Part I
Linear waves on Lorentzian manifolds

1 General techniques and basic statements

This section concerns the analysis of the covariant wave equation

$$\Box_g \psi = F \tag{1.1}$$

on an $(n+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$. Here, ψ and F are functions from $\mathcal{M}$ to $\mathbb{R}$, also called scalar fields, and $\Box_g$ is a differential operator defined by

$$\Box_g \psi := \nabla^\mu \nabla_\mu \psi = \frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \psi \right).$$

If $F \equiv 0$, we call (1.1) the homogeneous (scalar) wave equation on $\mathcal{M}$. Three important examples of inhomogeneous scalar waves arise from:

- adding a potential to the homogeneous wave equation:
$$(\Box_g - V)\psi = 0 \implies \Box_g \psi = V\psi;$$
- commuting the homogeneous wave equation with vector fields Z:
$$\Box_g \psi = 0 \implies \Box_g(Z\psi) = [\Box_g, Z]\psi + Z(\Box_g \psi) = [\Box_g, Z]\psi.$$
- nonlinearities: a general quasilinear wave equation
$$\Box_{g(\psi, \partial\psi)} \psi = N(\psi, \partial\psi)$$
can be written as an inhomogeneous wave equation
$$\Box_{g_0} \psi = F$$
with respect to the "background" metric $g_0 := g(0,0)$, where $F := N(\psi, \partial\psi) - (\Box_{g(\psi,\partial\psi)} - \Box_{g_0})\psi$.

The goal of the section is to give examples of how, using an entirely geometric language, we can derive analytical properties, like boundedness and decay in appropriate Sobolev (energy) spaces, of solutions to (1.1).

1.1 Symmetries and commutators

We first show that symmetries of the background spacetime interact well with the covariant wave equation (1.1). Recall the following definition:

Definition 1.1. Let X be a vector field on $(\mathcal{M}, g)$. Then, we define its deformation tensor by

$$^{(X)}\pi := \frac{1}{2}\mathcal{L}_X g \implies {}^{(X)}\pi_{\mu\nu} = \frac{1}{2}(\nabla_\mu X_\nu + \nabla_\nu X_\mu).$$

We say X is Killing if $^{(X)}\pi = 0$ and conformally Killing if there exists a scalar field λ such that $^{(X)}\pi = \lambda g$.

Lemma 1.2. *Let $(\mathcal{M}, g)$ be $(n+1)$-dimensional, and let Z be a smooth vector field on $\mathcal{M}$. Assume ψ satisfies (1.1) for some given F. Then, we have*

$$\Box_g(Z\psi) = Z(F) + 2{}^{(Z)}\pi^{\mu\nu}\nabla_\mu\nabla_\nu\psi + 2\nabla_\mu {}^{(Z)}\pi^{\mu\nu}\nabla_\nu\psi - \nabla_\nu {}^{(Z)}\pi^\mu_\mu \nabla^\nu \psi.$$

In particular, if Z is Killing, we have

$$\Box_g(Z\psi) = Z(\Box_g\psi) = Z(F);$$

if Z is conformally Killing, with $^{(Z)}\pi = \lambda g$, we have

$$\Box_g(Z\psi) = (Z + 2\lambda)(\Box_g\psi) + (1-n)\nabla_\alpha\lambda\nabla^\alpha\psi = Z(F) + 2\lambda F + (1-n)\nabla_\alpha\lambda\nabla^\alpha\psi.$$

Proof. We compute

$$\begin{aligned}
[\Box_g, Z]\psi &= \nabla^\mu\nabla_\mu(Z^\nu\nabla_\nu\psi) - Z^\nu\nabla_\nu\nabla^\mu\nabla_\mu\psi = \nabla^\mu(Z^\nu\nabla_\mu\nabla_\nu\psi + \nabla_\mu Z^\nu\nabla_\nu\psi) - Z^\nu\nabla_\nu\nabla^\mu\nabla_\mu\psi \\
&= \nabla^\mu Z^\nu(\nabla_\mu\nabla_\nu\psi + \nabla_\nu\nabla_\mu\psi) + \nabla^\mu\nabla_\mu Z^\nu\nabla_\nu\psi + Z^\nu(\nabla^\mu\nabla_\mu\nabla_\nu\psi - \nabla_\nu\nabla^\mu\nabla_\mu\psi) \\
&= 2{}^{(Z)}\pi^{\mu\nu}\nabla_\mu\nabla_\nu\psi + \nabla^\mu\nabla_\mu Z_\nu\nabla^\nu\psi + Z^\nu[\nabla_\mu, \nabla_\nu]\nabla^\mu\psi
\end{aligned}$$

using the fact that $[\nabla_\mu, \nabla_\nu]\psi = 0$ for a scalar field. Now note that

$$[\nabla_\mu, \nabla_\nu]\nabla^\mu\psi = R_{\lambda\nu}\nabla^\lambda\psi,$$

and, moreover, since ${}^{(Z)}\pi_\mu^\mu = \nabla^\alpha Z_\alpha$, we have

$$2\nabla^{\mu\,(Z)}\pi_{\mu\nu} = \nabla^\mu\nabla_\mu Z_\nu + \nabla_\nu\nabla_\mu Z^\mu + [\nabla_\mu, \nabla_\nu]Z^\mu = \nabla^\mu\nabla_\mu Z_\nu + \nabla_\nu{}^{(Z)}\pi_\mu^\mu + R_{\lambda\nu}Z^\lambda$$

Combining these identities, we obtain

$$[\Box_g, Z]\psi = 2^{(Z)}\pi^{\mu\nu}\nabla_\mu\nabla_\nu\psi + 2\nabla^{\mu\,(Z)}\pi_{\mu\nu}\nabla^\nu\psi - \nabla_\nu{}^{(Z)}\pi_\mu^\mu\nabla^\nu\psi, \tag{1.2}$$

since $R_{\lambda\nu}\left(Z^\nu\nabla^\lambda\psi - Z^\lambda\nabla^\nu\psi\right) = 0$ by the symmetries of the Ricci tensor. □

1.2 Energy-momentum tensor and multipliers

The homogeneous wave equation comes from a Lagrangian structure:

$$0 = \Box_g\psi\nabla_\nu\psi = \nabla^\mu T_{\mu\nu}[\psi], \tag{1.3}$$

where the energy-momentum tensor

$$T_{\mu\nu}[\psi] := \nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}\nabla^\alpha\psi\nabla_\alpha\psi \tag{1.4}$$

leads to the action

$$\mathcal{L}[\psi] := \int_{\mathcal{M}} g^{\mu\nu}\nabla_\mu\psi\nabla_\nu\psi \mathrm{dvol}_g.$$

We will make use of this Lagrangian structure heavily throughout these notes. However, as we have already seen, it is sometimes useful to think in terms of rescaled versions of ψ, such as the radiation field. Thus, let us introduce a nowhere vanishing function β on $\mathcal{M}$, and define the rescaling $\phi := \beta^{-1}\psi$.

Lemma 1.3. *Let F be given and assume ψ solves the wave equation (1.1). Then ϕ satisfies*

$$\beta^{-2}\nabla^\mu\left(\beta^2\nabla_\mu\phi\right) - U\phi = \beta^{-1}F, \qquad U = -\beta^{-1}\Box_g\beta.$$

Proof. We compute

$$\begin{aligned}
\Box_g\psi = \nabla^\mu\nabla_\mu\psi &= \nabla^\mu\left(\beta\nabla_\mu(\beta^{-1}\psi)\right) - \nabla^\mu(\beta\nabla_\mu\beta^{-1}\psi) \\
&= \beta^{-1}\nabla^\mu\left(\beta^2\nabla_\mu(\beta^{-1}\psi)\right) - (\beta^{-1}\nabla^\mu\beta)\beta\nabla_\mu(\beta^{-1}\psi) + \nabla^\mu(\beta^{-1}\psi\nabla_\mu\beta) \\
&= \beta^{-1}\nabla^\mu\left(\beta^2\nabla_\mu\phi\right) - \nabla^\mu\beta\nabla_\mu\phi + \nabla^\mu(\phi\nabla_\mu\beta) \\
&= \beta^{-1}\nabla^\mu\left(\beta^2\nabla_\mu\phi\right) + \Box_g\beta\phi = \beta\left[\beta^{-2}\nabla^\mu\left(\beta^2\nabla_\mu\phi\right) - U\phi\right],
\end{aligned}$$

which concludes the proof. □

Let us now generalize the definition in (1.4).

Definition 1.4. Take a scalar field ψ on $\mathcal{M}$. We define its twisted energy-momentum tensor by

$$\begin{aligned}
\tilde{T}_{\mu\nu}[\psi] &:= \beta\nabla_\mu(\beta^{-1}\psi)\beta\nabla_\nu(\beta^{-1}\psi) - \frac{1}{2}g_{\mu\nu}\left[\beta\nabla^\alpha(\beta^{-1}\psi)\beta\nabla_\alpha(\beta^{-1}\psi) + U\psi^2\right] \\
&= \beta^2\left[\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\alpha\phi\nabla_\alpha\phi + U\phi^2\right)\right].
\end{aligned} \tag{1.5}$$

Next, we generalize the divergence identity (1.3):

Lemma 1.5. *Let F be given and assume ψ solves (1.1). Then, its twisted energy-momentum tensor satisfies*

$$\nabla^\mu\tilde{T}_{\mu\nu}[\psi] = \beta F\nabla_\nu\phi + \tilde{S}_\nu[\psi], \tag{1.6}$$

where we have set

$$\tilde{S}_\nu[\psi] := -\beta\nabla_\nu\beta\nabla^\alpha\phi\nabla_\alpha\phi - \frac{1}{2}\nabla_\nu(U\beta^2)\phi^2.$$

Proof. We notice that

$$\tilde{T}_{\mu\nu}[\psi] = \beta^2 T_{\mu\nu}[\phi] - \frac{1}{2} g_{\mu\nu} U \phi^2 \beta^2,$$

and thus, using (1.3), we compute

$$\begin{aligned}
\nabla^\mu \tilde{T}_{\mu\nu}[\psi] &= \beta^2 \Box_g(\beta^{-1}\psi)\nabla_\nu\phi + \nabla^\mu(\beta^2) T_{\mu\nu}[\phi] - \frac{1}{2}\nabla_\nu(U\beta^2)\phi^2 - \frac{1}{2}\nabla_\nu(\phi^2) U\beta^2 \\
&= \beta^2 \left[(\Box_g \beta^{-1})\psi + 2\nabla^\mu\beta^{-1}\nabla_\mu\psi + \beta^{-1}\Box_g\psi\right]\nabla_\nu\phi + 2\beta\nabla^\mu\beta\nabla_\mu\phi\nabla_\nu\phi - \beta\nabla_\nu\beta\nabla^\alpha\phi\nabla_\alpha\phi \\
&\quad - \frac{1}{2}\nabla_\nu(U\beta^2)\phi^2 - \phi\nabla_\nu\phi U\beta^2 \\
&= \beta F\nabla_\nu\phi + \tilde{S}_\nu[\psi] + \left[\beta^2\left(\beta\Box_g\beta^{-1} - U\right)\phi + 2\beta\nabla^\mu\beta^{-1}\beta\nabla_\mu\psi + 2\beta\nabla^\mu\beta\nabla_\mu\phi\right]\nabla_\nu\phi \\
&= \beta F\nabla_\nu\phi + \tilde{S}_\nu[\psi] + \beta^2\left(\beta\Box_g\beta^{-1} + 2\nabla^\mu\beta\nabla_\mu\beta^{-1} - U\right)\phi\nabla_\nu\phi.
\end{aligned}$$

The term in round brackets vanishes; indeed,

$$\begin{aligned}
\beta\Box_g\beta^{-1} &= \beta\nabla^\mu\nabla_\mu\beta^{-1} = \nabla^\mu(\beta\nabla_\mu\beta^{-1}) - \nabla^\mu\beta\nabla_\mu\beta^{-1} = -\nabla^\mu(\beta^{-1}\nabla_\mu\beta) - \nabla^\mu\beta\nabla_\mu\beta^{-1} \\
&= -\beta^{-1}\nabla^\mu\nabla_\mu\beta - 2\nabla^\mu\beta\nabla_\mu\beta^{-1} = -\beta^{-1}\Box_g\beta - 2\nabla^\mu\beta\nabla_\mu\beta^{-1} = U - 2\nabla^\mu\beta\nabla_\mu\beta^{-1}.
\end{aligned}$$

This concludes the proof. □

Remark 1.6. In practice, by multiplying (1.1) by $\beta\nabla_\nu\phi$ and integrating by parts, one can identify the appropriate energy-momentum tensor $\tilde{T}[\psi]$ and one-form $\tilde{S}[\psi]$ which allow for identity (1.6) to hold.

The energy-momentum tensor for a scalar field naturally induces a notion of current associated to a vector field on $\mathcal{M}$. We jump ahead to introduce the most general form thereof that we will consider in these notes:

Definition 1.7. Take a scalar field ψ and a vector field X on $\mathcal{M}$. We define the twisted X-current of ψ as the one-form

$$\tilde{J}^X_\mu[\psi] := X^\nu \tilde{T}_{\mu\nu}[\psi]. \tag{1.7}$$

Lemma 1.8. *Let F be given and assume ψ solves (1.1). For any vector field X, the twisted X-current of ψ satisfies*

$$\nabla^\mu \tilde{J}^X_\mu[\psi] = \tilde{K}^X[\psi] + \tilde{\mathcal{E}}^X[\psi], \tag{1.8}$$

where we have set

$$\begin{aligned}
\tilde{K}^X[\psi] &:= {}^{(X)}\pi^{\mu\nu}\left(\tilde{T}_{\mu\nu}[\psi] - \beta\phi^2\nabla_\mu\nabla_\nu\beta\right) - \beta\left(\nabla_\nu{}^{(X)}\pi^{\mu\nu}\nabla_\mu\beta - \frac{1}{2}\nabla_\nu{}^{(X)}\pi^\mu_\mu\nabla^\nu\beta\right)\phi^2 \\
&\quad - \beta X(\beta)\left(\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{2}U\phi^2\right) + \frac{1}{2}\beta\Box_g\left(X(\beta)\right)\phi^2,
\end{aligned} \tag{1.9}$$

$$\tilde{\mathcal{E}}^X[\psi] := \beta F X(\phi). \tag{1.10}$$

In particular, if X is Killing and $X(\beta) = 0$, $\tilde{K}^X[\psi] = 0$.

Proof. We compute

$$\nabla^\mu \tilde{J}^X_\mu[\psi] = \nabla^\mu\left(X^\nu\tilde{T}_{\mu\nu}[\psi]\right) = \nabla^\mu X^\nu \tilde{T}_{\mu\nu}[\psi] + X^\nu\nabla^\mu\tilde{T}_{\mu\nu}[\psi] = {}^{(X)}\pi^{\mu\nu}\tilde{T}_{\mu\nu}[\psi] + X^\nu\nabla^\mu\tilde{T}_{\mu\nu}[\psi],$$

and then use (1.6) to obtain (1.8) with

$$\tilde{K}^X[\psi] = {}^{(X)}\pi^{\mu\nu}\tilde{T}_{\mu\nu}[\psi] + X^\nu\tilde{S}_\nu[\psi] = {}^{(X)}\pi^{\mu\nu}\tilde{T}_{\mu\nu}[\psi] - \beta X(\beta)\nabla^\alpha\phi\nabla_\alpha\phi - \frac{1}{2}X(U\beta^2)\phi^2.$$

To simplify the last term, we recall that

$$X(U\beta^2) = -X(\beta\Box_g\beta) = X(\beta)U\beta - \beta\Box_g\left(X(\beta)\right) + \beta[\Box_g, X]\beta,$$

and we apply (1.2), i.e.

$$[\Box_g, X]\beta = 2^{(X)}\pi^{\mu\nu}\nabla_\mu\nabla_\nu\beta + 2\nabla^{\mu(X)}\pi_{\mu\nu}\nabla^\nu\beta - \nabla_\nu{}^{(X)}\pi^\mu_\mu\nabla^\nu\beta,$$

to rewrite the commutator term. □

Remark 1.9. Note that the one-form $\tilde{J}^X[\psi]$ and the scalar $\tilde{K}^X[\psi]$ only depend on the 1-jet of ψ, i.e. ψ and its first derivatives. For solutions of the homogeneous wave equation, $\tilde{\mathcal{E}}^X[\psi] \equiv 0$ and thus (1.8) implies that $\tilde{J}^X[\psi]$ is a compatible current in the sense of [5].

For later applications, it will also be convenient to define an auxiliary current:

Definition 1.10. Take scalar fields ψ and w on $\mathcal{M}$. We define the w-Lagrangian current of ψ as the one-form

$$\tilde{J}^{\mathrm{aux},w}_\mu[\psi] := \beta^2 \left[w\phi\nabla_\mu\phi - \frac{1}{2}\phi^2\nabla_\mu w \right]. \tag{1.11}$$

Lemma 1.11. *Let F be given and assume ψ solves (1.1). For any scalar field w, the w-Lagrangian current of ψ satisfies*

$$\nabla^\mu \tilde{J}^{\mathrm{aux},w}_\mu[\psi] = \tilde{K}^{\mathrm{aux},w}[\psi] + \tilde{\mathcal{E}}^{\mathrm{aux},w}[\psi], \tag{1.12}$$

where we have set

$$\tilde{K}^{\mathrm{aux},w}[\psi] := \beta^2 w \left(U\phi^2 + \nabla^\mu\phi\nabla_\mu\phi\right) - \frac{1}{2}\phi^2\nabla^\mu(\beta^2\nabla_\mu w) \tag{1.13}$$

$$\tilde{\mathcal{E}}^{\mathrm{aux},w}[\psi] := \beta F w\phi. \tag{1.14}$$

Proof. We compute

$$\begin{aligned}
\nabla^\mu \tilde{J}^{\mathrm{aux},w}_\mu[\psi] &= \nabla^\mu(\beta^2)\left[w\phi\nabla_\mu\phi - \frac{1}{2}\phi^2\nabla_\mu w \right] + \beta^2 \left[w\nabla^\mu\phi\nabla_\mu\phi + w\phi\Box_g\phi - \frac{1}{2}\phi^2\Box_g w \right] \\
&= \nabla^\mu(\beta^2)\left[w\phi\nabla_\mu\phi - \frac{1}{2}\phi^2\nabla_\mu w \right] + \beta^2 w\nabla^\mu\phi\nabla_\mu\phi \\
&\quad + w\beta^2\phi\left[(\Box_g\beta^{-1})\psi + 2\nabla^\mu\beta^{-1}\nabla_\mu\psi + \beta^{-1}\Box_g\psi\right] \\
&\quad - \frac{1}{2}\phi^2\nabla^\mu(\beta^2\nabla_\mu w) + \frac{1}{2}\phi^2\nabla^\mu(\beta^2)\nabla_\mu w \\
&= \tilde{K}^{\mathrm{aux},w}[\psi] + \tilde{\mathcal{E}}^{\mathrm{aux},w}[\psi] + w\beta^2\phi\left[-U\phi + \beta\Box_g\beta^{-1}\phi - 2\beta^{-1}\nabla^\mu\beta\beta^{-1}\nabla_\mu\psi + 2\beta^{-1}\nabla^\mu\beta\nabla_\mu\phi\right],
\end{aligned}$$

where the term in square brackets vanishes by the same argument as in the previous lemma. □

Remark 1.12. In practice, by multiplying (1.1) by $\beta w\phi$ and integrating by parts, one can identify the appropriate one-form $\tilde{J}^{\mathrm{aux},w}[\psi]$ and scalar $\tilde{K}^{\mathrm{aux},w}[\psi]$ which allow for identity (1.12) to hold.

1.3 Stokes' theorem and volume forms

Finally, we can integrate the identities in Lemmas 1.8 and 1.11.

Proposition 1.13 (Stokes' theorem)**.** *Let F be given, and ψ be a solution to (1.1). Let $\mathcal{R} \subset \mathcal{M}$ be a time oriented domain in $\mathcal{M}$ with piecewise smooth boundary*

$$\partial\mathcal{R} \equiv \mathcal{T} \sqcup \mathcal{S}_+ \sqcup \mathcal{S}_- \sqcup \mathcal{N}_+ \sqcup \mathcal{N}_-$$

where each term denotes a possibly empty finite union of hypersurfaces, those in $\mathcal{T}$ are timelike, those in $\mathcal{S}_\pm$ are spacelike, and those in $\mathcal{N}_\pm$ are null. Let:

- *$n_\mathcal{T}$ denote the unit normal of $\mathcal{T}$ (i.e. a spacelike vector field satisfying $g(X, n_\mathcal{T}) = 0$ for all vector fields X tangent to $\mathcal{T}$ and normalized so that $g(n_\mathcal{T}, n_\mathcal{T}) = 1$) which is outward-pointing with respect to $\mathrm{int}(\mathcal{R})$; set $\mathrm{dvol}_\mathcal{T} = \mathrm{dvol}_{g|_\mathcal{T}}$;*
- *$n_{\mathcal{S}_\pm}$ denote the unit normal of $\mathcal{S}_\pm$ (i.e. a timelike vector field satisfying $g(X, n_{\mathcal{S}_\pm}) = 0$ for all vector fields X tangent to $\mathcal{S}_\pm$ and normalized so that $g(n_{\mathcal{S}_\pm}, n_{\mathcal{S}_\pm}) = -1$) which is future-directed; set $\mathrm{dvol}_{\mathcal{S}_\pm} = \mathrm{dvol}_{g|_{\mathcal{S}_\pm}}$;*
- *$\underline{n}_{\mathcal{N}_\pm}$ be a null generator of $\mathcal{N}_\pm$ (i.e. a null vector field tangent to the null generators of $\mathcal{N}_\pm$ and satisfying $g(X, \underline{n}_{\mathcal{N}_\pm}) = 0$ for all vector fields X tangent to $\mathcal{N}_\pm$) which is future-directed; then, choose $n_{\mathcal{N}_\pm}$ to be a future-directed transverse null vector field normalized so that $g(\underline{n}_{\mathcal{N}_\pm}, n_{\mathcal{N}_\pm}) = -1$, and set $\mathrm{dvol}_{\mathcal{N}_\pm} = -n_{\mathcal{N}_\pm} \lrcorner \mathrm{dvol}_g$.*

Then, we have

$$\begin{aligned}&\int_{\mathcal{T}} \tilde{J}^{\bigcirc}_\mu[\psi] n^\mu_{\mathcal{T}} \mathrm{dvol}_{\mathcal{T}} + \int_{\mathcal{S}_+} \tilde{J}^{\bigcirc}_\mu[\psi] n^\mu_{\mathcal{S}_+} \mathrm{dvol}_{\mathcal{S}_+} + \int_{\mathcal{N}_+} \tilde{J}^{\bigcirc}_\mu[\psi] \underline{n}^\mu_{\mathcal{N}_+} \mathrm{dvol}_{\mathcal{N}_+} + \int_{\mathcal{R}} \left(\tilde{K}^{\bigcirc}[\psi] + \tilde{\mathcal{E}}^{\bigcirc}[\psi]\right) \mathrm{dvol}_{\mathcal{R}} \\ &= \int_{\mathcal{S}_-} \tilde{J}^{\bigcirc}_\mu[\psi] n^\mu_{\mathcal{S}_-} \mathrm{dvol}_{\mathcal{S}_-} + \int_{\mathcal{N}_-} \tilde{J}^{\bigcirc}_\mu[\psi] \underline{n}^\mu_{\mathcal{N}_-} \mathrm{dvol}_{\mathcal{N}_-},\end{aligned} \tag{1.15}$$

where $\bigcirc$ *can be replaced by any vector field* X*, or by the notation* aux, w *for a scalar field* w*.*

Proof. Apply Stokes' theorem to the vector field $(\tilde{J}^{\bigcirc})^\sharp$, or equivalently to the three-form $(\tilde{J}^{\bigcirc})^\sharp \lrcorner \mathrm{dvol}_g$. Since

$$d\left((\tilde{J}^{\bigcirc})^\sharp \lrcorner \mathrm{dvol}_g\right) = \nabla_\mu (\tilde{J}^{\bigcirc})^{\sharp\mu} \,\mathrm{dvol}_g = \nabla^\mu \tilde{J}^{\bigcirc}_\mu \,\mathrm{dvol}_g = \left(\tilde{K}^{\bigcirc} + \tilde{\mathcal{E}}^{\bigcirc}\right) \mathrm{dvol}_g,$$

hence Stokes' theorem gives

$$\int_{\mathcal{R}} (\tilde{K}^{\bigcirc} + \tilde{\mathcal{E}}^{\bigcirc}) \,\mathrm{dvol}_g = \int_{\partial\mathcal{R}} (\tilde{J}^{\bigcirc})^\sharp \lrcorner \,\mathrm{dvol}_g.$$

With our $-+++$ signature, the pullback of $(\tilde{J}^{\bigcirc})^\sharp \lrcorner \,\mathrm{dvol}_g$ to a future spacelike or future null boundary component is the negative of the corresponding future-directed flux, whereas on a past spacelike or past null boundary component it is the positive of the corresponding future-directed flux. The timelike side boundary piece comes with the outward spacelike normal, as usual. Rearranging concludes the proof. □

End of Lecture 1

Start of Lecture 2

1.4 Application: local theory

We apply the geometric framework we developed to establish basic analysis results for (1.1) over $\mathcal{R}$ as in Figure 1.

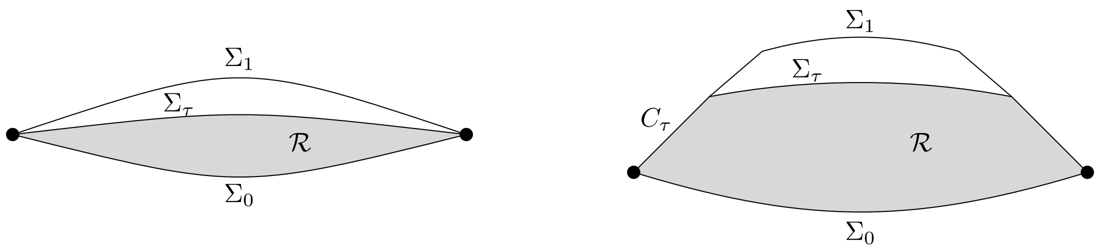


Figure 1: The regions $\mathcal{R}$ considered in Section 1.4.

1.4.1 First order energy estimates

In this section, we want to show the following a priori estimate:

Proposition 1.14 (First order a priori estimates)**.** *Let* F *be given, assume* ψ *is a smooth solution to* (1.1)*. Then, there exists a* C *depending only on the region* $\mathcal{R}$ *such that*

$$\|\psi\|^2_{H^1(\Sigma_\tau)} + \|n_{\Sigma_\tau}\psi\|^2_{L^2(\Sigma_\tau)} \le C\big(\|\psi\|^2_{H^1(\Sigma_0)} + \|n_{\Sigma_0}\psi\|^2_{L^2(\Sigma_0)} + \sup_{\tau'\in(0,\tau)} \|F\|^2_{L^2(\Sigma_{\tau'})}\big)$$

In particular, this implies uniqueness of solutions to (1.1) *with the same data* $\psi|_{\Sigma_0}$, $n_{\Sigma_0}\psi|_{\Sigma_0}$ *in the the domain of dependence of* Σ_0*.*

We will do this by employing the currents introduced in Section 1.2. The reason why one might hope to do so is contained in the following lemma

Lemma 1.15. *Suppose* β *is chosen such that* $U \geq 0$*. Then, the following hold.*

- *If X is a future-directed causal vector, $-\tilde{J}^X[\psi]$ is also future-directed and causal. Thus, if Y is also future-directed and causal,*
$$\tilde{J}^X_\mu[\psi]Y^\mu = \tilde{T}_{\mu\nu}[\psi]X^\nu Y^\mu \geq 0.$$
- *If X and Y are future-directed and timelike vector fields, then*
$$\tilde{J}^X_\mu[\psi]Y^\mu = \tilde{T}_{\mu\nu}[\psi]X^\nu Y^\mu \geq c\beta^2\Big[\sum_i(\partial_i\phi)^2 + U\phi^2\Big].$$
- *If X is a future-directed timelike vector field and Y is a future-directed null vector field, then*
$$\tilde{J}^X_\mu[\psi]Y^\mu = \tilde{T}_{\mu\nu}[\psi]X^\nu Y^\mu \geq c\beta^2\Big[(Y\phi)^2 + \sum_{i=1}^{n-1}(e_i\phi)^2 + U\phi^2\Big],$$
where we have $\mathrm{span}\{Y, e_1, \dots, e_{n-1}\} = Y^\perp$*. (Thus, $\tilde{J}^X_\mu[\psi]Y^\mu$ controls $Y\phi$ and all spacelike derivatives tangent to $Y^\perp$, but not the derivative along the null direction conjugate to Y.)*
- *If X and Y are linearly independent future-directed null vector fields, then*
$$\tilde{J}^X_\mu[\psi]Y^\mu = \tilde{T}_{\mu\nu}[\psi]X^\nu Y^\mu \geq c\beta^2\Big[\sum_{i=1}^{n-1}(e_i\phi)^2 + U\phi^2\Big],$$
where $\mathrm{span}\{e_1, \dots, e_{n-1}\} = X^\perp \cap Y^\perp$*. (Thus, $\tilde{J}^X_\mu[\psi]Y^\mu$ controls all derivatives of ϕ except for those along X and Y.)*

Proof. We have that
$$\tilde{T}_{\mu\nu}[\psi]X^\nu Y^\mu = \beta^2\left[\left((X\phi)(Y\phi) - \frac{1}{2}g(X,Y)g(\mathrm{d}\phi,\mathrm{d}\phi)\right) - \frac{1}{2}g(X,Y)U\phi^2\right],$$
so clearly the last term contributes non-negatively as long as $-g(X,Y) \geq 0$ and $U \geq 0$. For the term in square brackets, we decompose into two future-directed linearly independent null vectors L and $\underline{L}$ normalized so that $g(L,\underline{L}) = -2$: if X, Y are future-directed causal, then
$$X = aL + b\underline{L}, \quad Y = cL + d\underline{L},$$
for some $a, b, c, d \geq 0$. We compute
$$(X\phi)(Y\phi) - \frac{1}{2}g(X,Y)g(\mathrm{d}\phi,\mathrm{d}\phi) = ac(L\phi)^2 + bd(\underline{L}\phi)^2 + (ad+bc)\sum_{i=1}^{n-1}(e_i\phi)^2,$$
where $(e_1, \dots, e_{n-1})$ form a basis of the tangent space to $L^\perp \cap \underline{L}^\perp$. Note that this is always possible because the result is local, and we can always find a null frame $(e_1, \dots, e_{n-1}, L, \underline{L})$ at each point of the manifold. $\square$

Lemma 1.15 in some sense bridges the gap between the highly geometric language of the previous section and the analytical nature of the task at hand: looking back to Proposition 1.13, Lemma 1.15 says that for appropriate choices of multiplier X the last two terms on the left hand side of (1.15) amount to L^2 norms of some or all derivatives of ϕ along $\partial\mathcal{R}$.

Proof of Proposition 1.14. On $\mathcal{R}$, we can find a timelike vector field X. Let $\{\Sigma_\tau\}_{\tau\in[0,1]}$ be a foliation of $\mathcal{R}$. By Proposition 1.13, we have
$$\begin{aligned}&\int_{\Sigma_\tau} J^X_\mu[\psi]n^\mu_{\Sigma_\tau}\mathrm{dvol}_{\Sigma_\tau} + \int_{C_\tau} J^X_\mu[\psi]n^\mu_{C_\tau}\mathrm{dvol}_{C_\tau} - \int_{\Sigma_0} J^X_\mu[\psi]n^\mu_{\Sigma_0}\mathrm{dvol}_{\Sigma_0}\\ &= -\int_{\mathcal{R}}\left(K^X[\psi] + \mathcal{E}^X[\psi]\right)\mathrm{dvol}_{\mathcal{R}},\end{aligned} \tag{1.16}$$
where we have dropped tildes to indicate that we are setting $\beta \equiv 1$. By Lemma 1.15,
$$f(\tau) := \int_{\Sigma_\tau} J^X_\mu[\psi]n^\mu_{\Sigma_\tau}\mathrm{dvol}_{\Sigma_\tau} \geq c\|\phi\|^2_{\mathring{H}^1(\Sigma_\tau)} + c\|n_{\Sigma_\tau}\phi\|^2_{L^2(\Sigma_\tau)} \geq 0,$$
$$\int_{C_\tau} J^X_\mu[\psi]n^\mu_{C_\tau}\mathrm{dvol}_{C_\tau} \geq 0,$$

for all $\tau \in [0,1]$. Since X need not be Killing, we may have $K^X[\psi] \not\equiv 0$. However, since $K^X[\psi]$ only depends on ψ and its first order derivatives, by compactness of $\mathcal{R}$ we have

$$\int_{\mathcal{R}} \left(K^X[\psi] + \mathcal{E}^X[\psi]\right) \mathrm{dvol}_{\mathcal{R}} \le \int_0^\tau \|F\|^2_{L^2(\Sigma_{\tau'})} d\tau' + C\int_0^\tau \int_{\Sigma_{\tau'}} J^X_\mu[\psi] n^\mu_{\Sigma_{\tau'}} \mathrm{dvol}_{\Sigma_{\tau'}} d\tau'.$$

Thus, dropping the integral over C_τ due to its non-negativeness, (1.15) becomes

$$f(\tau) \le f(0) + D + C\int_0^\tau f(\tau')d\tau' \implies f(\tau) \le e^{C\tau}(f(0)+D), \quad D := \int_0^\tau \|F\|^2_{L^2(\Sigma_{\tau'})} d\tau',$$

by Grönwall's inequality. Hence, we obtain

$$\|\psi\|^2_{\mathring{H}^1(\Sigma_\tau)} + \|n_{\Sigma_\tau}\psi\|^2_{L^2(\Sigma_\tau)} \le C(\tau)\big(\|\psi\|^2_{\mathring{H}^1(\Sigma_0)} + \|n_{\Sigma_0}\psi\|^2_{L^2(\Sigma_0)} + \sup_{\tau'\in(0,\tau)} \|F\|^2_{L^2(\Sigma_{\tau'})}\big).$$

Finally, we need to show that the homogeneous $\mathring{H}^1$ norms can be replaced by inhomogeneous H^1 norms. To do so, first note that the fundamental theorem of calculus and the fact that $\mathcal{R}$ is bounded implies

$$\|\psi\|^2_{H^1(\Sigma_\tau)} \le C\|\psi\|^2_{\mathring{H}^1(\Sigma_\tau)} + C\|\psi\|^2_{L^2(\partial\Sigma_\tau)} = C\|\psi\|^2_{\mathring{H}^1(\Sigma_\tau)} + C\|\psi\|^2_{L^2(\partial\Sigma_0)}$$

and the fact that $\partial\Sigma_\tau = \partial\Sigma_0$ is a compact surface, implies, by the trace theorem

$$\|\psi\|_{L^2(\partial\Sigma_0)} \le C\|\psi\|_{H^1(\Sigma_0)},$$

concluding the proof. □

1.4.2 Higher order energy estimates

The estimates in the preceding section hold for higher order derivatives of ψ by the methods developed in Section 1.1. As an example, let us extend Proposition 1.14:

Proposition 1.16 (Higher order a priori estimates)**.** *Let F be given, assume ψ is a smooth solution to* (1.1)*. Then, for $s \ge 1$, there exists a constant C depending only on the region $\mathcal{R}$ and s such that*

$$\|\psi\|^2_{H^s(\Sigma_1)} + \|n_{\Sigma_1}\psi\|^2_{H^{s-1}(\Sigma_1)} \le C\big(\|\psi\|^2_{H^s(\Sigma_0)} + \|n_{\Sigma_0}\psi\|^2_{H^{s-1}(\Sigma_0)} + \sup_{\tau'\in(0,\tau)} \|F\|^2_{H^{s-1}(\Sigma_{\tau'})}\big). \quad (1.17)$$

Furthermore, if the inhomogeneity F and the data $\psi|_{\Sigma_0}$, $n_{\Sigma_0}\psi|_{\Sigma_0}$ are smooth, then ψ is the unique smooth solution to (1.1)*.*

Proof. Let X and Σ_τ be as in the proof of Proposition 1.14. Let us further introduce $\{Z^A\}_{A=0}^n$, a collection of vector fields spanning the tangent space at $\mathcal{M}$. Then,

$$\sum_{A=0}^n \int_{\Sigma_\tau} J^X_\mu[Z^A\psi] n^\mu_{\Sigma_\tau} \mathrm{dvol}_{\Sigma_\tau} \ge c\|\phi\|^2_{\mathring{H}^2(\Sigma_\tau)} + c\|n_{\Sigma_\tau}\phi\|^2_{\mathring{H}^1(\Sigma_\tau)} + \|n^2_{\Sigma_\tau}\phi\|^2_{L^2(\Sigma_\tau)} \ge 0,$$

and, as before,

$$\int_{\mathcal{R}} \sum_{A=0}^n K^X[Z^A\psi]\mathrm{dvol}_{\mathcal{R}} \le C\int_0^\tau \int_{\Sigma_{\tau'}} \sum_{A=0}^n J^X_\mu[Z^A\psi] n^\mu_{\Sigma_{\tau'}} \mathrm{dvol}_{\Sigma_{\tau'}} d\tau'.$$

Here we have again dropped the tildes from our notation associated to the X-currents of ψ as we have the twisting parameter to be $\beta \equiv 1$. Since Z^A need not commute with $\Box_g$, we will now have $\mathcal{E}^X[Z^A\psi] \not\equiv 0$ in general even if $F \equiv 0$; however, by Lemma 1.2, we can estimate

$$\int_{\mathcal{R}} \sum_{A=0}^n \mathcal{E}^X[Z^A\psi]\mathrm{dvol}_{\mathcal{R}} \le \int_0^\tau \|F\|^2_{\mathring{H}^1(\Sigma_{\tau'})} d\tau' + C\int_0^\tau \int_{\Sigma_{\tau'}} \sum_{A=0}^n J^X_\mu[Z^A\psi] n^\mu_{\Sigma_{\tau'}} \mathrm{dvol}_{\Sigma_{\tau'}} d\tau'.$$

From Proposition 1.13 and Grönwall's inequality, analogously to the proof of Proposition 1.14, we obtain

$$\begin{aligned}&\|\psi\|^2_{\mathring{H}^2(\Sigma_\tau)} + \|n_{\Sigma_\tau}\psi\|^2_{\mathring{H}^1(\Sigma_\tau)} + \|n^2_{\Sigma_\tau}\psi\|^2_{L^2(\Sigma_\tau)} \\ &\quad \le C(\tau)\big(\|\psi\|^2_{\mathring{H}^2(\Sigma_0)} + \|n_{\Sigma_0}\psi\|^2_{\mathring{H}^1(\Sigma_0)} + \|n^2_{\Sigma_0}\psi\|^2_{L^2(\Sigma_0)} + \sup_{\tau'\in(0,\tau)} \|F\|^2_{H^1(\Sigma_{\tau'})}\big).\end{aligned}$$

By combining with Proposition 1.14, we can change from homogeneous to inhomogenous Sobolev norms. Moreover, the wave equation (1.1) allows us to solve for $n^2_{\Sigma_\tau}\psi$ in terms of other quantities in the estimate; thus, we have

$$\|\psi\|^2_{H^2(\Sigma_\tau)} + \|n_{\Sigma_\tau}\psi\|^2_{H^1(\Sigma_\tau)} \le C(\tau)\big(\|\psi\|^2_{H^2(\Sigma_0)} + \|n_{\Sigma_0}\psi\|^2_{H^1(\Sigma_0)} + \sup_{\tau'\in(0,\tau)} \|F\|^2_{H^1(\Sigma_{\tau'})}\big).$$

Iterating the procedure above, we obtain the Sobolev estimates claimed in the statement. Finally, note that Sobolev inequalities

$$\sup_{\Sigma_\tau} \psi \lesssim \|\psi\|_{H^s(\Sigma_\tau)},$$

with $s - n/2 > k \geq 0$, allow us to pass from control of ψ in Sobolev norms to control of ψ in C^k norms. $\square$

1.4.3 Local well-posedness

Finally, we deduce:

Theorem 1.17 (Local well-posedness). *Let $(\mathcal{M}, g)$ be globally hyperbolic with Cauchy surface Σ, $s \geq 1$. The initial value problem for scalar wave equation (1.1) is locally well-posed. Concretely, given scalar fields $\uppsi \in H^s_{\mathrm{loc}}(\Sigma)$, $\mathring{\uppsi} \in H^{s-1}_{\mathrm{loc}}(\Sigma)$ and $F \in L^\infty_\tau(H^{s-1}_{\mathrm{loc}}(\Sigma_\tau))$, the problem*

$$\Box_g \psi = F, \qquad \psi|_\Sigma = \uppsi, \qquad n_\Sigma \psi|_\Sigma = \mathring{\uppsi}.$$

has a unique solution ψ in $\mathcal{M}$ which, on any spacelike hypersurface $S \subset \mathcal{M}$, satisfies $\psi \in H^s_{\mathrm{loc}}(S)$ and $n_S \psi \in H^{s-1}_{\mathrm{loc}}(S)$ and depends continuously on $(\uppsi, \mathring{\uppsi})$.

Proof. In Propositions 1.14 and 1.16, we showed the uniqueness of solutions in the classes discussed as well as the continuous dependence on the data. To prove existence, we restrict to a sufficiently small $\mathcal{R}$ that g can be well-approximated there by the Minkowski metric in some coordinate system. Then, Hahn–Banach can be used to produce a distributional solution to (1.1), which, using the equation and the a priori estimates (1.17), can be shown to have the right regularity. See, e.g. [15, Proof of Theorem 3.2]. $\square$

2 From Minkowski to the Schwarzschild exterior

In the previous section, we saw that for general globally hyperbolic spacetimes $(\mathcal{M}, g)$, we can solve the covariant wave equation (1.1) to the future of the Cauchy surface Σ. However, the control we have from these general results breaks down the further into the future of Σ we try to go. In other words, we have very little quantitative information on the long time dynamics of (1.1) for general $(\mathcal{M}, g)$. In this section, we focus on two cases where we can say a lot more in this direction: Minkowski space, and the exterior region of a Schwarzschild black hole (excluding the event horizon).

2.1 Primer on the geometry of Minkowski and Schwarzschild

Minkowski space and the Schwarzschild manifold are two examples of spherically symmetric solutions of the Einstein vacuum equations (i.e. are Ricci-flat Lorentzian manifolds). It will thus be useful to introduce some notation for the round sphere: $d\sigma_{\mathbb{S}^2}$ will denote here the round metric on $\mathbb{S}^2$; with the usual $(\theta, \varphi) \in (0, \pi) \times [0, 2\pi)$ polar coordinates[1] on $\mathbb{S}^2$, we write

$$|\mathring{\nabla} f|^2 := |\partial_\theta f|^2 + \frac{1}{\sin^2\theta}|\partial_\varphi f|^2, \qquad \mathring{\Delta} f := -\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta f) - \frac{1}{\sin^2\theta}\partial_\varphi^2 f,$$

for smooth scalar fields f on $\mathbb{S}^2$, and $\mathrm{d}\sigma = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ for the volume form associated to the round metric $d\sigma_{\mathbb{S}^2}$.

Minkowski space, in addition to being Ricci-flat, is (Riemann-)flat. It is defined by

$$\mathcal{M}_{\text{mink}} := \mathbb{R}_t \times \mathbb{R}^3_{(x,y,z)},$$
$$g_{\text{mink}} := -dt^2 + dx^2 + dy^2 + dz^2,$$

in Euclidean coordinates. In the ensuing sections we will in practice use other types of coordinates on this space: setting $r^2 = x^2 + y^2 + z^2$, we can use polar coordinates to describe Minkowski space (with the usual degeneration at $r = 0$) as

$$\mathcal{M}_{\text{mink}} = \mathbb{R}_t \times [0, \infty)_r \times \mathbb{S}^2,$$
$$g_{\text{mink}} = -dt^2 + dr^2 + r^2 d\sigma_{\mathbb{S}^2};$$

with null coordinates

$$u = \frac{1}{2}(t - r), \qquad v = \frac{1}{2}(t + r),$$

we have (with the usual degeneration at $r(u, v) = 0 \iff u = v$)

$$\mathcal{M}_{\text{mink}} = \{(u, v) \in \mathbb{R}^2 \colon v \geq u\} \times \mathbb{S}^2,$$
$$g_{\text{mink}} = -4dudv + r^2(u, v)d\sigma_{\mathbb{S}^2}, \qquad r(u, v) = v - u.$$

We illustrate the Minkowski manifold, as well as some foliations of it which we will use in the ensuing sections, in Figure 2.

The exterior region of a Schwarzschild black hole (excluding the event horizon) can be described by

$$\mathcal{M}_{\text{Schw}(M)} := \mathbb{R}_t \times (r_+, \infty)_r \times \mathbb{S}^2, \qquad M > 0, \qquad r_+ = 2M,$$
$$g_{\text{Schw}(M)} := -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\sigma_{\mathbb{S}^2}, \qquad \Delta = r^2 - 2Mr.$$

In the ensuing sections, we will also find it convenient to introduce the tortoise coordinate $r^* = r^*(r)$

$$\frac{dr^*}{dr} = \frac{r^2}{\Delta}, \quad r^*(3M) = 0,$$

with which we have

$$\mathcal{M}_{\text{Schw}(M)} = \mathbb{R}_t \times \mathbb{R}_{r^*} \times \mathbb{S}^2, \qquad M > 0,$$
$$g_{\text{Schw}(M)} := \frac{\Delta}{r^2}\left[-dt^2 + (dr^*)^2\right] + r^2(r^*)d\sigma_{\mathbb{S}^2},$$

and the corresponding null coordinates

$$u = \frac{1}{2}(t - r^*), \qquad v = \frac{1}{2}(t + r^*),$$

[1] Recall that it is understood that $\varphi = 0$ and $\varphi = 2\pi$ are identified; then, these coordinates are global except for the usual degeneration at $\theta = 0, \pi$.

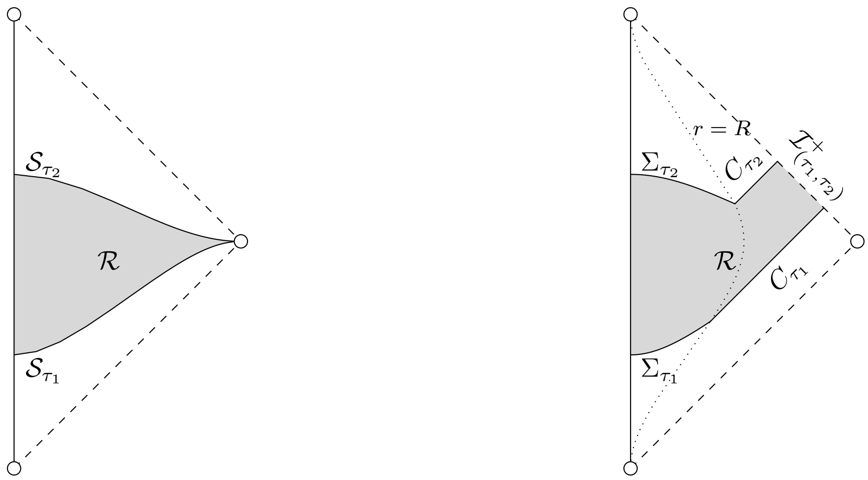


Figure 2: Minkowski space visualized by its Penrose diagram. The left panel illustrates a foliation by an $SO(3)$-invariant uniformly spacelike hypersurfaces $\{\mathcal{S}_\tau\}_\tau$ which terminate at spacelike infinity i^0, such as $\mathcal{S}_\tau = \{(t, r, \theta, \varphi) \in \mathcal{M}_{\text{mink}} : t = \tau\}$. The right panel illustrates a foliation by piecewise smooth hypersurfaces $\{\Sigma_\tau\}_\tau$ which terminate at future null infinity $\mathcal{I}^+$: each Σ_τ consists of a spacelike piece $\mathcal{S}_\tau \cap \{r \le R\}$, for some $R > 0$, and an outgoing null piece C_τ emanating from $S_\tau \cap \{r = R\}$.

with which we have

$$\mathcal{M}_{\text{Schw}(M)} = \mathbb{R}_u \times \mathbb{R}_v \times \mathbb{S}^2, \qquad M > 0,$$
$$g_{\text{Schw}(M)} := -4\frac{\Delta}{r^2} du dv + r^2(u, v) d\sigma_{\mathbb{S}^2}, \qquad r^*(u, v) = v - u.$$

Furthermore, to extend the manifold to include the future event horizon, it is helpful to introduce a horizon-penetrating time coordinate

$$t^* = 2v - r,$$

with which

$$\mathcal{M}_{\text{Schw}(M)} = \mathbb{R}_{t^*} \times (2M, \infty)_r \times \mathbb{S}^2, \qquad M > 0,$$
$$g_{\text{Schw}(M)} = -\frac{\Delta}{r^2}(dt^*)^2 + \frac{4M}{r} dt^* dr + \left(1 + \frac{2M}{r}\right) dr^2 + r^2 d\sigma_{\mathbb{S}^2},$$

can be extended to include $\mathcal{H}^+ = \{r = 2M\}$. We illustrate the Schwarzschild exterior manifold, as well as some foliations of it which we will use in the ensuing sections, in Figure 3.

Exercise 2.1. A common source of confusion in the above equivalent descriptions of the Schwarzschild exterior manifold is that, although the notation r is kept throughout several changes of variables, the coordinate vector field denoted by ∂_r depends on the chart being used.

Compare the coordinate vector field ∂_r in the $(t^*, r, \theta, \varphi)$ chart with the coordinate vector field ∂_r in the (t, r, θ, φ) chart on $\mathcal{M}_{\text{Schw}(M)}$.

Explain why the notation r is nevertheless kept in both charts by assigning a geometric meaning to the scalar function r on $\mathcal{M}_{\text{Schw}(M)}$.

2.2 First order energy estimate

In this section, we re-do the estimates of Section 1.4.1 for general manifolds now on Minkowski and on the Schwarzschild exterior away from the horizon. The crucial point is that for these special spacetimes there is a privileged choice of timelike vector field to use as multiplier: the vector field $T = \partial_t$ which is, moreover, Killing. Another difference is that the hypersurfaces in our foliations of the manifold will not be compact; hence, we will take $\beta^{-1} = r$ nontrivial, so that $\phi := r\psi$ is the radiation field.

Proposition 2.2 (Energy estimates on Minkowski)**.** *Let $g = g_{\text{mink}}$, and fix some $R > 0$. Fix foliations $\{\mathcal{S}_\tau\}_\tau$ and $\{\Sigma_\tau\}_\tau$ of $\mathcal{M}_{\text{mink}}$ as follows: $\{\mathcal{S}_\tau\}_\tau$ is an $SO(3)$-invariant, T-invariant uniformly spacelike foliation by smooth hypersurfaces which terminate at spacelike infinity (such as the constant-t foliation); $\{\Sigma_\tau\}_\tau$ is a*

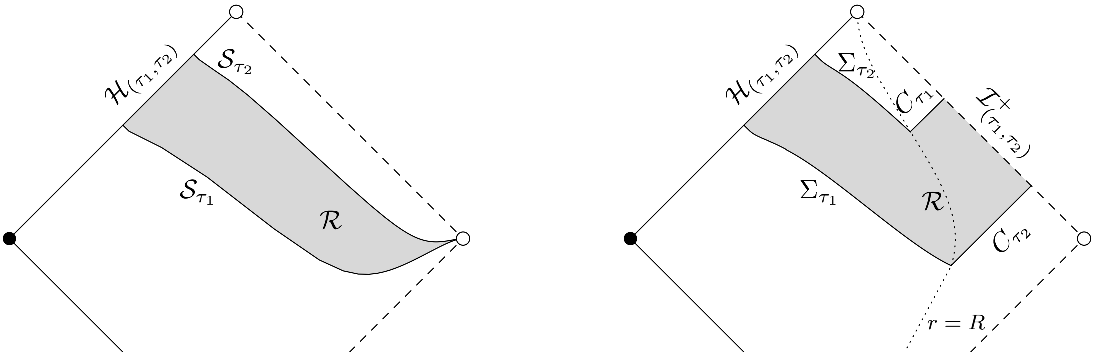


Figure 3: The Schwarzschild exterior, with event horizons attached, visualized by its Penrose diagram. The left panel illustrates a foliation by an $SO(3)$-invariant uniformly spacelike hypersurfaces $\{\mathcal{S}_\tau\}_\tau$ which connect the future event horizon $\mathcal{H}^+$ with spacelike infinity i^0, such as $\mathcal{S}_\tau = \{(t^*, r, \theta, \varphi) \in \mathcal{M}_{\mathrm{Schw}(M)} : t = \tau\}$. The right panel illustrates a foliation by piecewise smooth hypersurfaces $\{\Sigma_\tau\}_\tau$ which connect the future event horizon $\mathcal{H}^+$ with future null infinity $\mathcal{I}^+$: each Σ_τ consists of a spacelike piece $\mathcal{S}_\tau \cap \{r \leq R\}$, for some $R > 0$, and an outgoing null piece C_τ emanating from $S_\tau \cap \{r = R\}$.

foliation by piecewise smooth hypersurfaces which terminate at future null infinity and consist of a spacelike piece $\mathcal{S}_\tau \cap \{r \leq R\}$ and an outgoing null piece C_τ emanating from $\mathcal{S}_\tau \cap \{r = R\}$ (see Figure 2 for an illustration).

Assume ψ is a smooth scalar field which solves the homogeneous wave equation (1.1). Then, ψ has constant T-energy across $\{\mathcal{S}_\tau\}_\tau$, and non-increasing T-energy across a foliation $\{\Sigma_\tau\}_\tau$: for any $0 \leq \tau_1 \leq \tau_2 < \infty$,

$$\begin{aligned} \int_{\mathcal{S}_{\tau_2}} \tilde{J}^T_\mu[\psi] n^\mu_{\mathcal{S}_{\tau_2}} \mathrm{dvol}_{\mathcal{S}_{\tau_2}} &= \int_{\mathcal{S}_{\tau_1}} \tilde{J}^T_\mu[\psi] n^\mu_{\mathcal{S}_{\tau_1}} \mathrm{dvol}_{\mathcal{S}_{\tau_1}}, \\ \int_{\Sigma_{\tau_2}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} &\leq \int_{\Sigma_{\tau_1}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}. \end{aligned} \tag{2.1}$$

We note the upper and lower bounds

$$\begin{aligned} \tilde{J}^T_\mu[\psi] n^\mu_{\mathcal{S}_\tau} \mathrm{dvol}_{\mathcal{S}_\tau} &\sim \left(|\partial_t \phi|^2 + |\partial_r \phi|^2 + \frac{1}{r^2} |\mathring{\nabla}\phi|^2 \right) \mathrm{d}r\mathrm{d}\sigma \\ &\lesssim \left(|\partial_t \psi|^2 + |\partial_r \psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + |r^{-1}\psi|^2 \right) r^2 \mathrm{d}r\mathrm{d}\sigma, \\ \tilde{J}^T_\mu[\psi] \underline{n}^\mu_{C_\tau} \mathrm{dvol}_{C_\tau} &\sim \left(|\partial_v \phi|^2 + \frac{1}{r^2} |\mathring{\nabla}\phi|^2 \right) \mathrm{d}v\mathrm{d}\sigma, \end{aligned} \tag{2.2}$$

for any $\tau \in \mathbb{R}$. It follows that there are universal constants C such that the following Sobolev estimates hold for any $0 \leq \tau_1 \leq \tau_2 < \infty$:

$$\begin{aligned} &\int_{\mathcal{S}_{\tau_2}} \left(|\partial_t \psi|^2 + |\partial_r \psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + |(r+1)^{-1}\psi|^2 \right) r^2 \mathrm{d}r\mathrm{d}\sigma \\ &\quad \leq C \int_{\mathcal{S}_{\tau_1}} \left(|\partial_t \psi|^2 + |\partial_r \psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + |(r+1)^{-1}\psi|^2 \right) r^2 \mathrm{d}r\mathrm{d}\sigma, \\ &\int_{\mathcal{S}_{\tau_2} \cap \{r \leq R\}} \left(|\partial_t \psi|^2 + |\partial_r \psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + |(r+1)^{-1}\psi|^2 \right) r^2 \mathrm{d}r\mathrm{d}\sigma \\ &\quad + \int_{C_{\tau_2}} \left(|\partial_v (r\psi)|^2 + |(r^{-1}\mathring{\nabla})(r\psi)|^2 + |\psi|^2 \right) \mathrm{d}v\mathrm{d}\sigma \\ &\quad \leq C \int_{\mathcal{S}_{\tau_1} \cap \{r \leq R\}} \left(|\partial_t \psi|^2 + |\partial_r \psi|^2 + |(r^{-1}\mathring{\nabla})\psi|^2 + |(r+1)^{-1}\psi|^2 \right) r^2 \mathrm{d}r\mathrm{d}\sigma \\ &\quad + \int_{C_{\tau_1}} \left(|\partial_v (r\psi)|^2 + |(r^{-1}\mathring{\nabla})(r\psi)|^2 + |\psi|^2 \right) \mathrm{d}v\mathrm{d}\sigma. \end{aligned} \tag{2.3}$$

Proof. The assumptions on the spacelike hypersurfaces $\mathcal{S}_\tau$ imply that these can be parametrized as graphs $\{(t, r, \theta, \varphi) : t = \tau + h(r)\}$ with $1 - (\partial_r h)^2$ bounded above and below away from zero uniformly in r; thus,

$\mathrm{dvol}_{\mathcal{S}_\tau} \sim r^2 \mathrm{d}r\mathrm{d}\sigma$. The assumptions on the null hypersurfaces C_τ imply that these can be parametrized in terms of the (v,θ,φ) coordinates and $\underline{n}_{C_\tau}\mathrm{dvol}_{C_\tau} \sim \mathrm{d}v\mathrm{d}\sigma$. These observations, when combined with Lemma 1.15, yield (2.2), since $U = -r^{-1}\Box_{g_{\mathrm{mink}}} r = 0$.

Lemma 1.15 also yields that

$$\int_{\mathcal{I}^+_{(\tau_1,\tau_2)}} \tilde{J}^T_\mu[\psi]\underline{n}^\mu_{\mathcal{I}^+}\mathrm{dvol}_{\mathcal{I}^+} := \lim_{V\to\infty}\int_{\underline{C}_V} \tilde{J}^T_\mu[\psi]\underline{n}^\mu_{\underline{C}_V}\mathrm{dvol}_{\underline{C}_V} \geq 0,$$

where $\underline{C}_V$ is defined as the portion of $\{(u,v,\theta,\varphi)\in\mathcal{M}_{\mathrm{mink}}\colon v = V\}$ between C_{τ_1} and C_{τ_2}. Since T is Killing and $T(\beta)=0$, from Proposition 1.13, we obtain (2.1).

To conclude the proof, we establish that

$$\|(r+1)^{-1}\psi\|^2_{L^2(\mathcal{S}_\tau)} \lesssim \int_{\mathcal{S}_\tau} \tilde{J}^T_\mu[\psi]n^\mu_{\mathcal{S}_\tau}\mathrm{dvol}_{\mathcal{S}_\tau},$$

$$\|(r+1)^{-1}\psi\|^2_{L^2(\Sigma_\tau\cap\{r\leq R\})} + \int_{C_\tau}\frac{|\phi|^2}{(r+1)^2}\mathrm{d}v\mathrm{d}\sigma \lesssim \int_{\Sigma_\tau}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_\tau}\mathrm{dvol}_{\Sigma_\tau}.$$

for any $\tau\in\mathbb{R}$; this amounts to a Hardy estimate. Since

$$\begin{aligned}\frac{1}{(r+1)^2}|\phi|^2 &= \frac{d}{dr}\left(-\frac{1}{(r+1)}\right)|\phi|^2\\ &= (\partial_r + \partial_r h(r)\partial_t)\left(-\frac{1}{(r+1)}|\phi|^2\right) + \frac{2}{(r+1)}\mathrm{Re}[(\partial_r+\partial_r h(r)\partial_t)\phi\overline{\phi}];\end{aligned}$$

after integrating over $\mathcal{S}_\tau$, we have

$$\begin{aligned}\int_{\mathcal{S}_\tau}\frac{1}{(r+1)^2}|\phi|^2\mathrm{d}r\mathrm{d}\sigma &\leq \int_{\mathcal{S}_\tau}\frac{2\mathrm{Re}[(\partial_r+\partial_r h\partial_t)\phi\overline{\phi}]}{(r+1)}\mathrm{d}r\mathrm{d}\sigma\\ &\leq \frac{1}{2}\int_{\mathcal{S}_\tau}\frac{1}{(r+1)^2}|\phi|^2\mathrm{d}r\mathrm{d}\sigma + 2\int_{\mathcal{S}_\tau}|(\partial_r+\partial_r h\partial_t)\phi|^2\mathrm{d}r\mathrm{d}\sigma\\ \implies \int_{\mathcal{S}_\tau}|(r+1)^{-1}\psi|^2 r^2\mathrm{d}r\mathrm{d}\sigma &\leq 4\int_{\mathcal{S}_\tau}|(\partial_r+\partial_r h\partial_t)\phi|^2\mathrm{d}r\mathrm{d}\sigma \lesssim \int_{\mathcal{S}_\tau}\left(|\partial_r\phi|^2+|\partial_t\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma.\end{aligned}$$

Using the assumptions on $\mathcal{S}_\tau$, one easily concludes the proof of the top estimate in (2.3). Similarly, using the conclusion of Exercise 2.3, one easily completes the proof of the bottom estimate in (2.3). $\square$

Exercise 2.3 (Hardy estimate I)**.** Prove the Hardy estimate

$$\int_{\mathcal{S}_\tau\cap\{r\leq R\}}\frac{|\phi|^2}{(r+1)^2}\mathrm{d}r\mathrm{d}\sigma + \int_{C_\tau}\frac{|\phi|^2}{(r+1)^2}\mathrm{d}v\mathrm{d}\sigma \lesssim_R \int_{\mathcal{S}_\tau\cap\{r\leq R\}}\left(|\partial_r\phi|^2+|\partial_t\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma + \int_{C_\tau}|\partial_v\phi|^2\mathrm{d}v\mathrm{d}\sigma,$$

carefully justifying why there are no boundary integrals over corner 2-spheres needed on the right hand side.

Proposition 2.4 (Degenerate-at-$\mathcal{H}^+$ energy estimates on Schwarzschild)**.** *Fix $M>0$, and let $g = g_{\mathrm{Schw}(M)}$. Fix $R>2M$ and fix foliations $\{\mathcal{S}_\tau\}_\tau$ and $\{\Sigma_\tau\}_\tau$ of $\mathcal{M}_{\mathrm{Schw}(M)}$ as follows: $\{\mathcal{S}_\tau\}_\tau$ is an $SO(3)$-invariant, T-invariant, uniformly spacelike foliation by smooth hypersurfaces which connect the future event horizon to spacelike infinity (such as the constant-t^* foliation); $\{\Sigma_\tau\}_\tau$ is a foliation by piecewise smooth hypersurfaces which connect the future event horizon to future null infinity and consist of a spacelike piece $\mathcal{S}_\tau\cap\{r\leq R\}$ and an outgoing null piece C_τ emanating from $\mathcal{S}_\tau\cap\{r=R\}$ (see Figure 3 for an illustration).*

Assume ψ is a smooth scalar field which solves the homogeneous wave equation (1.1). Then, ψ has non-increasing T-energy across both foliations $\mathcal{S}_\tau$ and Σ_τ: for any $0\leq\tau_1\leq\tau_2<\infty$,

$$\begin{aligned}\int_{\mathcal{S}_{\tau_2}}\tilde{J}^T_\mu[\psi]n^\mu_{\mathcal{S}_{\tau_2}}\mathrm{dvol}_{\mathcal{S}_{\tau_2}} &\leq \int_{\mathcal{S}_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\mathcal{S}_{\tau_1}}\mathrm{dvol}_{\mathcal{S}_{\tau_1}},\\ \int_{\Sigma_{\tau_2}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} &\leq \int_{\Sigma_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}}.\end{aligned} \tag{2.4}$$

We note the upper and lower pointwise bounds

$$\begin{aligned}\tilde{J}^T_\mu[\psi]n^\mu_{\mathcal{S}_\tau}\mathrm{dvol}_{\mathcal{S}_\tau} &\sim \left(|\partial_v\phi|^2+\frac{r^2}{\Delta}|\partial_u\phi|^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2+\frac{1}{r^3}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma\\ &\sim \left(|\partial_{t^*}\phi|^2+\frac{\Delta}{r^2}|\partial_r\phi|^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2+\frac{1}{r^3}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma, \\ \tilde{J}^T_\mu[\psi]\underline{n}^\mu_{C_\tau}\mathrm{dvol}_{C_\tau} &\sim \left(|\partial_v\phi|^2+\frac{1}{r^2}|\mathring{\nabla}\phi|^2+\frac{1}{r^3}|\phi|^2\right)\mathrm{d}v\mathrm{d}\sigma.\end{aligned}\tag{2.5}$$

It follows that, for any $\epsilon>0$, there are universal $C(\epsilon)$ functions with the property that $C(\epsilon)\to\infty$ as $\epsilon\to 0$, so that the following Sobolev estimates hold for any $0\le\tau_1\le\tau_2<\infty$:

$$\begin{aligned}&\int_{\mathcal{S}_{\tau_2}\cap\{r\ge r_+(1+\epsilon)\}}\left(|\partial_{t^*}\psi|^2+|\partial_r\psi|^2+|(r^{-1}\mathring{\nabla})\psi|^2+|r^{-1}\psi|^2\right)r^2\mathrm{d}r\mathrm{d}\sigma\\ &\quad\le C(\epsilon)\int_{\mathcal{S}_{\tau_1}}\left(|\partial_{t^*}\psi|^2+\frac{\Delta}{r^2}|\partial_r\psi|^2+|(r^{-1}\mathring{\nabla})\psi|^2+|r^{-1}\psi|^2\right)r^2\mathrm{d}r\mathrm{d}\sigma,\\ &\int_{\mathcal{S}_{\tau_2}\cap\{r_+(1+\epsilon)\le r\le R\}}\left(|\partial_{t^*}\psi|^2+|\partial_r\psi|^2+|(r^{-1}\mathring{\nabla})\psi|^2+|r^{-1}\psi|^2\right)r^2\mathrm{d}r\mathrm{d}\sigma\\ &\quad+\int_{C_{\tau_2}}\left(|\partial_v(r\psi)|^2+|(r^{-1}\mathring{\nabla})(r\psi)|^2+|\psi|^2\right)\mathrm{d}v\mathrm{d}\sigma\\ &\le C(\epsilon)\int_{\mathcal{S}_{\tau_1}\cap\{r\le R\}}\left(|\partial_{t^*}\psi|^2+\frac{\Delta}{r^2}|\partial_r\psi|^2+|(r^{-1}\mathring{\nabla})\psi|^2+|r^{-1}\psi|^2\right)r^2\mathrm{d}r\mathrm{d}\sigma\\ &\quad+C(\epsilon)\int_{C_{\tau_1}}\left(|\partial_v(r\psi)|^2+|(r^{-1}\mathring{\nabla})(r\psi)|^2+|\psi|^2\right)\mathrm{d}v\mathrm{d}\sigma.\end{aligned}\tag{2.6}$$

Proof. We repeat the same steps as in the proof of Proposition 2.2, *mutatis mutandis*. There are three main differences worth emphasizing:

- In applying Lemma 1.15, we now have $U=-\beta^{-1}\Box_{g_{\mathrm{Schw}(M)}}\beta=\frac{2M}{r^3}>0$. Furthermore, the fact that $T=\partial_{t^*}$, which is timelike for $r>2M$, becomes null as $r\to 2M$ leads to a degeneration in the Sobolev estimates.
- In our application of Proposition 1.13, for both types of foliation we see an additional term
$$\int_{\mathcal{H}^+_{(0,\tau)}}\tilde{J}^T_\mu[\psi]\underline{n}^\mu_{\mathcal{H}^+}\mathrm{dvol}_{\mathcal{H}^+}\ge 0,$$
where the inequality comes from Lemma 1.15. As a consequence, the top estimate in (2.4) is an inequality, cf. the top line of (2.1), in Minkowski space, which is an equality.
- In our derivation of a Hardy estimate, the key identity for the $\{\mathcal{S}_\tau\}_\tau$ foliation is
$$\frac{\Delta}{r^4}|\phi|^2-U|\phi|^2=(\partial_r+\partial_rh(r)\partial_{t^*})\left(-\frac{\Delta}{r^3}|\phi|^2\right)+\frac{2\Delta}{r^3}\mathrm{Re}[(\partial_r+\partial_rh(r)\partial_{t^*})\phi\overline{\phi}],$$
so that, when integrating over $\mathcal{S}_\tau$, the boundary term at $\mathcal{S}_\tau\cap\{r=2M\}$ vanishes. Note, however, that the presence of the zeroth order term in (2.5) allows us to remove the horizon degeneracy from the L^2 term in the final estimates.

This concludes the proof. □

Exercise 2.5 (Degenerate energy estimates on Reissner-Nordström)**.** The Reissner-Nordström black hole exterior with parameters $M>0$ and $|e|\le M$ can be defined as $(\mathcal{M}_{\mathrm{Schw}(M)},g_{\mathrm{Schw}(M)})$ but with Δ replaced by $\Delta=r^2-2Mr+e^2$ and $r_+=M+\sqrt{M^2-e^2}$. Show that Proposition 2.4 holds for this spacetime with the following modifications: (a) the r^{-3} coefficient in $|\phi|^2$ term in (2.5) must be replaced by $(r-M)/r^4$, (b) the $|r^{-1}\psi|^2$ term on the right hand side of (2.6) must be replaced by $(1-M/r)|r^{-1}\psi|^2$.

Exercise 2.6 (Nondegenerate-at-horizon energy estimates)**.** On the Schwarzschild exterior manifold, let $Y:=\frac{r^2}{\Delta}\partial_u$. By considering a suitable smooth cutoff function $0\le\chi_H\le 1$, show that applying the vector field multiplier $X = \chi_H Y$ to the homogeneous scalar wave equation (1.1) and applying Proposition 2.4 leads to the estimate

$$\begin{aligned}
&\int_{\mathcal{S}_{\tau_2}} \left(|\partial_{t^*}\phi|^2 + |\partial_r\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{1}{r^3}|\phi|^2 \right) \mathrm{d}r\mathrm{d}\sigma \\
&\quad\lesssim \int_{\mathcal{S}_{\tau_1}} \left(|\partial_{t^*}\phi|^2 + |\partial_r\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{1}{r^3}|\phi|^2 \right) \mathrm{d}r\mathrm{d}\sigma \\
&\qquad + \int_{\mathcal{R}_{(\tau_1,\tau_2)}\cap\{r\le r_+ + \epsilon\}} \left[|\partial_{r^*}\phi|^2 + |\partial_t\phi|^2 + |\mathring{\nabla}\phi|^2 + |\phi|^2 \right] \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau.
\end{aligned}$$

End of Lecture 2

Start of Lecture 3

2.3 First order Morawetz estimate

The goal of this section is to obtain spacetime estimates for (derivatives of) ψ solving the homogeneous wave equation (1.1) on Minkowski space and the Schwarzschild exterior away from $\mathcal{H}^+$.

Since the estimates we seek are on spacetime rather than on hypersurfaces, we want to choose multipliers X and Lagrangian functions w which generate good bulk, $\tilde{K}^X + \tilde{K}^{\mathrm{aux},w}$, (rather than boundary, $\tilde{J}^X + \tilde{J}^{\mathrm{aux},w}$) terms. In particular, X must *not* be a Killing field, see Lemma 1.8. Two choices of multipliers and Lagrangian functions will be particularly important to us: $X = \mathfrak{f}(r^*)\partial_{r^*}$ and $w = 0$ or $w = \frac{1}{2}\mathfrak{f}$ for some scalar $\mathfrak{f}$.

2.3.1 Minkowski space

We start with the easier case of Minkowski space:

Theorem 2.7 (Morawetz estimate on Minkowski)**.** *Consider the setting of Proposition 2.2. Then there is a universal constant $c > 0$ such that, for any $0 \le \tau_1 \le \tau_2 \le \infty$,*

$$c\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left(\frac{|\partial_r\phi|^2}{(r+1)^2} + \frac{|\partial_t\phi|^2}{(r+1)^2} + \frac{|\mathring{\nabla}\phi|^2}{(r+1)^3} + \frac{|\phi|^2}{(r+1)^4} \right) \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \le \int_{\Sigma_{\tau_1}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}. \tag{2.7}$$

Remark 2.8. Estimate (2.7) is sometimes called an integrated local energy decay estimate. This is because, if we restrict to $r \le R_{\mathrm{large}} < \infty$, we have

$$\begin{aligned}
&\int_0^\infty \int_{\Sigma_\tau\cap\{r\le R_{\mathrm{large}}\}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_\tau} \mathrm{dvol}_{\Sigma_\tau}\mathrm{d}\tau \\
&\quad\le C(R_{\mathrm{large}}) \int_{\mathcal{R}_{(0,\infty)}\cap\{r\le R_{\mathrm{large}}\}} \left(\frac{|\partial_r\phi|^2}{(r+1)^2} + \frac{|\partial_t\phi|^2}{(r+1)^2} + \frac{|\mathring{\nabla}\phi|^2}{(r+1)^3} + \frac{|\phi|^2}{(r+1)^4} \right) \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \\
&\quad\le C(R_{\mathrm{large}}) \int_{\Sigma_0} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0},
\end{aligned}$$

which implies that the T-energy of ψ must, at least on average, decay in τ: e.g. for each $n \in \mathbb{N}$, there exists $\tau_n \in [2^n, 2^{n+1}]$ such that

$$\int_{\Sigma_{\tau_n}\cap\{r\le R_{\mathrm{large}}\}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_n}} \mathrm{dvol}_{\Sigma_{\tau_n}} \le \frac{C(R_{\mathrm{large}})}{2^n} \int_{\Sigma_0} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}.$$

Proof of Theorem 2.7. There are several ways of proving this theorem. We follow an approach which is similar to that we will see for black holes in the next section: using twisted multiplier currents with twisting parameter $\beta^{-1} = r$. We recall that, with this choice of β, recall that $\phi := r\psi$ is the radiation field and $U = 0$. We also recall that, without loss of generality, there we may assume there is a smooth function h such that $\mathcal{S}_\tau = \{(t, r, \theta, \varphi) : t = \tau + h(r)\}$, with $1 - (\partial_r h)^2$ bounded above and below away from zero uniformly in r.

Step 1: bulk term. Setting $X = 2y(r)\partial_r$, we compute

$$\tilde{K}^X[\psi]\mathrm{dvol}_g = \left[\partial_r y \left(|\partial_r\phi|^2 + |\partial_t\phi|^2\right) - \partial_r\left(\frac{y}{r^2}\right)|\mathring{\nabla}\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau,$$

from the definition in (1.9), and

$$\begin{aligned}
|\tilde{J}^X_\mu[\psi]n^\mu_{\mathcal{S}_\tau}|\mathrm{dvol}_{\mathcal{S}_\tau} &= |y|\left|2\mathrm{Re}[\partial_t\phi\,\overline{\partial_r\phi}] + \partial_r h(r)\left(|\partial_t\phi|^2 + |\partial_r\phi|^2 - r^{-2}|\mathring{\nabla}\phi|^2\right)\right|\mathrm{d}r\mathrm{d}\sigma\\
&\lesssim |y|\left(|\partial_t\phi|^2 + |\partial_r\phi|^2 + r^{-2}|\mathring{\nabla}\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma,\\
|\tilde{J}^X_\mu[\psi]\underline{n}^\mu_{C_\tau}|\mathrm{dvol}_{C_\tau} &\leq |y|\left(|\partial_v\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2\right)\mathrm{d}v\mathrm{d}\sigma,
\end{aligned}$$

from the definition in (1.7). We choose $y = 1-(r+1)^{-1}$, so that

$$|y| \leq 1, \qquad \partial_r y = \frac{1}{(r+1)^2}, \qquad -\partial_r\left(\frac{y}{r^2}\right) = \frac{1}{r^3}\frac{r(2r+1)}{(r+1)^2} \geq \frac{1}{(r+1)^3}.$$

Then, from Proposition 1.13, we obtain

$$\begin{aligned}
&\int_{\mathcal{R}_{(\tau_1,\tau_2)}}\left[\frac{1}{(r+1)^2}|\partial_r\phi|^2 + \frac{1}{(r+1)^2}|\partial_t\phi|^2 + \frac{1}{(r+1)^3}|\mathring{\nabla}\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau\\
&\quad\lesssim \int_{\mathcal{R}_{(\tau_1,\tau_2)}}\tilde{K}^X[\psi]\mathrm{dvol}_g \lesssim \int_{\Sigma_{\tau_2}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}},
\end{aligned}$$

Using the conclusion of Exercise 2.9, we at last establish control of the bulk term by flux terms:

$$\begin{aligned}
&\int_{\mathcal{R}_{(\tau_1,\tau_2)}}\left[\frac{1}{(r+1)^2}|\partial_r\phi|^2 + \frac{1}{(r+1)^2}|\partial_t\phi|^2 + \frac{1}{(r+1)^3}|\mathring{\nabla}\phi|^2 + \frac{1}{(r+1)^4}|\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau\\
&\quad\lesssim \int_{\mathcal{R}_{(\tau_1,\tau_2)}}\tilde{K}^X[\psi]\mathrm{dvol}_g \lesssim \int_{\Sigma_{\tau_2}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}}.
\end{aligned}\tag{2.8}$$

Step 2: boundary term. In (2.8), we estimate the right hand side using the energy estimates in Proposition 2.2. □

Exercise 2.9 (Hardy estimate II)**.** Let X be as in the proof of Theorem 2.7. Prove a Hardy estimate which establishes control of the zeroth order term in (2.8) from the terms in $\tilde{K}^X[\psi]$, i.e. show

$$\int_{\mathcal{R}_{(\tau_1,\tau_2)}}\frac{1}{(r+1)^4}|\phi|^2\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \lesssim \int_{\mathcal{R}_{(\tau_1,\tau_2)}}\tilde{K}^X[\psi]\mathrm{dvol}_g.$$

Remark 2.10. By choosing $y = 1-(r+1)^{-\delta}$ with $\delta\in(0,1]$, we can improve the large r-weights on the left hand side of (2.7) to

$$c(\delta)\int_{\mathcal{R}_{(\tau_1,\tau_2)}}\left[\frac{1}{(r+1)^{1+\delta}}|\partial_r\phi|^2 + \frac{1}{(r+1)^{1+\delta}}|\partial_t\phi|^2 + \frac{1}{(r+1)^3}|\mathring{\nabla}\phi|^2 + \frac{1}{(r+1)^{3+\delta}}|\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau.$$

2.3.2 Schwarzschild exterior

Now we turn to a much more involved case:

Theorem 2.11 (Degenerate-at-$\mathcal{H}^+$ Morawetz estimates on Schwarzschild)**.** *Consider the setting of Proposition 2.4. Then, there is a universal constant $c = c(M) > 0$ such that, for any $0 \leq \tau_1 \leq \tau_2 \leq \infty$,*

$$\begin{aligned}
&c(M)\int_{\mathcal{R}_{(\tau_1,\tau_2)}}\left[\frac{1}{r^2}|\phi'|^2 + \left(1-\frac{3M}{r}\right)^2\left(\frac{1}{r^2}|T\phi|^2 + \frac{1}{r^3}|\mathring{\nabla}\phi|^2\right) + \frac{1}{r^4}|\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau\\
&\quad\leq \int_{\Sigma_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}},
\end{aligned}\tag{2.9}$$

where prime denotes a derivative with respect to r^*.

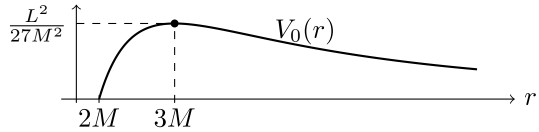
Remark 2.12. Estimate (2.9) is degenerate in two ways. First, it loses control of some derivatives of ψ as $r\to r_+$. This should be familiar from Proposition 2.4 and, once again, it arises from our use of T energies and the fact that T becomes null at $\mathcal{H}^+$. Combining Theorem 2.11 with Exercise 2.6, the degeneracy of our energy estimates at $\mathcal{H}^+$ can be removed; and, in fact, by revisiting Exercise 2.6 it turns out that the same is true for our Morawetz estimate (2.9).

Second, estimate (2.9) loses control of time and angular derivatives of ψ as $r \to 3M$. This is a novelty! It is related to a geometric feature associated with the curved Schwarzschild geometry, the trapping of null geodesics at the so-called photon sphere $\{r = 3M\}$; some loss at trapping *cannot* be avoided [13]. Estimate 2.9 is sometimes called an integrated local energy decay estimate with derivative loss. Note that, were it not for the degeneration at $r = 3M$, Remark 2.8 would apply mutatis mutandis if one were to restrict the integrals $\{r_+ + \epsilon \leq r \leq R_{\text{large}}\}$, i.e. one could deduce that the T-energy of ψ decays in τ on average. As we will understand in Corollary **??**, one can salvage this conclusion by losing a time derivative: for each $n \in \mathbb{N}$, there is $\tau_n \in [2^n, 2^{n+1}]$ such that

$$\int_{\Sigma_{\tau_n}\cap\{r_+ +\epsilon\leq r\leq R_{\text{large}}\}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_n}} \mathrm{dvol}_{\Sigma_{\tau_n}} \leq \frac{C(\epsilon, R_{\text{large}})}{2^n}\int_{\Sigma_0} (\tilde{J}^T_\mu[T\psi] + \tilde{J}^T_\mu[\psi]) n^\mu_{\Sigma_0} \mathrm{dvol}_{\Sigma_0}.$$

Exercise 2.13 (Trapped null geodesics)**.** Let $\gamma(s) = (t(s), r(s), \theta(s), \varphi(s))$ be an affinely parametrized null geodesic. Explain why, by spherical symmetry, we can restrict $\theta(s) = \pi/2$ without loss of generality. Show that

$$\begin{cases} \dot{r}^2 = E^2 - V_0(r) \\ \ddot{r} = -\dfrac{1}{2}\partial_r V_0(r) \end{cases}, \qquad V_0(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2},$$



where, since $T = \partial_t$ and $Z = \partial_\varphi$ are Killing vector fields,

$$E := -g(T, \dot{\gamma}) = \left(1 - \frac{2M}{r}\right)\dot{t},$$
$$L := g(Z, \dot{\gamma}) = r^2 \sin^2\theta\dot{\varphi} = r^2\dot{\varphi}$$

are conserved quantities along γ. Show that, for $L \neq 0$, $V_0(r)$ attains a unique maximum in (r_+, ∞) given at a point $r = 3M$. We say γ is future trapped if $\lim_{s\to\infty}(r(s), \dot{r}(s)) = (3M, 0)$, and past trapped if $\lim_{s\to-\infty}(r(s), \dot{r}(s)) = (3M, 0)$. Show that a necessary condition for γ to be future or past trapped is

$$\frac{\dot{r}}{E} = \pm\left(1 - \frac{3M}{r}\right)\left(1 + \frac{6M}{r}\right)^{1/2}.$$

There are several ways of proving Theorem 2.11. Here we will follow the strategy of [11]. We apply twisted multiplier currents with twisting parameter $\beta^{-1} = r$, so that

$$U = \frac{2M}{r^3} \tag{2.10}$$

and $\phi := r\psi$ is the radiation field. For readability, we will break down the non-trivial steps of the proof as lemmas.

Lemma 2.14. *Consider the setting of Theorem 2.11. Let $\phi_0 := r\psi_0$ and $\phi_{\geq 1} := r\psi_{\geq 1}$, where*

$$\psi_0 = \frac{1}{4\pi}\int_{\mathbb{S}^2}\psi d\sigma \qquad \psi_{\geq 1} := \psi - \psi_0$$

(We call ψ_0 the spherical mean of ψ, ϕ_0 the spherical mean of ϕ.) There is a suitable choice of bounded smooth function $f = f(r^*)$ such that

$$\int_{\mathcal{R}_{(\tau_1,\tau_2)}}\left(\frac{1}{r^3}|\phi'|^2 + \frac{1}{r^3}\left(1 - \frac{3M}{r}\right)^2|\mathring{\nabla}\phi_{\geq 1}|^2 + \frac{1}{r^3}|\phi_{\geq 1}|^2 + \frac{1}{r^4}|\phi_0|^2\right)\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau$$
$$\lesssim \int_{\mathcal{R}_{(\tau_1,\tau_2)}}\left(\tilde{K}^{2f(r^*)\partial_{r^*}}[\psi] + \tilde{K}^{\mathrm{aux},f'}[\psi]\right)\mathrm{dvol}_g \lesssim \int_{\Sigma_{\tau_2}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}},$$

with implicit constants depending only on M.

Proof. Recall that $'$ denotes a derivative with respect to r^*. From (1.7) and (1.11), we can estimate

$$\begin{aligned}
&\left|\left(\tilde{J}^{2f\partial_{r^*}}_\mu[\psi]+\tilde{J}^{\mathrm{aux},f'}_\mu[\psi]\right)n^\mu_{\mathcal{S}_\tau}\right|\mathrm{dvol}_{\mathcal{S}_\tau}\\
&\quad\lesssim\left((|f|+1)|\partial_{t^*}\phi|^2+(|f|+1)\frac{\Delta}{r^2}|\partial_r\phi|^2+\frac{|f|}{r^2}|\mathring{\nabla}\phi|^2+\frac{|f|+r^5\Delta^{-1}(|f'|^2+|f''|)}{r^3}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma,\\
&\left|\left(\tilde{J}^{2f\partial_{r^*}}_\mu[\psi]+\tilde{J}^{\mathrm{aux},f'}_\mu[\psi]\right)\underline{n}^\mu_{C_\tau}\right|\mathrm{dvol}_{C_\tau}\\
&\quad\lesssim\left((|f|+1)|\partial_v\phi|^2+\frac{|f|}{r^2}|\mathring{\nabla}\phi|^2+\frac{|f|+r^3(|f'|^2+|f''|)}{r^3}|\phi|^2\right)\mathrm{d}v\mathrm{d}\sigma.
\end{aligned}$$

Thus, a choice of function f such that f, $r^5\Delta^{-1}f'^2$ and $r^5\Delta^{-1}f''$ are bounded uniformly will lead to, via Proposition 1.13,

$$\begin{aligned}
&\int_{\mathcal{R}_{(\tau_1,\tau_2)}}(\tilde{K}^{2f(r^*)\partial_{r^*}}[\psi]+\tilde{K}^{\mathrm{aux},f'}[\psi])\mathrm{dvol}_g\\
&\quad=-\int_{\Sigma_{\tau_2}}(\tilde{J}^{2f\partial_{r^*}}_\mu[\psi]+\tilde{J}^{\mathrm{aux},f'}_\mu[\psi])n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}}+\int_{\Sigma_{\tau_1}}(\tilde{J}^{2f\partial_{r^*}}_\mu[\psi]+\tilde{J}^{\mathrm{aux},f'}_\mu[\psi])n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}}\\
&\quad\lesssim\int_{\Sigma_{\tau_2}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}}+\int_{\Sigma_{\tau_1}}\tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}}. \qquad (2.11)
\end{aligned}$$

To obtain the result, it remains to establish a lower bound on the left hand side. First, from (1.9) and (1.13), we computed exactly

$$(\tilde{K}^{2f(r^*)\partial_{r^*}}[\psi]+\tilde{K}^{\mathrm{aux},f'}[\psi])\mathrm{dvol}_g=\frac{r^2}{\Delta}\tilde{\mathcal{K}}^f[\psi]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau,$$

where we introduce the notation

$$\tilde{\mathcal{K}}^f[\psi]:=2f'|\phi'|^2-f\left(\frac{\Delta}{r^4}\right)'|\mathring{\nabla}\phi|^2-\left(f\left(U\frac{\Delta}{r^2}\right)'+\frac{1}{2}f'''\right)|\phi|^2. \qquad (2.12)$$

Motivated by the geodesic flow, see Exercise 2.13, we take

$$f(r^*)=\left(1-\frac{3M}{r}\right)\sqrt{1+\frac{6M}{r}},$$

as our ansatz. Indeed, for this choice $|f|+r^5\Delta^{-1}(|f'|^2+|f''|)\lesssim 1$, and in the next two steps we prove it leads to a positive, coercive, bulk term for ϕ.

Step 1: bulk estimate for $\phi_{\geq}1$. The smallest nonzero eigenvalue of $\mathring{\Delta}$ is 2, hence

$$\int_{\mathbb{S}^2}\tilde{\mathcal{K}}^f[\psi_{\geq 1}]\mathrm{d}\sigma\geq\int_{\mathbb{S}^2}\left[2f'|\phi'_{\geq 1}|^2-\left(f\left(\frac{2\Delta}{r^4}+U\frac{\Delta}{r^2}\right)'+\frac{1}{2}f'''\right)|\phi_{\geq 1}|^2\right]\mathrm{d}\sigma$$

By construction, we have that

$$\begin{aligned}
\partial_r f&=\frac{27M^2}{r^3\sqrt{1+\frac{6M}{r}}},\\
-f\partial_r\left(\frac{\Delta}{r^4}\right)&=\frac{2}{r^3}\left(1-\frac{3M}{r}\right)^2\sqrt{1+\frac{6M}{r}},
\end{aligned}$$

are positive, and that

$$-\left(f\left(\frac{2\Delta}{r^4}+U\frac{\Delta}{r^2}\right)'+\frac{1}{2}f'''\right)=\frac{1-\frac{2M}{r}}{2r^9\left(1+\frac{6M}{r}\right)^{\frac{5}{2}}}S_1(r),$$

where, using (2.10), we have

$$\begin{aligned}
S_1(r)&=8r^6+108Mr^5-104M^2r^4-3342M^3r^3+2889M^4r^2+36180M^5r-54108M^6\\
&=r\left(1-\frac{2M}{r}\right)\left[(8r^4+32Mr^3-224M^2r^2-478M^3r+2278M^4)(r+23M/2)+3545M^5\right]+5376M^6\\
&\geq 5376M^6,
\end{aligned}$$

is also positive for $r \in [2M, \infty)$. We deduce

$$\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left(\frac{27M^2}{r^3}|\phi'_{\geq 1}|^2 + \frac{1}{r^3}\left(1 - \frac{3M}{r}\right)^2 |\mathring{\nabla}\phi_{\geq 1}|^2 + \frac{1}{4r^3}|\phi_{\geq 1}|^2 \right) \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau$$
$$\leq \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left[2f'|\phi'_{\geq 1}|^2 - \left(f\left(\frac{2\Delta}{r^4} + U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right)|\phi_{\geq 1}|^2 \right] \mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau$$
$$= \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{\mathcal{K}}^f[\psi_{\geq 1}]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau. \tag{2.13}$$

Step 2: bulk estimate for ϕ_0. In this case,

$$-\left(f\left(U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right) = \frac{1 - \frac{2M}{r}}{2r^9\left(1 + \frac{6M}{r}\right)^{\frac{5}{2}}} S_0(r),$$
$$S_0(r) = 12Mr^5 - 176M^2r^4 - 1182M^3r^3 + 5481M^4r^2 + 20628M^5r - 54108M^6,$$

is not globally positive for $r \in [2M, \infty)$; for instance, $S_0(2M) = -2816M^6$. However, we can exploit the positivity of $f'|\phi'_0|^2$ to compensate for the bad sign in $-\left(f(Ur^{-2}\Delta)' + \frac{1}{2}f'''\right)|\phi_0|^2$. In our proof of Theorem 2.7 for Minkowski, we implemented this idea through a Hardy estimate in Exercise 2.9. Here, we will pursue the same idea with a more precise method.

We note that

$$\tilde{\mathcal{K}}^f[\psi_0] = 2f'|\phi'_0|^2 - \left(f\left(U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right)|\phi_0|^2$$
$$= 2f'|\phi'_0 - g\phi_0|^2 + 4f'g\mathrm{Re}[\phi'_0\overline{\phi_0}] - \left(2f'g^2 + f\left(U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right)|\phi_0|^2$$
$$= 2f'|\phi'_0 - g\phi_0|^2 + (2f'g|\phi_0|^2)' - \left(2f'g^2 + 2(f'g)' + f\left(U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right)|\phi_0|^2,$$

for any smooth function $g = g(r^*)$. By choosing

$$g = \frac{1}{2r}\left[\left(1 - \frac{2M}{r}\right) - \frac{M^2}{r^2} \right],$$

we obtain

$$-\left(2(f'g)' + 2f'g^2 + f\left(U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right) = \frac{\left(1 - \frac{2M}{r}\right)}{2r^{11}\left(1 + \frac{6M}{r}\right)^{\frac{5}{2}}} M\tilde{S}_0(r),$$

where we can compute

$$\tilde{S}_0(r) = 12r^7 + 13Mr^6 - 48M^2r^5 + 1215M^3r^4 + 702M^4r^3 - 16551M^5r^2 + 21060M^6r - 972M^7$$
$$= \frac{3177426317M^7}{80000000} + \left(r - \frac{M}{20}\right)\hat{S}_0(r)$$
$$\hat{S}_0(r) = 12(r-2M)^6 + \frac{788}{5}M(r-2M)^5 + \frac{20217}{25}M^2(r-2M)^4 + \frac{1649037}{500}M^3(r-2M)^3$$
$$+ \frac{108707557}{10000}M^4(r-2M)^2 + \frac{593405277}{200000}M^5(r-2M) + \frac{1625194197}{4000000}M^6.$$

Since all terms in $\hat{S}_0(r)$ are non-negative for $r \geq 2M$, we have

$$\tilde{S}_0(r) \geq \frac{3177426317M^7}{80000000} > 30M^7.$$

is clearly positive for $r \geq 2M$. Carefully keeping track of constants, we can obtain, for instance,

$$\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left(\frac{M^2}{8r^3}|\phi'_0|^2 + \frac{13M}{256r^4}|\phi_0|^2 \right) \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau$$
$$\leq \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left[2f'|\phi'_0|^2 - \left(f\left(U\frac{\Delta}{r^2}\right)' + \frac{1}{2}f''' \right)|\phi_0|^2 \right] \mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau = \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{\mathcal{K}}^f[\psi_0]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau. \tag{2.14}$$

Step 3: conclusion. From the orthogonality of the spherical mean ψ_0 to $\psi_{\geq 1}$ in L^2 on the sphere, we have

$$\int_{\mathbb{S}^2} \tilde{\mathcal{K}}^f[\psi]\mathrm{d}\sigma = \int_{\mathbb{S}^2} \tilde{\mathcal{K}}^f[\psi_{\geq 1}]\mathrm{d}\sigma + \int_{\mathbb{S}^2} \tilde{\mathcal{K}}^f[\psi_0]\mathrm{d}\sigma.$$

Thus, by combining (2.13) and (2.14), we have

$$\begin{aligned}
&\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left[\frac{M^2}{8r^3}|\phi'|^2 + \frac{1}{r^3}\left(1-\frac{3M}{r}\right)^2|\mathring{\nabla}\phi|^2 + \frac{1}{4r^3}|\phi_{\geq 1}|^2 + \frac{13M}{256r^4}|\phi_0|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \\
&\quad \leq \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{\mathcal{K}}^f[\psi]\mathrm{d}r^*\mathrm{d}\sigma\mathrm{d}\tau = \int_{\mathcal{R}_{(\tau_1,\tau_2)}} (\tilde{K}^{2f(r^*)\partial_{r^*}}[\psi] + \tilde{K}^{\mathrm{aux},f'}[\psi])\mathrm{dvol}_g;
\end{aligned}$$

appealing to (2.11) concludes the proof. □

Remark 2.15. In the proof of Lemma 2.14, treating ϕ_0 requires an additional argument compared to the $\phi_{\geq 1}$. This argument is not necessary if one replaces the covariant wave equation (1.1) by an appropriate wave equation arising from the linearized Einstein equations around Schwarzschild, such as the Regge–Wheeler equation, as those equations are supported on higher angular modes (so that $\phi_0 \equiv 0$). In that sense, one can say that the tensorial linearized Einstein equations are *better* than the scalar "toy problem" (1.1). See, for instance, [6].

Exercise 2.16 (An alternative f current on Schwarzschild)**.** Repeat the proof of Lemma 2.14, choosing instead

$$f = \left(1-\frac{3M}{r}\right)\left(1+\frac{M}{r}\right),$$

This is not as well-motivated by the geodesic flow, but is a lot easier computationally!

Note that Lemma 2.14 fails to provide control over $T\phi$ derivatives. To recover these, we need an additional multiplier current argument.

Lemma 2.17. *Consider the setting of Theorem 2.11. There is a suitable choice of smooth bounded function $y = y(r^*)$ such that we have the estimate*

$$\begin{aligned}
&\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left[\frac{1}{r^2}|\phi'|^2 + \frac{1}{r^2}\left(1-\frac{3M}{r}\right)^2|T\phi|^2 + \frac{1}{r^4}|\phi|^2\right]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau \\
&\lesssim \int_{\Sigma_{\tau_2}} \tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}} \tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}} + \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left(\tilde{K}^{2f(r^*)\partial_{r^*}}[\psi] + \tilde{K}^{\mathrm{aux},f'}[\psi]\right)\mathrm{dvol}_g,
\end{aligned}$$

with implicit constants depending only on M.

Proof. From (1.7) and (1.11), we can estimate

$$\begin{aligned}
\left|\tilde{J}^{2y\partial_{r^*}}_\mu[\psi]n^\mu_{\mathcal{S}_\tau}\right|\mathrm{dvol}_{\mathcal{S}_\tau} &\lesssim |y|\left(|\partial_{t^*}\phi|^2 + \frac{\Delta}{r^2}|\partial_r\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{1}{r^3}|\phi|^2\right)\mathrm{d}r\mathrm{d}\sigma, \\
\left|\tilde{J}^{2y\partial_{r^*}}_\mu[\psi]\underline{n}^\mu_{C_\tau}\right|\mathrm{dvol}_{C_\tau} &\lesssim |y|\left(|\partial_v\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{1}{r^3}|\phi|^2\right)\mathrm{d}v\mathrm{d}\sigma;
\end{aligned}$$

it follows from Proposition 1.13 that

$$\begin{aligned}
\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{K}^{2y\partial_{r^*}}[\psi]\mathrm{dvol}_g &= -\int_{\Sigma_{\tau_2}} \tilde{J}^{2y\partial_{r^*}}_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}} \tilde{J}^{2y\partial_{r^*}}_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}} \\
&\lesssim \int_{\Sigma_{\tau_2}} \tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_2}}\mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}} \tilde{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_1}}\mathrm{dvol}_{\Sigma_{\tau_1}},
\end{aligned} \tag{2.15}$$

as long as y is uniformly bounded.

We turn to the left hand side of the above estimate. From (1.9), we compute exactly

$$\tilde{K}^{2y(r^*)\partial_{r^*}}[\psi]\mathrm{dvol}_g = \frac{r^2}{\Delta}\tilde{\mathcal{K}}^y[\psi]\mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau, \tag{2.16}$$

where we introduce the notation

$$\tilde{\mathcal{K}}^y[\psi] := y'\left(|\phi'|^2 + |T\phi|^2\right) - \left(yU\frac{\Delta}{r^2}\right)'|\phi|^2 - \left(\frac{\Delta}{r^4}y\right)'|\mathring{\nabla}\phi|^2. \tag{2.17}$$

We choose

$$y = \left(1 - \frac{3M}{r}\right)^3 \left(1 + \frac{3M}{r}\right),$$

and then compute

$$\begin{aligned}
y' &= \frac{6M\Delta}{r^4}\left(1 - \frac{3M}{r}\right)^2\left(1 + \frac{6M}{r}\right),\\
-\left(y\frac{\Delta}{r^4}\right)' &= \frac{2\Delta}{r^5}\left(1 - \frac{3M}{r}\right)^2\left(1 - \frac{6M}{r} - \frac{21M^2}{r^2} + \frac{63M^3}{r^3}\right)\\
&\geq -\frac{2\Delta}{567r^5}\left(250\sqrt{7} - 91\right)\left(1 - \frac{3M}{r}\right)^2,\\
-\left(yU\frac{\Delta}{r^2}\right)' &= \frac{6M\Delta}{r^6}\left(1 - \frac{3M}{r}\right)^2\left(1 - \frac{14M}{3r} - \frac{17M^2}{r^2} + \frac{48M^3}{r^3}\right) \geq -\frac{4M\Delta}{r^6}\left(1 - \frac{3M}{r}\right)^2.
\end{aligned}$$

Hence, we deduce that

$$\begin{aligned}
\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{\mathcal{K}}^y[\psi] \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau \geq& \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \frac{6M}{r^2}\left(1 - \frac{3M}{r}\right)^2\left(1 + \frac{6M}{r}\right)\left(|\phi'|^2 + |T\phi|^2\right) \mathrm{d}r \mathrm{d}\sigma \mathrm{d}\tau\\
&- \frac{1024}{13}\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \tilde{\mathcal{K}}^f[\psi] \mathrm{d}r^* \mathrm{d}\sigma \mathrm{d}\tau,
\end{aligned}$$

concluding the proof. □

Remark 2.18. As for the Minkowski case, we can improve the large r-weights on the left hand side of the estimate in Lemma 2.17 as described in Remark 2.10 by changing our choice of y at very large r so that $y = 1 - r^{-\delta} + o(r^{-\delta})$ as $r \to \infty$, $\delta \in (0, 1]$.

We are ready to conclude the proof of the ILED estimate for waves on Schwarzschild:

Proof. By combining Lemmas 2.14 and 2.17, we have

$$\begin{aligned}
&\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \left[\frac{1}{r^2}|\phi'|^2 + \left(1 - \frac{3M}{r}\right)^2\left(\frac{1}{r^2}|T\phi|^2 + \frac{1}{r^3}|\mathring{\nabla}\phi|^2\right) + \frac{1}{r^4}|\phi|^2\right] \mathrm{d}r \mathrm{d}\sigma \mathrm{d}\tau\\
&\lesssim \int_{\Sigma_{\tau_2}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_2}} \mathrm{dvol}_{\Sigma_{\tau_2}} + \int_{\Sigma_{\tau_1}} \tilde{J}^T_\mu[\psi] n^\mu_{\Sigma_{\tau_1}} \mathrm{dvol}_{\Sigma_{\tau_1}}.
\end{aligned}$$

Indeed, notice that, for the $|\phi'|^2$ term, we can improve on the degeneracy given by the y multiplier (Lemma 2.17) using the f multiplier (Lemma 2.14), and we can improve on the large r-weights given by the f multiplier (Lemma 2.14) using the y multiplier (Lemma 2.17). Finally, we control the right hand side of the above estimate using the energy estimates of Proposition 2.4. □

Exercise 2.19 (Trapping degeneracy in Morawetz estimates on Reissner-Nordström)**.** Repeat Exercise 2.13 to show that, for the Reissner-Nordström exterior with $|e| \leq M$, there are null geodesics which asymptote toward $\{r = r_{\text{trap}}\}$, with

$$r_{\text{trap}} = 3M + \frac{3M}{2}\left(\sqrt{1 - \frac{8e^2}{9M^2}} - 1\right),$$

and a necessary condition for this to occur along a null geodesic is that

$$\frac{\dot{r}}{E} = \pm f(r), \qquad f(r) = \frac{1}{r}\left(1 - \frac{r_{\text{trap}}}{r}\right)\left(r^2 + 2r r_{\text{trap}} - \frac{r_{\text{trap}}^2 e^2}{r_{\text{trap}}^2 - 2M r_{\text{trap}} + e^2}\right)^{1/2}.$$

With

$$g(r) = \frac{1}{2r}\left(\frac{\Delta}{r^2} - \left(1 - \frac{e^2}{Mr}\right)\frac{M^2}{r^2}\right),$$

or otherwise, derive the analogue of Lemma 2.14 for the Reissner-Nordström exterior spacetime with $|e| \leq M$. [Note: you may use either the approach of our proof of Theorem 2.11 or that of Exercise 2.16.] Deduce the analogue of Theorem 2.11 for wave on this background.

End of Lecture 3

Start of Problem Session XX

3 From the Schwarzschild exterior to the Kerr exterior

In the previous section, we saw that, for the scalar wave equation (1.1) on a stationary, asymptotically flat spacetime, (energy, higher order Sobolev, or pointwise) boundedness and decay estimates through a suitable spacelike foliation of the spacetime can be deduced, by an essentially black-box method (see Sections **??** and **??**), from first order energy boundedness, such as Proposition 2.4 in the case of Schwarzschild, and suitable Morawetz estimates, such as Theorem 2.11 in the case of Schwarzschild. Thus, we have essentially reduced the study of (1.1) on stationary, asymptotically flat spacetimes to obtaining first order energy and Morawetz estimates.

In this section, we will briefly sketch how we can obtain such estimates for scalar waves (1.1) on the Kerr family of black holes, with $M > 0$ and $|a| \leq M$, in the exterior region away from $\mathcal{H}^+$. The exterior region of a Kerr black hole (excluding the event horizon) can be described by

$$\begin{aligned}
\mathcal{M}_{\mathrm{Kerr}(a,M)} &:= \mathbb{R}_t \times (r_+,\infty)_r \times \mathbb{S}^2, \qquad M>0, \qquad r_+ = M+\sqrt{M^2-a^2},\\
g_{\mathrm{Kerr}(a,M)} &:= -\frac{\Delta}{\rho^2}(\mathrm{d}t - a\sin^2\theta \mathrm{d}\varphi)^2 + \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)\mathrm{d}\varphi - a\mathrm{d}t]^2 + \frac{\rho^2}{\Delta}\mathrm{d}r^2 + \rho^2(\mathrm{d}\theta)^2,\\
\Delta &:= r^2 - 2Mr + a^2, \qquad \rho^2 := r^2 + a^2\cos^2\theta.
\end{aligned}$$

Note that $(\mathcal{M}_{\mathrm{Schw}(M)}, g_{\mathrm{Schw}(M)}) = (\mathcal{M}_{\mathrm{Kerr}(0,M)}, g_{\mathrm{Kerr}(0,M)})$. As in the former case, we often work with the tortoise coordinate $r^* = r^*(r)$, here defined via

$$\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta}, \qquad r^*(3M) = 0.$$

It will also be useful to have the notation

$$T := \partial_t, \quad Z := \partial_\varphi, \quad K := T + \omega_+ Z, \quad \omega_+ := \frac{a}{2Mr_+}.$$

Note that, since a Kerr with $a = 0$ reduces to Schwarzschild, the comparisons we will draw will be with waves on the Schwarzschild exterior, away from $\mathcal{H}^+$, which have been exhaustively analyzed in the previous section. A remark about the choice of foliations in this section is in order:

Remark 3.1. Our energy boundedness and Morawetz estimates will be cast in terms of the foliation $\{\mathcal{S}_\tau\}_{\tau\geq 0}$ of $\mathcal{R}$ whose leaves terminate at i_0, rather than the foliation $\{\Sigma_\tau\}_{\tau\geq 0}$ which connects $\mathcal{H}^+$ to $\mathcal{I}^+$. Though to implement the r^p method of (energy) decay in Section **??**, strictly speaking we need such estimates to be obtained for the $\{\Sigma_\tau\}$ foliation, writing these estimates in terms of the foliation $\{\mathcal{S}_\tau\}$ is less cumbersome notationally. Furthermore, having obtained energy boundedness and Morawetz estimates on $\{\mathcal{S}_\tau\}$ it is not too hard to understand how to derive analogous estimates on $\{\Sigma_\tau\}_{\tau\geq 0}$ if $R \gg_M 1$, as then the difference between $\mathcal{S}_\tau$ and Σ_τ is in the asymptotically flat region (where Kerr looks like Schwarzschild).

3.1 Introduction

Let us first try to apply the physical space methods of Sections 2.2 and 2.3 to the Kerr family, as this will naturally lead us to identify some obstacles in that approach.

3.1.1 Energy estimates and the ergoregion

It turns out that, for $a \neq 0$, though T is still a Killing vector field, it is not causal in the entire Kerr exterior, due to the presence of a so-called ergoregion:

Exercise 3.2 (Ergoregion and causal vectors)**.** Consider the Kerr black hole family with parameters $M > 0$ and $|a| \leq M$. Show that ∂_t is timelike for $r > 2M$, spacelike in the ergoregion $\mathscr{E} := \{r \geq r_+ : \Delta - a^2\sin^2\theta < 0\}$ and null on $\partial\mathscr{E}$. Show that $\mathrm{span}\{T, Z\}$ yields a timelike subspace of the tangent space to the exterior everywhere except at $\mathcal{H}^+$, where it is null; for instance,

$$V = T + \frac{2Mar}{(r^2+a^2)^2} Z$$

is timelike on $\mathcal{M}\backslash\mathcal{H}^+$ and null at $\mathcal{H}^+$. Furthermore, show that if $|a| \leq a_0 < M$, then there is an $\epsilon_0 = \epsilon_0(a_0)$ such that $T + \omega_+ Z$ is timelike for $r \in (r_+, r_+ + \epsilon_0)$.

From Exercise 3.2, we see that we can replace the use of T in Proposition 2.4 for $a = 0$ with a more complicated vector field in $\mathrm{span}\{T, Z\}$. We then obtain the following first order energy estimate:

Proposition 3.3 (Conditional degenerate-at-$\mathcal{H}^+$ energy estimates on Kerr)**.** *Let $g = g_{\mathrm{Kerr}(a,M)}$ for some $M > 0$ and $|a| \leq M$. Assume ψ is a smooth scalar field which solves the homogeneous wave equation (1.1).*

- *If ψ is such that $\mathrm{Re}[T\phi\overline{K\phi}]|_{\mathcal{H}^+} \geq 0$, then, for any $0 \leq \tau_1 \leq \tau_2 \leq \infty$, we have the energy estimate*
$$\int_{\mathcal{S}_{\tau_2}} \tilde{J}^T_\mu[\psi] n^\mu_{\mathcal{S}_{\tau_2}} \mathrm{dvol}_{\mathcal{S}_{\tau_2}} \leq \int_{\mathcal{S}_{\tau_1}} \tilde{J}^T_\mu[\psi] n^\mu_{\mathcal{S}_{\tau_1}} \mathrm{dvol}_{\mathcal{S}_{\tau_1}}. \tag{3.1}$$
- *Let $0 \leq \chi_Z \leq 1$ be a smooth function chosen so $\chi_Z(r_+) = 1$, $\lim_{r\to\infty} \chi_Z = 0$ and so that $T + \omega_+ \chi_Z Z$ is timelike on $\mathcal{R}\backslash\mathcal{H}^+$ and null at $\mathcal{H}^+$. Then, for any $0 \leq \tau_1 \leq \tau_2 \leq \infty$, we have the energy estimate*
$$\begin{aligned}\int_{\mathcal{S}_{\tau_2}} \tilde{J}^{T+\omega_+\chi_Z Z}_\mu[\psi] n^\mu_{\mathcal{S}_{\tau_2}} \mathrm{dvol}_{\mathcal{S}_{\tau_2}} \leq& \int_{\mathcal{S}_{\tau_1}} \tilde{J}^{T+\omega_+\chi_Z Z}_\mu[\psi] n^\mu_{\mathcal{S}_{\tau_1}} \mathrm{dvol}_{\mathcal{S}_{\tau_1}} \\ &- \frac{a}{2Mr_+} \int_{\mathcal{R}_{(\tau_1,\tau_2)}} \chi'_Z \mathrm{Re}\left[\phi' \overline{Z\phi}\right] \mathrm{d}r \mathrm{d}\sigma \mathrm{d}\tau,\end{aligned} \tag{3.2}$$
where we use $'$ to denote r^* derivatives, and we note that
$$\begin{aligned}\tilde{J}^{T+\omega_+\chi_Z Z}_\mu[\psi] n^\mu_{\mathcal{S}_\tau} \mathrm{dvol}_{\mathcal{S}_\tau} &\sim \tilde{J}^V_\mu[\psi] n^\mu_{\mathcal{S}_\tau} \mathrm{dvol}_{\mathcal{S}_\tau} \\ &\sim \left(|\partial_t\phi|^2 + \frac{\Delta}{r^2}|\partial_r\phi|^2 + \frac{1}{r^2}|\mathring{\nabla}\phi|^2 + \frac{r-M}{r^4}|\phi|^2\right) \mathrm{d}r\mathrm{d}\sigma.\end{aligned}$$

Proof. The proof is identical to that of Proposition 2.4: we invoke Proposition 1.13 and then use the assumptions on χ_Z or ψ, together with Lemma 1.15, to drop the flux term over $\mathcal{H}^+$. □

We refer to (3.2) as a conditional energy estimate because the presence of the last term on the right hand side shows us that, for $a \neq 0$, we cannot hope to close energy estimates for ψ unless we control bulk terms, i.e. a Morawetz estimate, in ψ. This is quite different from the case of $a = 0$, where the energy estimate (2.4) could be closed completely independently of the Morawetz estimate (2.9)!

From Proposition 3.3, we deduce:

Corollary 3.4 (of Proposition 3.3)**.** *Consider the setting of Proposition 3.3. The $T + \omega_+\chi_Z Z$-energy flux of ψ on $\mathcal{S}_\tau$ can grow at most exponentially as $\tau \to \infty$: there is a constant $C(M) > 0$ such that*
$$\int_{\mathcal{S}_\tau} \tilde{J}^{T+\omega_+\chi_Z Z}_\mu[\psi] n^\mu_{\mathcal{S}_\tau} \mathrm{dvol}_{\mathcal{S}_\tau} \leq \exp\left(\frac{|a|}{M} C(M)\tau\right) \int_{\mathcal{S}_0} \tilde{J}^{T+\omega_+\chi_Z Z}_\mu[\psi] n^\mu_{\mathcal{S}_0} \mathrm{dvol}_{\mathcal{S}_0},$$
for all $\tau \geq 0$.

Proof. For a suitable choice of χ_Z,
$$\left|\frac{a}{2Mr_+}\int_{\mathcal{R}_{(\tau_1,\tau_2)}} \chi'_Z \mathrm{Re}\left[\phi'\overline{Z\phi}\right] \mathrm{d}r\mathrm{d}\sigma\mathrm{d}\tau\right| \lesssim_M \frac{|a|}{M}\int_{\tau_1}^{\tau_2}\int_{\mathcal{S}_\tau} \tilde{J}^{T+\omega_+\chi_Z Z}_\mu[\psi] n^\mu_{\mathcal{S}_\tau} \mathrm{dvol}_{\mathcal{S}_\tau} d\tau,$$
and so we can apply Grönwall's inequality to conclude. □

Corollary 3.4 tells us that the worst possible behavior for solutions of the homogeneous scalar wave equation (1.1) on Kerr is exponential *growth*. Comparing to the $a = 0$ case where, in Proposition **??**, we showed energy *decay*, this seems very bad indeed. To improve on Corollary 3.4, we need to obtain better bounds on the last term in the right hand side of (3.2).

3.1.2 Morawetz estimates and trapping

In view of the previous comments, we should be motivated to try to prove a Morawetz estimate for Kerr. In the case $a = 0$ studied in Section 2.3.2 above, we gathered intuition for this estimate by studying the null geodesic flow of the Schwarzschild exterior. Let us try the same approach here, following [9, Section 6]:

Exercise 3.5 (Null geodesic flow on Kerr I)**.** Let $\gamma(s) = (t(s), r(s), \theta(s), \varphi(s))$ be a null geodesic. Show that the following quantities are conserved along γ:

$$\begin{aligned} E &:= -g\left(\dot{\gamma}, T\right) = \left(1 - \frac{2Mr}{\rho^2}\right)\dot{t} + \frac{2Mra\sin^2\theta}{\rho^2}\dot{\varphi}, \\ L &:= g\left(\dot{\gamma}, Z\right) = -\frac{2Mra\sin^2\theta}{\rho^2}\dot{t} + \sin^2\theta\frac{(r^2+a^2)^2 - a^2\sin^2\theta\Delta}{\rho^2}\dot{\varphi}, \\ \mathcal{K} &:= \rho^4\dot{\theta}^2 + \frac{L^2}{\sin^2\theta} + a^2E^2\sin^2\theta. \end{aligned}$$

Here, $\mathcal{K}$ is a conserved quantity associated not to a Killing field but to the Carter Killing tensor. Show that the evolution equation for $r(s)$ can be written as

$$\frac{\rho^4}{(r^2+a^2)^2}\left(\dot{r}\right)^2 = E^2 - V_0(r), \qquad V_0(r) = \frac{\Delta\mathcal{K} + 4MarEL - a^2L^2}{(r^2+a^2)^2}. \tag{3.3}$$

A future-directed geodesic with $E < 0$ is called superradiant. Show that, for such geodesics, $aL < 0$ and, moreover, $aL(E - \omega_+ L) \leq 0$. *Solution: see [4, page 367].*

Notice that, unlike in the $a = 0$ case studied in Exercise 2.13, here V_0 depends on E! This makes plotting $V_0(r)$ and E^2 much more complicated see Figs.

Proposition 3.6 (Null geodesic flow on Kerr II)**.** *For all admissible $(E, L, \mathcal{K}) \in \mathbb{R}^2 \times \mathbb{R}_0^+$ and for $r \in [r_+, \infty)$, $V_0(r)$ has either (i) no critical points, (ii) a maximum at $r = r^0_{\max}$, or (iii) a maximum at $r = r^0_{\max}$ and a minimum at $r^0_{\min} < r^0_{\max}$. Furthermore:*

- *Stable future-trapping of γ cannot occur except, possibly, if $E = \omega_+ L$.*

  More precisely, we have $V_0(r_{\min}) - E^2 \geq 0$ with equality if and only if $E = \omega_+ L$.

- *Unstable future-trapping of γ can occur and its location depends nontrivially on $(E, L, \mathcal{K})$.*

  More precisely, there exist parameters $(E, L, \mathcal{K}) \in \mathbb{R}^2 \times \mathbb{R}_0^+$ such that $V_0(r^0_{\max}) - E^2 = 0$, where $r^0_{\max} = r^0_{\max}(E, L, \mathcal{K})$ but satisfies uniform upper and lower bounds: if $|a| \leq a_0 < M$, we have constants $c = c(a_0, M) > 0$, with $\lim_{a_0 \to M} c(a_0, M) = 0$, and $C = C(M) > 0$ such that $c(a_0, M) \leq r^0_{\max} - r_+ \leq C(M)$.

- *If γ is superradiant then it cannot be (stably or unstably) future-trapped except, possibly, if $E = \omega_+ L$ and $|a| = M$.*

  More precisely, if $|a| \leq a_0 < M$, then there is a constant $\alpha = \alpha(a_0, M) > 0$, with $\lim_{a_0 \to M} \alpha(a_0, M) = 0$ such that

  $$0 < E^2 \leq EL\omega_+ + \alpha(a_0, M)\mathcal{K} \implies V_0(r^0_{\max}) - E^2 \geq c(a_0, M)\mathcal{K},$$

  for some constant $c = c(a_0, M) > 0$ satisfying $\lim_{a_0 \to M} c(a_0, M) = 0$.

Proof. It is easy to see that, because

$$\mathcal{K} \geq \frac{L^2}{\sin^2\theta} + a^2E^2\sin^2\theta,$$

we must have $\mathcal{K} \geq 2|aLE|$ and $\mathcal{K} \geq L^2$. These conditions restrict the allowable values of $(E, L, \mathcal{K}) \in \mathbb{R}^3$, and we refer to the triples $(E, L, \mathcal{K}) \in \mathbb{R}^3$ which meet them as admissible.

We direct the reader to [9, Sections 6.2 to 6.4] for the proofs of the statements. It is useful, moreover, to keep in mind the analysis of the classical textbook [4, Chapter 63]. There, Chandrasekhar computes $r^0_{\max}$ as the solution to the parametric equations

$$\begin{cases} \dfrac{aE}{L} = \dfrac{a^2(r-M)}{M(r^2-a^2) - r\Delta}\Big|_{r=r^0_{\max}} \\[2ex] \dfrac{E^2}{\mathcal{K} - a^2E^2 - L^2} = \dfrac{a^2(r-M)^2}{r^3[4M\Delta - r(r-M)^2]}\Big|_{r=r^0_{\max}} \end{cases}.$$

From the first equation we see that $r^0_{\max} = r^0_{\max}(E, L)$, which is slightly stronger than we claimed in the second bullet point. However, one can see that $\mathcal{K}$ constrains the allowable $r^0_{\max}$: by an analysis of the θ

motion, Chandrasekhar determines that a further admissibility condition is $\mathcal{K} - a^2E^2 - L^2 \geq 0$, and thus

$$\begin{aligned} 0 &\leq \frac{\mathcal{K} - a^2E^2 - L^2}{E^2} \\ &\implies M\left[2 - \cos\left(\frac{1}{3}\arg x\right) - \sqrt{3}\sin\left(\frac{1}{3}\arg x\right)\right] \leq r^0_{\max} \leq M\left[2 - \cos\left(\frac{1}{3}\arg x\right) + \sqrt{3}\sin\left(\frac{1}{3}\arg x\right)\right], \\ &\qquad x := \sqrt[3]{\frac{2a^2}{M^2} - 1 + \frac{2i|a|}{M}\sqrt{1 - \frac{a^2}{M^2}}}. \end{aligned}$$

One can then deduce (though this does not appear to be explicitly written in [4]) that

$$\frac{aE}{L} = \frac{a^2(r-M)}{M(r^2-a^2) - r\Delta}\Big|_{r=r^0_{\max}} \leq \frac{a^2}{2Mr_+},$$

with equality only if $|a| = M$ and $\mathcal{K} = a^2E^2 + L^2$. The latter condition implies that the geodesic is contained in the equatorial plane. $\square$

Though Proposition 3.6 has several interesting statements, the more important one for now is that, unlike for $a = 0$ (see Exercise 2.13), there isn't a single r-value to which future-trapped null geodesics asymptote: instead of a photon sphere, we have a photon "shell", with each radius in the shell corresponding to a choice of suitable triple $(E, L, \mathcal{K})$. This seems problematic for finding a vector field multiplier to capture trapping, and indeed it is; see [1].

End of Lecture 4