StAR summer school

Waves on Black Holes – problem session 1

1 Preliminaries for the lectures

Exercise 1.1. Consider a Lorentzian manifold $(\mathcal{M}, g)$ and scalar fields ψ, F, β on it. Multiply the equation $\Box_g \psi = F$ by $\beta \nabla_\nu \phi$, where $\phi = \beta^{-1}\psi$, and show

$$F\beta\nabla_\nu\phi = \nabla^\mu \tilde{T}_{\mu\nu}[\phi] - \tilde{S}_\nu[\phi],$$

for some tensors $\tilde{T}_{\mu\nu}[\phi], \tilde{S}_\nu[\phi]$ on $\mathcal{M}$ that you should specify and which have the property that they depend only on ϕ and its first derivatives.

Exercise 1.2. Recall the descriptions of the Schwarzschild exterior manifold $(\mathcal{M}_{\mathrm{Schw}(M)}, g_{\mathrm{Schw}(M)})$ in lectures with charts $(t^*, r, \theta, \varphi)$ and (t, r, θ, φ).

(i) Compare the coordinate vector field ∂_r in the $(t^*, r, \theta, \varphi)$ chart with the coordinate vector field ∂_r in the (t, r, θ, φ) chart on $\mathcal{M}_{\mathrm{Schw}(M)}$.

(ii) Explain why the notation r is nevertheless kept in both charts by assigning a geometric meaning to the scalar function r on $\mathcal{M}_{\mathrm{Schw}(M)}$.

2 Consolidating the lectures

Problem 1.3. Prove the Hardy estimate

$$\int_{\mathcal{S}_\tau\cap\{r\le R\}} \frac{|\phi|^2}{(r+1)^2}\mathrm{d}r\mathrm{d}\sigma + \int_{C_\tau} \frac{|\phi|^2}{(r+1)^2}\mathrm{d}v\mathrm{d}\sigma \lesssim_R \int_{\mathcal{S}_\tau\cap\{r\le R\}} \left(|\partial_r\phi|^2 + |\partial_t\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma + \int_{C_\tau} |\partial_v\phi|^2 \mathrm{d}v\mathrm{d}\sigma,$$

carefully justifying why there are no boundary integrals over corner 2-spheres needed on the right hand side.

Problem 1.4. The Reissner-Nordström black hole exterior with parameters $M > 0$ and $|e| \le M$ can be defined in a similar fashion to the Schwarzschild exterior $(\mathcal{M}_{\mathrm{Schw}(M)}, g_{\mathrm{Schw}(M)})$, but with Δ replaced by $\Delta = r^2 - 2Mr + e^2$ and $r_+ = M + \sqrt{M^2 - e^2}$. Show that (degenerate-at-$\mathcal{H}^+$) energy estimate hold for the scalar wave equation on this spacetime:

$$\int_{\Sigma_{\tau_2}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} \le \int_{\Sigma_{\tau_1}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}},$$

where

$$\tilde{J}^T_\mu[\psi] n^\mu_{\mathcal{S}_\tau} \mathrm{dvol}_{\mathcal{S}_\tau} \sim \left(|\partial_{t^*}\phi|^2 + \frac{\Delta}{r^2}|\partial_r\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{r-M}{r^4}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma,$$

$$\tilde{J}^T_\mu[\psi] \underline{n}^\mu_{C_\tau} \mathrm{dvol}_{C_\tau} \sim \left(|\partial_v\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{1}{r^3}|\phi|^2\right)\mathrm{d}v\mathrm{d}\sigma.$$

Using a suitable Hardy estimate, deduce that

$$\begin{aligned}
&\int_{\mathcal{S}_{\tau_2}\cap\{r_+(1+\epsilon)\le r\le R\}} \left(|\partial_{t^*}\psi|^2 + |\partial_r\psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + |r^{-1}\psi|^2\right) r^2\mathrm{d}r\mathrm{d}\sigma \\
&\quad + \int_{C_{\tau_2}} \left(|\partial_v(r\psi)|^2 + |(r^{-1}\mathring{\nabla})(r\psi)|^2 + |\psi|^2\right)\mathrm{d}v\mathrm{d}\sigma \\
&\le C(\epsilon)\int_{\mathcal{S}_{\tau_1}\cap\{r\le R\}} \left(|\partial_{t^*}\psi|^2 + \frac{\Delta}{r^2}|\partial_r\psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + (1-M/r)|r^{-1}\psi|^2\right) r^2\mathrm{d}r\mathrm{d}\sigma \\
&\quad + C(\epsilon)\int_{C_{\tau_1}} \left(|\partial_v(r\psi)|^2 + |(r^{-1}\mathring{\nabla})(r\psi)|^2 + |\psi|^2\right)\mathrm{d}v\mathrm{d}\sigma.
\end{aligned}$$

3 Beyond the lectures

Problem 3.5. On Minkowski space, prove higher order energy estimates for solutions ψ of the scalar wave equation:

$$\sum_{0\le k_1+k_2+k_3\le K}\int_{\mathcal{S}_{\tau_2}} \tilde{J}^T_\mu[\partial_t^{k_1}\partial_r^{k_2}(r^{-1}\mathring{\nabla})^{k_3}\psi]n^\mu_{\mathcal{S}_{\tau_2}}\mathrm{dvol}_{\mathcal{S}_{\tau_2}} \lesssim \sum_{k=0}^{K}\int_{\mathcal{S}_{\tau_1}} \tilde{J}^T_\mu[T^k\psi]n^\mu_{\mathcal{S}_{\tau_1}}\mathrm{dvol}_{\mathcal{S}_{\tau_1}}, \tag{3.1}$$

with implicit constant depending only on K. To do so, start with $K=1$:

(i) Show that the energy identity from lectures holds with $T\psi$ in place of ψ. Take note of any derivatives you must still control to obtain the left hand side of the estimate above.

(ii) Rewrite the wave equation as $(r^{-2}\mathring{\Delta}+\partial_r^2)\phi=\partial_t^2\phi$. Use this identity to control over all the missing derivatives you identified in (i).

Prove the analogous statement on the Schwarzschild exterior manifold.

Problem 3.6. Recall the Sobolev interpolation inequality

$$\|f\|^2_{L^\infty(\mathbb{R}^3)} \lesssim \left(\|f\|^2_{\dot{H}^1}\|f\|^2_{\dot{H}^2}\right)^{\frac{1}{2}}.$$

On Minkowski space, use this to show that

$$\|\psi|_{L^\infty(\{t=\tau\})} \lesssim \int_{\{t=\tau\}}\left(\int_{\{t=\tau\}} \tilde{J}^T_\mu[\psi]n^\mu_{\{t=\tau\}}\mathrm{dvol}_{\{t=\tau\}}\right)^{1/2}\left(\int_{\{t=\tau\}} \tilde{J}^T_\mu[T\psi]n^\mu_{\{t=\tau\}}\mathrm{dvol}_{\{t=\tau\}}\right)^{1/2},$$

where the implicit constant is independent of τ. Deduce that solutions of the scalar wave equation on Minkowski space arising from suitably regular data remain uniformly bounded in time. What would you change to obtain a similar statement for the Schwarzschild exterior (away from the event horizon) instead of Minkowski?

Problem 3.7. On the Schwarzschild exterior manifold, let $Y:=\frac{r^2}{\Delta}\partial_u$. By considering a suitable smooth cutoff function $0\le\chi_H\le 1$, show that the vector field multiplier $X=\chi_H Y$, when combined with the degenerate energy estimates from lectures, leads to the following inequality for a non-degenerate energy:

$$\begin{aligned}
&\int_{\mathcal{S}_{\tau_2}}\left(|\partial_{t^*}\phi|^2+|\partial_r\phi|^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2+\frac{1}{r^3}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma\\
&\lesssim \int_{\mathcal{S}_{\tau_1}}\left(|\partial_{t^*}\phi|^2+|\partial_r\phi|^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2+\frac{1}{r^3}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma\\
&\quad+\int_{\mathcal{R}_{(\tau_1,\tau_2)}\cap\{r\le r_++\epsilon\}}\left[|\partial_{r^*}\phi|^2+|\partial_t\phi|^2+|\mathring{\nabla}\phi|^2+|\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau,
\end{aligned}$$

with implicit constant independent of τ_1,τ_2.

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StAR summer school

Waves on Black Holes – problem session 2

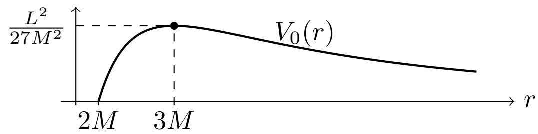
1 Preliminaries for the lectures

Exercise 1.1. Let $y = 1-(r+1)^{-1}$ as in the proof of the Morawetz estimate on Minkowski space. Prove the following Hardy estimate on Minkowski space:

$$\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \frac{1}{(r+1)^4}|\phi|^2 \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \lesssim \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{K}^{2y\partial_r}[\psi]\mathrm{dvol}_g.$$

Exercise 1.2. Let $\gamma(s) = (t(s), r(s), \theta(s), \varphi(s))$ be an affinely parametrized null geodesic on the Schwarzschild exterior manifold. Explain why, by spherical symmetry, we can restrict $\theta(s) = \pi/2$ without loss of generality. Then, show that

$$\begin{cases} \dot{r}^2 = E^2 - V_0(r) \\ \ddot{r} = -\dfrac{1}{2}\partial_r V_0(r) \end{cases}, \qquad V_0(r) = \left(1-\frac{2M}{r}\right)\frac{L^2}{r^2},$$



where, since $T=\partial_t$ and $Z=\partial_\varphi$ are Killing vector fields,

$$E := -g(T,\dot\gamma) = \left(1-\frac{2M}{r}\right)\dot{t},$$
$$L := g(Z,\dot\gamma) = r^2\sin^2\theta\dot\varphi = r^2\dot\varphi$$

are conserved quantities along γ. Show that, for $L \neq 0$, $V_0(r)$ attains a unique maximum in (r_+, ∞) given at a point $r = 3M$. We say γ is future trapped if $\lim_{s\to\infty}(r(s), \dot r(s)) = (3M, 0)$, and past trapped if $\lim_{s\to-\infty}(r(s),\dot r(s)) = (3M,0)$. Show that a necessary condition for γ to be future or past trapped is

$$\frac{\dot r}{E} = \pm\left(1-\frac{3M}{r}\right)\left(1+\frac{6M}{r}\right)^{1/2}.$$

Exercise 1.3. Repeat the previous exercise for the Reissner-Nordström exterior with $|e|\leq M$. In particular, show that there are null geodesics which asymptote toward $\{r = r_{\mathrm{trap}}\}$, with

$$r_{\mathrm{trap}} = 3M + \frac{3M}{2}\left(\sqrt{1-\frac{8e^2}{9M^2}} - 1\right),$$

and that a necessary condition for this to occur along a null geodesic is that

$$\frac{\dot r}{E} = \pm f(r), \qquad f(r) = \frac{1}{r}\left(1-\frac{r_{\mathrm{trap}}}{r}\right)\left(r^2 + 2rr_{\mathrm{trap}} - \frac{r_{\mathrm{trap}}^2 e^2}{r_{\mathrm{trap}}^2 - 2Mr_{\mathrm{trap}} + e^2}\right)^{1/2}.$$

Exercise 2.4 Consider the Kerr black hole family with parameters $M>0$ and $|a|\leq M$. Show that ∂_t is timelike for $r>2M$, spacelike in the ergoregion $\mathscr{E} := \{r \geq r_+ \colon \Delta - a^2\cos^2\theta < 0\}$ and null on $\partial\mathscr{E}$. Show that span$\{T, Z\}$ yields a timelike subspace of the tangent space to the exterior everywhere except at $\mathcal{H}^+$, where it is null; for instance,

$$V = T + \frac{2Mar}{(r^2+a^2)^2}Z$$

is timelike on $\mathcal{M}\backslash\mathcal{H}^+$ and null at $\mathcal{H}^+$. Furthermore, show that if $|a|\leq a_0 < M$, then there is an $\epsilon_0 = \epsilon_0(a_0)$ such that $T + \omega_+ Z$ is timelike for $r \in (r_+, r_+ + \epsilon_0)$.

2 Consolidating the lectures

[There is a heavy computational element to some of these problems: it may be helpful to use Mathematica or your favorite software for symbolic computations.]

Problem 2.5. Repeat the proof of the (f-current bit of the) Morawetz estimate on the Schwarzschild exterior, choosing instead

$$f = \left(1 - \frac{3M}{r}\right)\left(1 + \frac{M}{r}\right),$$

This is not as well-motivated by the geodesic flow, but is a lot easier computationally!

Problem 2.6. Obtain (f-current bit of the) Morawetz estimates on the Reissner-Nordström exterior with $|e| \leq M$. Comment on any degenerations in the resulting estimates.
Hint: It is useful to recall that $r_{\text{trap}}(M,e)$ from Exercise 1.3. You may wish to try either

$$f(r) = \frac{1}{r}\left(1 - \frac{r_{\text{trap}}}{r}\right)\left(r^2 + 2rr_{\text{trap}} - \frac{r_{\text{trap}}^2 e^2}{r_{\text{trap}}^2 - 2Mr_{\text{trap}} + e^2}\right)^{1/2},$$

from Exercise 1.3, or

$$f = \left(1 - \frac{r_{\text{trap}}}{r}\right)\left(1 + \frac{M}{r}\right),$$

by analogy with Problem 2.4, as well as

$$g(r) = \frac{1}{2r}\left(\frac{\Delta}{r^2} - \left(1 - \frac{e^2}{Mr}\right)\frac{M^2}{r^2}\right).$$

3 Beyond the lectures

Problem 3.7. On the Schwarzschild exterior manifold, prove a version of the Morawetz estimate from lectures which is **not** degenerate at trapping, i.e. show

$$\begin{aligned} c(M)\int_{\mathcal{R}_{(\tau_1,\tau_2)}} &\left[\frac{1}{r^2}|\phi'|^2 + \frac{1}{r^2}|T\phi|^2 + \frac{1}{r^3}|\mathring{\nabla}\phi|^2 + \frac{1}{r^4}|\phi|^2\right] \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \\ \leq \int_{\Sigma_{\tau_1}} &\left(\tilde{J}^T_\mu[T\psi] + \tilde{J}^T_\mu[\psi]\right) n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}. \end{aligned}$$

Hint: Observe that, using the Morawetz estimate from lectures, is is enough to show that

$$\int_{\mathcal{R}_{(\tau_1,\tau_2)}\cap\{\frac{5}{2}M\leq r\leq\frac{7}{2}M\}} \left(|T\phi|^2 + |\mathring{\nabla}\phi|^2\right) \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau$$

is controlled by the right hand side of the above estimate.

Problem 3.8. r^p

Email comments and suggestions to `rita.t.costa@dpmms.cam.ac.uk`