t)\Phi_q$

Project onto mode $n$

$$\sum_q \langle\langle \Phi_n, \mathcal{O} c_q(t)\Phi_q\rangle\rangle + \langle\langle \Phi_n, c_q(t)\delta V\Phi_q\rangle\rangle = 0$$

Assume $\dot{c} \sim \delta V, \ddot{c} \sim \delta V^2$

$$\boxed{2i\omega_n\langle\langle \Phi_n, \Phi_n\rangle\rangle + O(\delta V^2) = \sum_q c_q(t)\langle\langle \Phi_n, \delta V\Phi_q\rangle\rangle}$$

cf. QM time-dependent perturbation theory

# Example: Frequency shifts

- Consider (Newtonian, flat space) self-gravity of a superradiant mode (from Siemonson+ 2023)

$$\delta V(r) = -2\mu^2 \left[ \frac{1}{r} \int_{r_+}^{r} d^3r' T^t_t + \int_r^\infty d^3r' \frac{T^t_t}{r'} \right]$$

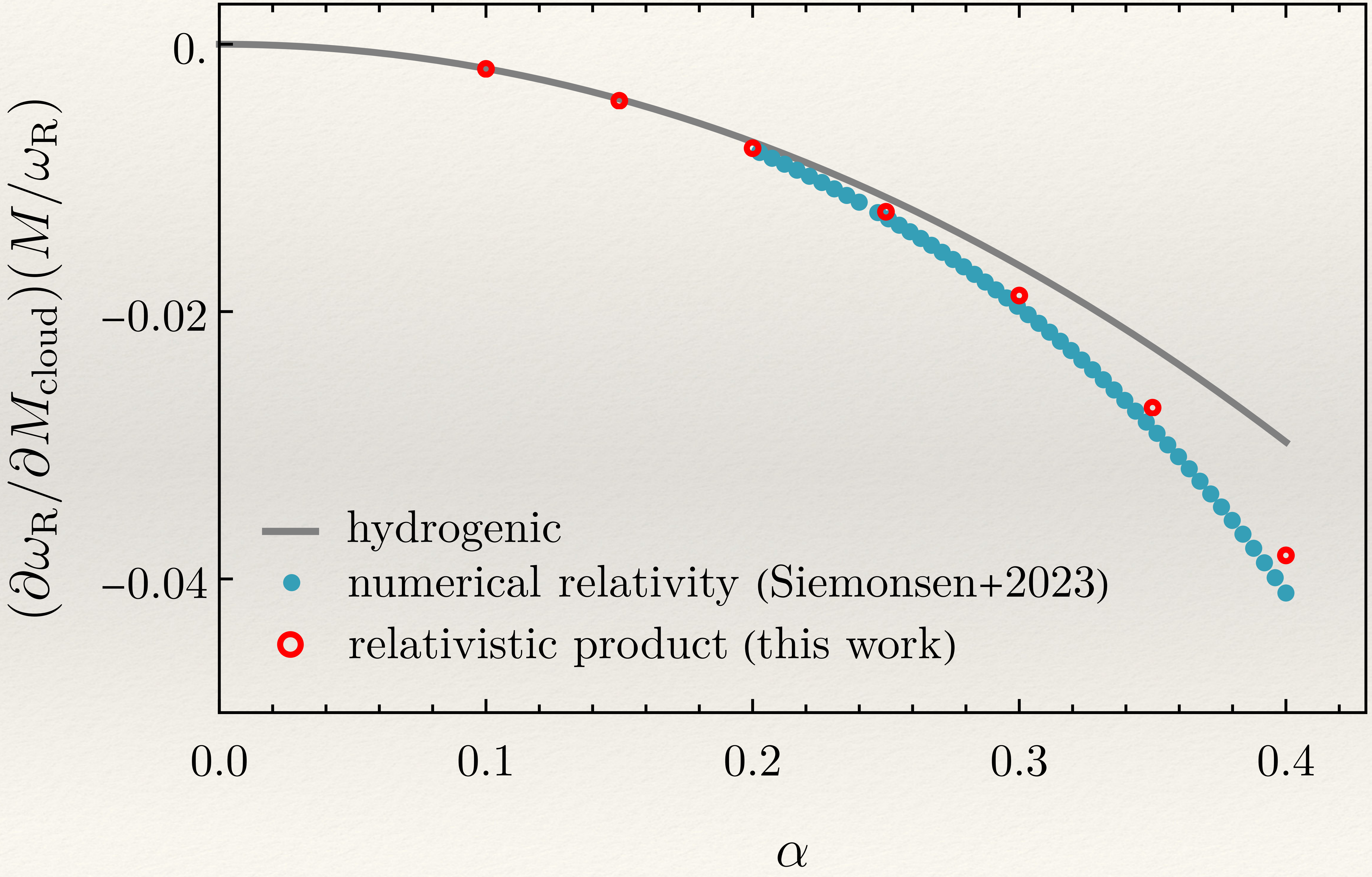
stress-energy of $\Phi$

- For a single mode frequency-shift, $c_n(t) \propto e^{-i\delta\omega_n t}$

$$2i\omega_n \langle\langle \Phi_n, \Phi_n \rangle\rangle + O(\delta V^2) = \sum_q c_q(t) \langle\langle \Phi_n, \delta V \Phi_q \rangle\rangle \longrightarrow \delta\omega_n = -\frac{\langle\langle \Phi_n, \delta V \Phi_n \rangle\rangle}{2\omega_n \langle\langle \Phi_n, \Phi_n \rangle\rangle}$$

- Compute $\ell = m = 1$ mode frequency shifts for BH spins close to the superradiant bound.
  - Compare to hydrogenic approximation and numerical relativity (Siemonson+ 2023) for a variety of $\alpha$.

# Frequency shifts




✅ Excellent match to NR!

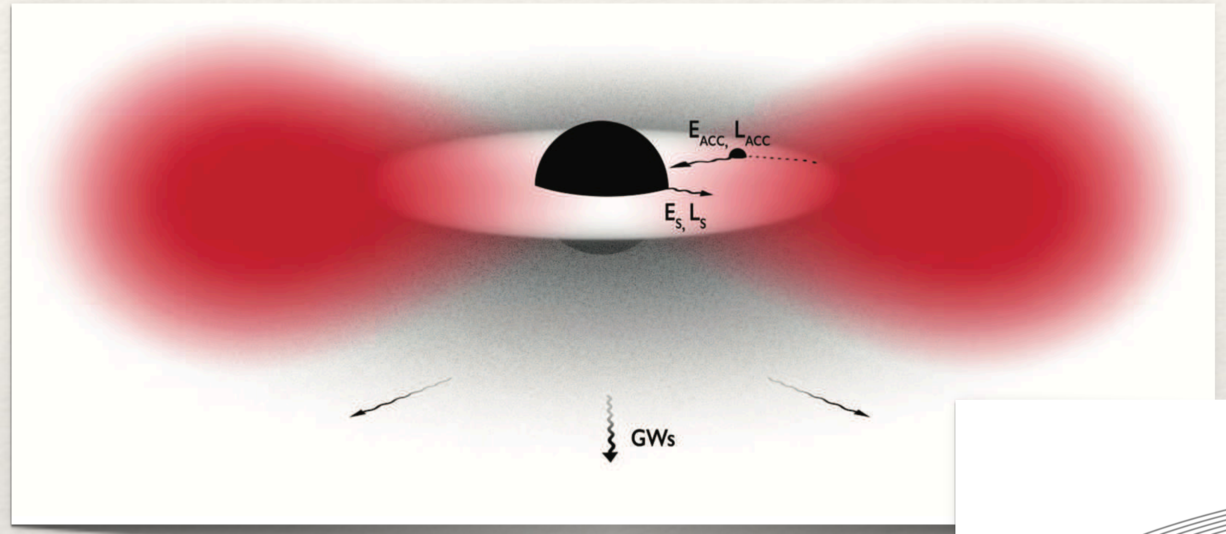
# Conclusions

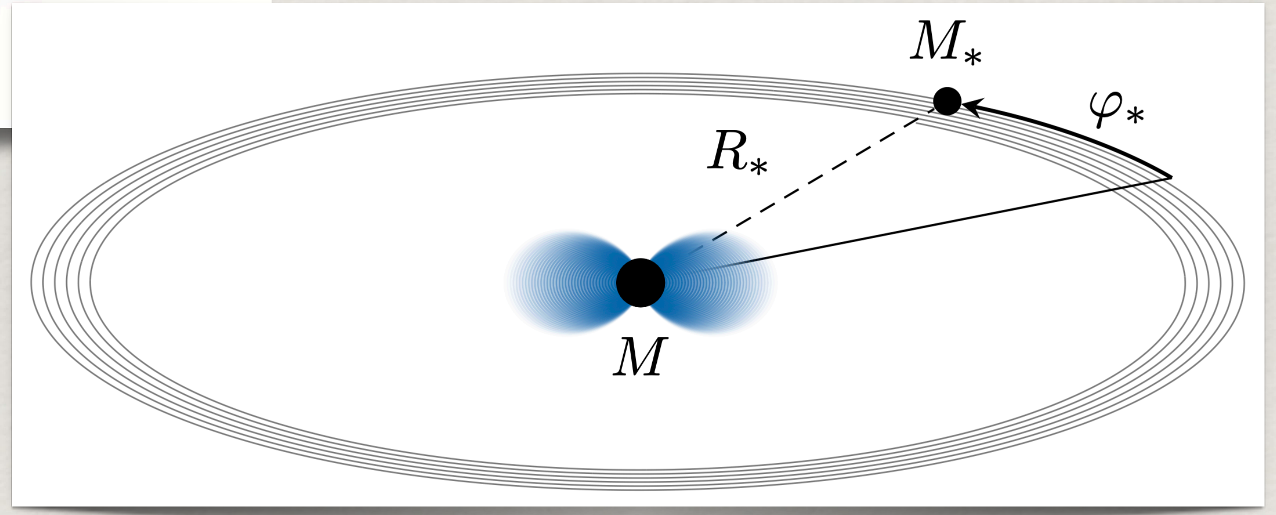
- The bilinear form generalizes the quantum mechanical inner product for black holes.
- QNMs and QBSs are orthogonal for different $\omega$.
- Perturbation theory based on our relativistic product gives greatly improved agreement with numerical relativity.

# Further directions

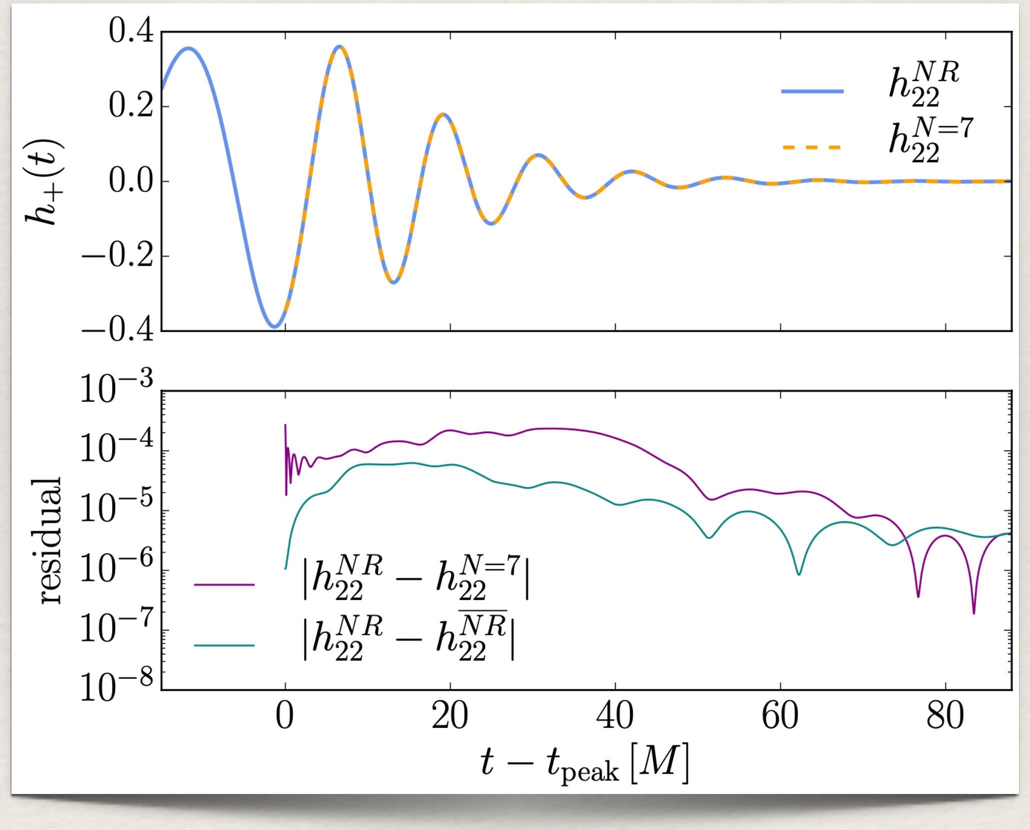
The relativistic product opens **many directions of research:**

- Boson clouds (self-interactions, tidal perturbers, …)

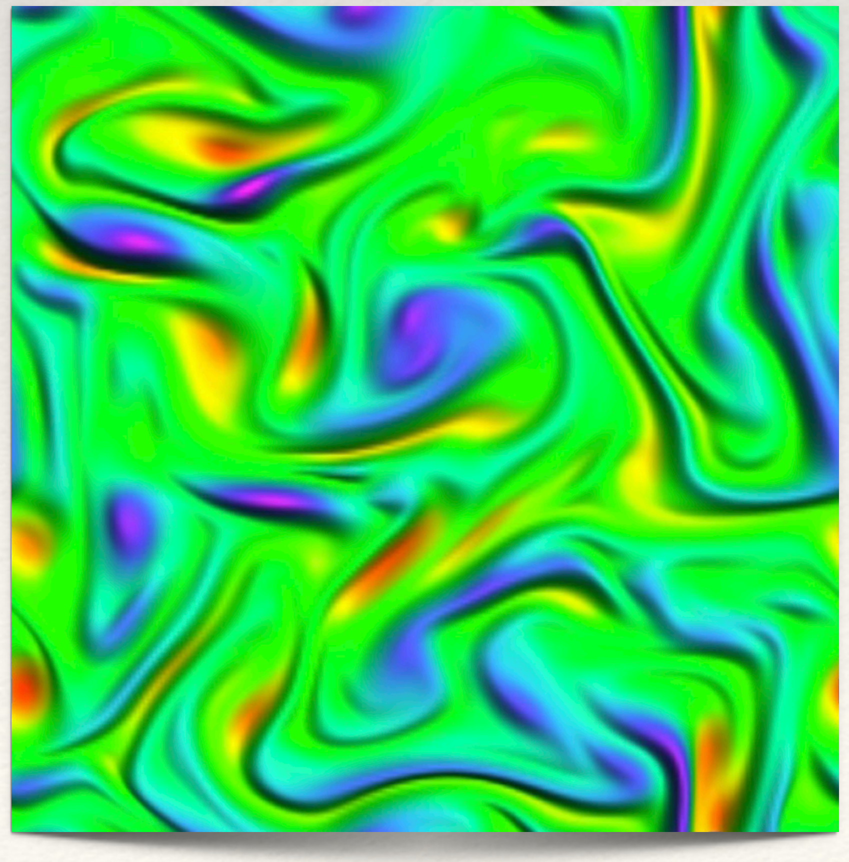



- Extension to Proca fields?
- Nonlinear ringdown




- Extension to hyperbolic slices
- Gravitational turbulence? (talk of S. Hollands)



Thank you!