Lattice Models (Michaelmas 2022)

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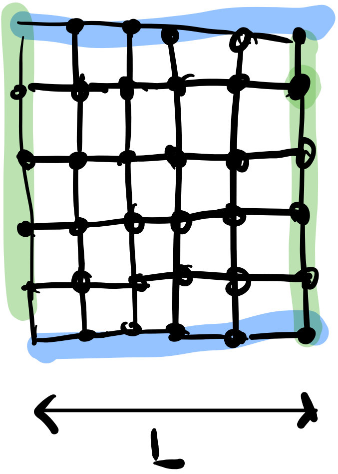
1. The Ising model — first properties

1.1. Overview. $G=(\Lambda,E)$ finite graph (always connected)



Usually:

Vertices $\Lambda\subset\mathbb{Z}^d$

Edges $E=\{x,y\in\Lambda : |x-y|_1=1\}$

or torus: $E=\{x,y\in\Lambda : |x^i-y^i|\in\{1,L\}\ \forall i=1,\dots,d\}$

Defn. Given $\beta>0$ and $h\in\mathbb{R}$, the Ising model on G is the prob. measure on $\{\pm1\}^\Lambda$ given by

$$\mathbb{P}^\Lambda_{\beta,h}(\sigma)=\frac{1}{Z_\beta}e^{\beta\sum_{xy\in E}\sigma_x\sigma_y+h\sum_{x\in\Lambda}\sigma_x}$$

$$=-\frac{\beta}{2}\sum_{xy\in E}(\sigma_x-\sigma_y)^2+\beta|E|$$

$$=-\frac{\beta}{4}\sum_{x,y\in\Lambda}1_{x\sim y}(\sigma_x-\sigma_y)^2+\beta|E|$$

Denote expectation by $\langle\cdot\rangle^\Lambda_{\beta,h}$:

$$\langle F\rangle^\Lambda_{\beta,h}=\sum_\sigma\mathbb{P}^\Lambda_{\beta,h}(\sigma)F(\sigma)$$

Rk. • $\beta=\frac{1}{T}$ is inverse temperature

$>0\to$ ferromagnetic: spins want to align.

• h is external field: bias in $+$ or $-$ direction

• No boundary conditions = free boundary cond.

<u>Main question:</u> Do spins order over large distances?

Let $\Lambda_L = \{x \in \mathbb{Z}^d : x_i \in \{-\frac{L}{2}, \dots, \frac{L}{2}\} \ \forall i = 1, \dots, d\}$ if L is even

$\Lambda_L = \{x \in \mathbb{Z}^d : x_i \in \{-\frac{L-1}{2}, \dots, \frac{L+1}{2}\} \ \forall i = 1, \dots, d\}$ if L is odd

<u>Thm.</u> Let $d \geq 2$. There is $\beta_c \in (0, \infty)$ s.t.

$$\sup_L \sum_{x \in \Lambda_L} \langle \sigma_0 \sigma_x \rangle_{\beta, 0}^{\Lambda_L} < \infty \quad \text{if } \beta < \beta_c$$
$$= \infty \quad \text{if } \beta > \beta_c$$

and the limit $\underbrace{\lim_{L \to \infty} \langle \sigma_0 \sigma_x \rangle_{\beta, h}^{\Lambda_L}}_{\langle \sigma_0 \sigma_x \rangle_{\beta, h}^{\mathbb{Z}^d}}$ exists for all $x, y \in \mathbb{Z}^d$.

<u>Three steps:</u>

- $\exists \beta_0 > 0$ s.t. have $< \infty$: high temp. exp.
- $\exists \beta_1 < \infty$ s.t. have $= \infty$: low temp. exp.
- $\langle \sigma_0 \sigma_x \rangle_{\beta, h}^{\Lambda}$ is monotone in $\beta \geq 0$, $h \geq 0$, Λ: Griffiths ineq.

Much more precise things can be said (later).

Rk. If $\beta=0$ then σ_x are i.i.d, so if $h=0$,

$$\Rightarrow \sum_{x\in\Lambda} \langle \sigma_0\sigma_x\rangle_{0,0} = \langle \sigma_0^2\rangle = 1$$

If $\beta\to\infty$ with Λ fixed then only const. configurations contribute: $\sigma_x=+1$ $\forall x$ or $\sigma_x=-1$ $\forall x$

$$\Rightarrow \sum_{x\in\Lambda} \langle \sigma_0\sigma_x\rangle_{\beta,h} \xrightarrow{\beta\to\infty} |\Lambda|$$

Exercise: Let $d=1$. Then $\sup_L \sum_{x\in\Lambda_L} \langle\sigma_0\sigma_x\rangle < \infty$ for all $\beta>0$ "$\beta_c=+\infty$."

Later:
- What happens at/near $\beta=\beta_c$?
- Related models, e.g.,

  $\sigma_x\in\{\pm1\} \to \sigma_x\in S^n$ O(n) model

  $\to \sigma_x\in\{1,\dots,q\}$ Potts model
- Connections of such models to percolation and interacting random walks.
- Such models as approximations to Euclidean QFT: $\Lambda \mapsto \varepsilon\Lambda$, $\varepsilon\to 0$.
- ...

1.2. High temperature expansion

Since $\sigma_x \sigma_y \in \{\pm 1\}$,

$$e^{\beta \sigma_x \sigma_y} = \left(1 + \tanh(\beta)\sigma_x\sigma_y\right)\cosh(\beta)$$

$$\Rightarrow Z_{\beta,0} = (\cosh\beta)^{|E|} \sum_{\sigma\in\{\pm1\}^\Lambda} \underbrace{\prod_{xy\in E} (1+\tanh(\beta)\sigma_x\sigma_y)}_{\sum_{E'\subset E} (\tanh\beta)^{|E'|} \prod_{xy\in E'} \sigma_x\sigma_y}$$

Ex. Show $\prod_{i\in I}(1+a_i) = \sum_{S\subset I} \prod_{i\in S} a_i$

$$= (\cosh\beta)^{|E|} \sum_{E'\subset E} (\tanh\beta)^{|E'|} \underbrace{\sum_{\sigma} \prod_{xy\in E'} \sigma_x\sigma_y}_{2^{|\Lambda|} 1_{\partial E' = \emptyset}}$$

($E' = \emptyset$ is permitted)

where $\partial E' = \{x\in\Lambda :$ x is in odd number of edges in $E'\}$

$$\Rightarrow Z = 2^{|\Lambda|} \cosh(\beta)^{|E|} \sum_{\substack{E'\subset E \\ \partial E' = \emptyset}} (\tanh\beta)^{|E'|}$$

Rk. Subgraphs with $\partial E' = \emptyset$ are called even.

Let $Z[F] = \sum_{\sigma} e^{-H(\sigma)} F(\sigma)$.

The same computation gives

$$Z[\sigma_a \sigma_b] = 2^{|\Lambda|} (\cosh \beta)^{|E|} \sum_{\substack{E' \subset E \\ \partial E' = \{a,b\}}} (\tanh \beta)^{|E'|}$$

$$\Rightarrow \boxed{\langle \sigma_a \sigma_b \rangle = \frac{Z[\sigma_a \sigma_b]}{Z[1]} = \frac{\displaystyle\sum_{\substack{E' \subset E \\ \partial E' = \{a,b\}}} (\tanh \beta)^{|E'|}}{\displaystyle\sum_{\substack{E' \subset E \\ \partial E' = \emptyset}} (\tanh \beta)^{|E'|}}}$$

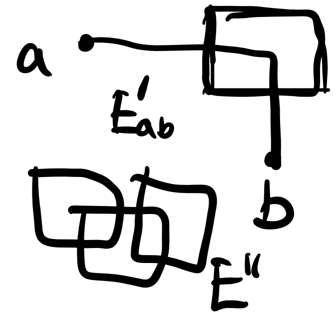
This is called the high temperature expansion.

<u>Theorem</u>. Let $\beta < 1/(4d)^2$. Then there are $C, c > 0$:

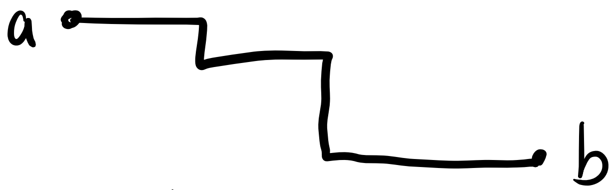
$$\langle \sigma_a \sigma_b \rangle^{\Lambda}_{\beta,0} \leq C e^{-c|a-b|} \quad \forall a, b \in \Lambda.$$

<u>Proof</u>. Decompose E' with $\partial E' = \{a,b\}$ as $E' = E'_{ab} \cup E''$ with E'_{ab} connected, $\partial E'_{ab} = \{a,b\}$ and $\partial E'' = \emptyset$ contained in $E \setminus E'_{ab}$.

$$\Rightarrow \langle \sigma_a \sigma_b \rangle \leq \sum_{\substack{E'_{ab} \subset E \text{ connected} \\ \partial E'_{ab} = \{a,b\}}} (\tanh \beta)^{|E'_{ab}|}$$



How many such E'_{ab} with n edges exist?
Certainly need $n \geq |a-b|_1$:



<u>Exercise</u>: For each connected graph G and $x \in G$, there is a path from x crossing each edge exactly twice.

Thus the number of E'_{ab} with n edges is bounded by number of paths with $2n$ edges from a. There are at most $(2d)^{2n}$ such paths.

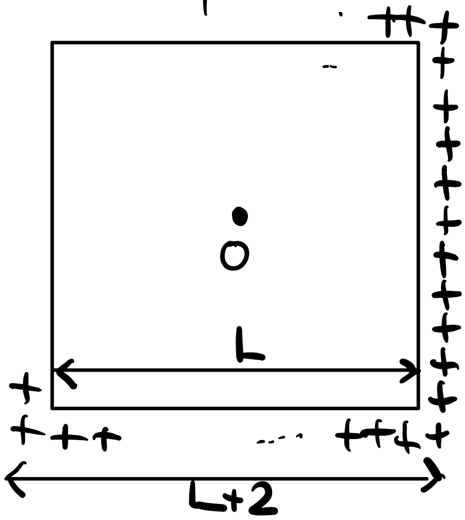
$$\Rightarrow \langle \sigma_a \sigma_b \rangle \leq \sum_{n \geq |a-b|_1} (4d^2 \tan \beta)^n$$

Thus if $\beta < 1/(4d^2)$ then $4d^2 \tanh \beta < e^{-c} < 1$,

$$\Rightarrow \langle \sigma_a \sigma_b \rangle \leq \sum_{n \geq |a-b|_1} e^{-cn} \leq C e^{-c|a-b|_1}.$$

<u>Exercise:</u> For + boundary conditions show

$$\langle \sigma_0 \rangle_{\beta,0}^{\Lambda_L,+} \longrightarrow 0. \qquad (L \to \infty).$$



Here $\bar{\Lambda}_L = \Lambda_{L+2}$ and $\sigma \in \{\pm 1\}^{\Lambda_L}$ is extended to $\sigma \in \{\pm 1\}^{\bar{\Lambda}_L}$ by setting $\sigma_x = +1$ for $x \in \bar{\Lambda}_L \setminus \Lambda_L$ and

$$\mathbb{P}_{\beta,h}^{\Lambda_L,+}(\sigma) = \frac{1}{Z_{\beta,h}^{\Lambda_L,+}} e^{\beta \sum_{xy \in \bar{\Lambda}_L} \sigma_x \sigma_y + h \sum_{x \in \Lambda_L} \sigma_x}$$

<u>Rk.</u> Versions of the above high temperature expansion apply very generally to many models.

<u>Rk.</u> Simple alternative arguments also exist. For example, one can show that, for all $a \neq b$,

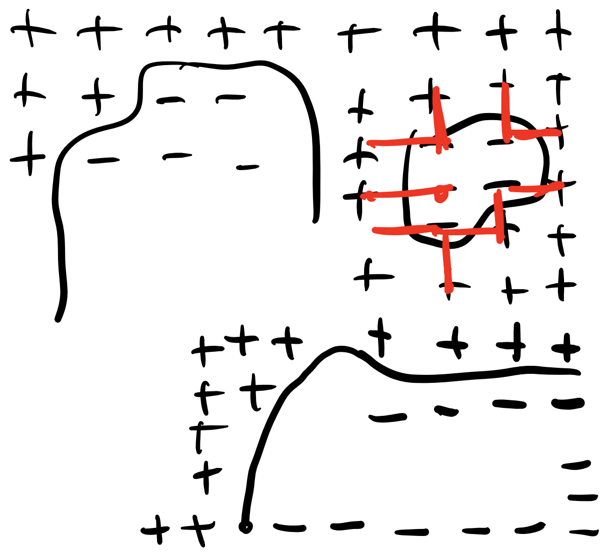
$$\langle \sigma_a \sigma_b \rangle_{\beta,0} \leq \sum_x J_{xb} \langle \sigma_a \sigma_x \rangle_{\beta,0}, \quad J_{xy} = \beta \mathbb{1}_{x \sim y}$$

$\uparrow$ $xy \in E$

<u>Exercise:</u> If $\sup_x \sum_y J_{xy} < 1$ then $\langle \sigma_a \sigma_b \rangle$ decays exp. in $|a-b|$.

1.3. Low temperature — Peierls' argument

Given a spin configuration in $\mathbb{Z}^2$, consider its domain walls or contours:



$$\Gamma = \{ xy \in E : \sigma_x \sigma_y = -1 \}$$

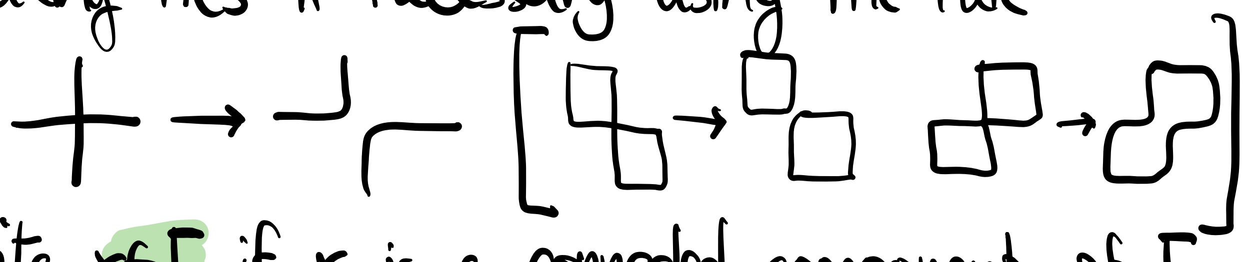
$$\Rightarrow \quad H(\sigma) = 2\beta \underbrace{|\Gamma|}_{\#\text{ edges in } \Gamma} - \beta \underbrace{|E|}_{\text{constant}}$$

Fact: There is a bijection

$$\sigma \mapsto (\Gamma, \sigma_0)$$

Assume $\sigma_0 = +1$ (say).

Decompose Γ into connected components breaking ties if necessary using the rule



Write $\gamma \in \Gamma$ if γ is a connected component of Γ.

Thus γ is either a closed contour or ends at the boundary (open).

Note $p(\gamma) = P(\gamma \in \Gamma)$

$\gamma \cup \Gamma$ is a contour

$$= \frac{\sum_{\Gamma : \gamma \in \Gamma} e^{-2\beta|\Gamma|}}{\sum_{\Gamma} e^{-2\beta|\Gamma|}} = e^{-2\beta|\gamma|} \frac{\sum_{\Gamma : \Gamma \nsim \gamma} e^{-2\beta|\Gamma|}}{\sum_{\Gamma} e^{-2\beta|\Gamma|}} \leq e^{-2\beta|\gamma|}$$

This is called the low temperature expansion.

Thm. Let $d=2$. There are $c, C > 0$ s.t.

$$\langle \sigma_a \sigma_b \rangle_{\beta,0}^{\Lambda_L} \geq 1 - Ce^{-c\beta} \quad \forall\, a,b \in \Lambda_{L/2}.$$

Proof. Note that $\langle \sigma_a \sigma_b \rangle = 1 - 2\langle \mathbb{1}_{\sigma_a \neq \sigma_b} \rangle$.

$\sigma_a \neq \sigma_b \Rightarrow$ (I) a or b is surrounded by a closed contour γ

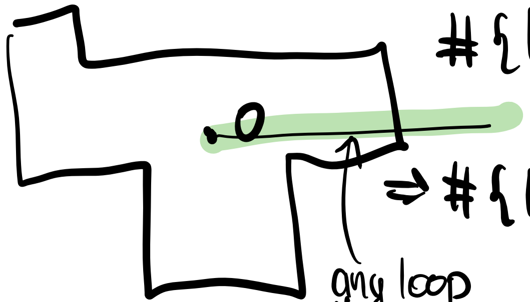
or

(II) a and b are on opposite sides of an open contour γ

$$\langle \sigma_a \sigma_b \rangle \geq 1 - 2 \sum_{\gamma : \text{(I) or (II)}} p(\gamma)$$

$$= 1 - 2 \sum_n e^{-2\beta n} \underbrace{\#\left\{\gamma : |\gamma| = n \text{ with (I) or (II)}\right\}}_{\text{Claim: } \leq 100 n \cdot 4^n.}$$

(I): $\#\{\gamma$ surrounding a, $|\gamma| = n\}$?



$\#\{$loops of length n passing through a fixed vertex$\} \leq 4^n$

$\Rightarrow \#\{$loops surrounding a, $|\gamma| = n\} \leq n 4^n$

any loop necessarily contains a vertex on this segment of length $\leq n$

(II): $\#\{$open γ separating a and b, $|\gamma| = n\}$?

Since γ passes through the boundary and $a, b \in \Lambda_{L/2}$, necessarily $n \geq L/2$ (or $\# = 0$).

$$\Rightarrow \#\{\cdots\} \leq 1_{n \geq L/2} \underbrace{4L}_{\#\text{points on the boundary}} 4^n \leq 8n 4^n$$

<u>Rk</u>. Extension to $d \geq 3$ possible. Alternative below.

1.4. Griffiths' inequalities

<u>Prop</u>. (1st Griffiths inequality). Let $h \geq 0$ (and as always $\beta \geq 0$). Then for any set $S \subset \Lambda$,

$$\Big\langle \underbrace{\prod_{x \in S} \sigma_x}_{} \Big\rangle_{\beta,h} \geq 0.$$

Notation: σ^S (or σ_S)

<u>Proof</u>.

$$\langle \sigma^S \rangle = \frac{1}{Z_{\beta,h}} \sum_{\sigma} \sigma^S \sum_{k=0}^{\infty} \frac{1}{k!} \Big(\beta \sum_{x \sim y} \sigma_x \sigma_y + h \sum_x \sigma_x\Big)^k$$

The claim now follows from: for any $x_1, \dots, x_n \in \Lambda$,

$$\sum_{\sigma} \prod_{n=1}^{N} \sigma_{x_i} \geq 0.$$

The RHS $=0$ if any σ_{x_i} appears an odd number of times, and is a sum of squares otherwise.

<u>Rk</u>. This is a use of the symmetry $\mathbb{Z}_2$.

<u>Cor</u>. For all $h \in \mathbb{R}$, $\langle \sigma_x \sigma_y \rangle_{\beta,h} \geq 0$.

<u>Proof</u>. If $h \geq 0$ then the claim is immediate from the above. If $h < 0$ consider $\tilde{\sigma} = -\sigma$.

Prop. (2nd Griffiths inequality). Let $h \geq 0$. Then for any $S, S' \subset \Lambda$,

$$\underbrace{\langle \sigma^S \sigma^{S'} \rangle - \langle \sigma^S \rangle \langle \sigma^{S'} \rangle}_{} \geq 0.$$

Notation: $\langle \sigma^S ; \sigma^{S'} \rangle_{\beta,h} = \mathrm{Cov}_{\beta,h}(\sigma^S, \sigma^{S'})$

Proof. Let $\tilde{\sigma}$ be an independent copy of σ:

$$\langle \sigma^S ; \sigma^{S'} \rangle = \tfrac{1}{2} \langle (\sigma^S - \tilde{\sigma}^S)(\sigma^{S'} - \tilde{\sigma}^{S'}) \rangle.$$

On the RHS, $\langle \cdot \rangle$ is now the product measure:

$$\langle F \rangle \propto \sum_{\sigma, \tilde{\sigma}} e^{-[H(\sigma) + H(\tilde{\sigma})]} F(\sigma, \tilde{\sigma}).$$

Expanding the exponential $e^{-[H+\tilde{H}]}$, it suffices to check

$$\sum_{\sigma, \tilde{\sigma}} \prod_i (\sigma^{S_i} \pm \tilde{\sigma}^{S_i}) \geq 0$$

where the product is over finitely many $S_i \subset \Lambda$ and '$\pm$' denotes a sign that can depend on i.

By repeated use of the identities

$$ab + \tilde{a}\tilde{b} = \tfrac{1}{2}(a + \tilde{a})(b + \tilde{b}) + \tfrac{1}{2}(a - \tilde{a})(b - \tilde{b})$$

$$ab - \tilde{a}\tilde{b} = \tfrac{1}{2}(a + \tilde{a})(b - \tilde{b}) + \tfrac{1}{2}(a - \tilde{a})(b + \tilde{b})$$

this inequality is reduced to

$$\sum_{\sigma,\tilde{\sigma}} \prod_i (\sigma_{x_i} \pm \tilde{\sigma}_{x_i}) \geq 0$$

where the product is over finitely many $x_i \in \Lambda$ and again '$\pm$' can depend on i.

Change variables:

$$\alpha^+_x = \sigma_x + \tilde{\sigma}_x \,, \quad \alpha^-_x = \sigma_x - \tilde{\sigma}_x$$

i.e.

$$\begin{aligned} (+1,+1) &\mapsto (2,0) \\ (-1,-1) &\mapsto (-2,0) \\ (+1,-1) &\mapsto (0,2) \\ (-1,+1) &\mapsto (0,-2) \end{aligned}$$

$$\Rightarrow \sum_{\substack{\sigma,\tilde{\sigma} \\ \in (+1,+1),(-1,-1),\dots}} \prod_i (\sigma_{x_i} \pm \tilde{\sigma}_{x_i}) = \sum_{\substack{\alpha^+,\alpha^- \\ \in (2,0),(-2,0),\dots}} \prod_i \alpha^{\pm}_{x_i} \geq 0.$$

<u>Cor</u>. $\langle \sigma_a \sigma_b \rangle^{\Lambda}_{\beta,h}$ is monotone in $\beta \geq 0$ and $|h|$.

<u>Proof</u>. Again, since $\langle \sigma_a \sigma_b \rangle^{\Lambda}_{\beta,h} = \langle \sigma_a \sigma_b \rangle^{\Lambda}_{\beta,-h}$, we may assume that $h \geq 0$ and apply Griffiths' inequalities. Then (2nd Griffiths inequality):

$$\frac{\partial}{\partial\beta}\langle\sigma_a\sigma_b\rangle^{\Lambda}_{\beta,h} = \langle\sigma_a\sigma_b\,;\,\sum_{xy}\sigma_x\sigma_y\rangle \geq 0$$

$$\frac{\partial}{\partial h}\langle\sigma_a\sigma_b\rangle^{\Lambda}_{\beta,h} = \langle\sigma_a\sigma_b\,;\,\sum_{x}\sigma_x\rangle \geq 0$$

The same proof shows monotonicity in spatially varying coupling constants, i.e., consider the Ising model

$$\mathbb{P}^{\Lambda}_{J,g}(\sigma) \propto e^{\sum_{xy\in E} J_{xy}\sigma_x\sigma_y + \sum_{x\in\Lambda} g_x\sigma_x}$$

with coupling constants $(J_{xy})_{xy\in E}$ and $(g_x)_{x\in\Lambda}$.
More precisely:

Cor. $\langle\sigma_a\sigma_b\rangle_{J,g}$ is monotone in each J_{xy} and g_x when all g_x have the same sign.

Exercise: Show $\langle\sigma_a\sigma_b\rangle^{\Lambda}_{\beta,h}$ is increasing in Λ.
Show $\beta_c(d)$ is decreasing in d.
(The same holds for σ^S instead of $\sigma_a\sigma_b$ if $|S|$ is even or if $h\geq 0$.)

Show $\langle\sigma_a\sigma_b\rangle^{\Lambda,+}_{\beta,h}$ is decreasing in Λ if $h\geq 0$.

1.5. Generalised Ising models and boundary cond.

It is often useful to consider generalised Ising models of (continuous scalar) spins with expectation given for $F : \mathbb{R}^\Lambda \to \mathbb{R}$ (bounded) by

$$\langle F \rangle^\Lambda_{H,\mu} \propto \int_{\mathbb{R}^\Lambda} F(\varphi)\, e^{-H(\varphi)} \prod_{x \in \Lambda} \mu(d\varphi_x)$$

The measure μ is an (even) Borel measure on $\mathbb{R}$ called the single spin measure.

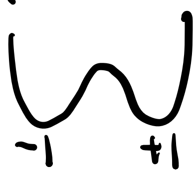
The Ising model is the special case

$$H(\varphi) = -\beta \sum_{xy \in E} \varphi_x \varphi_y - h \sum_{x \in \Lambda} \varphi_x$$

$$\mu(d\varphi) = \delta_{+1}(d\varphi) + \delta_{-1}(d\varphi).$$

Exercise: Let $H(\varphi) = -\beta \sum_{xy \in E} \varphi_x \varphi_y - h \sum_{x \in \Lambda} \varphi_x$

$$\mu(d\varphi) = e^{-g(\varphi^2 - 1)^2}$$



Show that $\langle F \rangle^\Lambda_{H,\mu} \xrightarrow{g \to \infty} \langle F \rangle^\Lambda_{\beta,h}$ for F bounded continuous (where the RHS is the Ising expect.)

There is also a partial converse direction (later).

This generalises further the spatially dependent coupling constants $J_{xy} \geq 0$, $g_x \geq 0$ already discussed.

Boundary conditions: Given $\xi \in \{\pm 1\}^{\mathbb{Z}^d}$, spin configurations with boundary condition ξ are

$$\sigma : \mathbb{Z}^d \to \{\pm 1\}$$

$$\sigma|_{\Lambda^c} = \xi|_{\Lambda^c}$$

The Ising model with boundary condition ξ is

$$\mathbb{P}^{\Lambda,\xi}_{J,g}(\sigma) \propto e^{\sum_{xy \in \bar{E}} J_{xy} \sigma_x \sigma_y + \sum_{x \in \Lambda} g_x \sigma_x}$$

where $\bar{E} = E \cup \{xy : x \in \Lambda, y \notin \Lambda, J_{xy} \neq 0\}$

An analogous definition applies to generalised Ising models.

1.6. FKG inequality

Thm. Consider the continuous spin Ising model $\propto e^{-H(\varphi)}\,d\varphi$ on $\mathbb{R}^\Lambda$ ($d\varphi$ is Lebesgue measure) with $H \in C^2(\mathbb{R}^\Lambda)$ and

$$\partial_{\varphi_x}\partial_{\varphi_y} H(\varphi) \le 0 \quad \forall x \neq y.$$

Then the measure (*) is positively associated:

$$\underbrace{\langle FG\rangle - \langle F\rangle\langle G\rangle}_{\langle F;G\rangle} \ge 0$$

for all $F, G : \mathbb{R}^\Lambda \to \mathbb{R}$ that are increasing, i.e.,

$$F(\varphi) \ge F(\varphi') \quad \text{if} \quad \underbrace{\varphi_x \ge \varphi'_x \;\; \forall x}_{\varphi \ge \varphi'}.$$

Cor. (of statement or proof). Same for Ising models for all $\beta \ge 0$ and $h \in \mathbb{R}$ (or $J_{xy} \ge 0$, $g_x \in \mathbb{R}$).

Exercise: Show that the condition $\partial_{\varphi_x}\partial_{\varphi_y} H \le 0 \;\forall x \neq y$ is equivalent to: $g(\varphi) = \exp(-H(\varphi))$ satisfies

$$g(\varphi \wedge \varphi')\, g(\varphi \vee \varphi') \ge g(\varphi)\, g(\varphi')$$

(FKG lattice condition).

Proof. WLOG $\Lambda = \{1, \dots, N\}$. For $N=1$,

$$\langle F; G\rangle = \frac{1}{2}\langle (F(\varphi) - F(\tilde{\varphi}))(G(\varphi) - G(\tilde{\varphi}))\rangle \geq 0$$

since F, G are increasing.

Here $\tilde{\varphi}$ is an independent copy of φ.

By induction, assume the FKG inequality for $N-1$ and write $\varphi = (\varphi', \varphi_N)$. Conditioning on φ_N,

$$\langle FG\rangle = \langle\langle FG | \varphi_N\rangle\rangle \geq \langle\langle F|\varphi_N\rangle\langle G|\varphi_N\rangle\rangle$$

by the inductive assumption which implies that

$$\langle \cdot | \varphi_N\rangle \propto \int_{\mathbb{R}^{N-1}} e^{-H(\varphi', \varphi_N)} (\cdot)\, d\varphi'$$

satisfies the FKG inequality for each $\varphi_N \in \mathbb{R}$.

Claim: $\underbrace{\langle F|\varphi_N\rangle}_{\langle F(\varphi', \varphi_N)|\varphi_N\rangle}$ is increasing in φ_N (& same for G)

Since F is increasing, it suffices to show the claim with $F(\varphi', \varphi_N)$ replaced by $\tilde{F}(\varphi')$ for all all $\tilde{F}$ increasing in φ' (and not depending on φ_N).

To see this, differentiate:

$$\frac{\partial}{\partial \varphi_N} \langle \tilde{F} | \varphi_N \rangle = \left\langle -\frac{\partial H}{\partial \varphi_N} \tilde{F} \,\middle|\, \varphi_N \right\rangle + \left\langle \frac{\partial H}{\partial \varphi_N} \,\middle|\, \varphi_N \right\rangle \langle \tilde{F} | \varphi_N \rangle$$
$$= \left\langle -\frac{\partial H}{\partial \varphi_N} ; \tilde{F} \,\middle|\, \varphi_N \right\rangle \geq 0$$

where used that $-\frac{\partial H}{\partial \varphi_N}$ is increasing and again that $\langle \cdot | \varphi_N \rangle$ satisfies the FKG inequality.

Finally, using FKG on 1 variable (on φ_N),
$$\langle \langle F | \varphi_N \rangle \langle G | \varphi_N \rangle \rangle \geq \langle \langle F | \varphi_N \rangle \rangle \langle \langle G | \varphi_N \rangle \rangle = \langle F \rangle \langle G \rangle.$$

Cor. Let F be increasing and $\Lambda_1 \subset \Lambda_2$. Then for all $\beta \geq 0$ and $h \in \mathbb{R}$,
$$\langle F \rangle_{\beta,h}^{\Lambda_1,+} \geq \langle F \rangle_{\beta,h}^{\Lambda_2,+}.$$

We will need the Markov property (Exercise):
$$\langle F \rangle_{\beta,h}^{\Lambda_1,\xi} = \langle F \,|\, \sigma|_{\Lambda_1^c} = \xi|_{\Lambda_1^c} \rangle_{\beta,h}^{\Lambda_2,\xi}$$
for any $\Lambda_1 \subset \Lambda_2$ and $\xi \in \{\pm 1\}^{\mathbb{Z}^d}$

Proof. $\langle F\rangle_{\beta,h}^{\Lambda_1,+} \overset{\text{Mark.}}{=} \langle F \mid \sigma_x=+1 \ \forall x\notin\Lambda_1\rangle_{\beta,h}^{\Lambda_2,+}$

$$= \frac{\langle F\,\mathbb{1}\{\sigma|_{\Lambda_1^c}=+1\}\rangle_{\beta,h}^{\Lambda_2,+}}{\langle \mathbb{1}\{\sigma|_{\Lambda_1^c}=+1\}\rangle_{\beta,h}^{\Lambda_2,+}} \overset{\text{FKG}}{\geq} \langle F\rangle_{\beta,h}^{\Lambda_2,+}$$

since $G(\sigma)=\mathbb{1}\{\sigma|_{\Lambda_1^c}=+1\}$ is increasing.

Cor. Let F be increasing. Then for any $\beta\geq 0$, $h\in\mathbb{R}$, and any boundary condition $\xi\in\{\pm1\}^{\mathbb{Z}^d}$:

$$\langle F\rangle_{\beta,h}^{\Lambda,-} \leq \langle F\rangle_{\beta,h}^{\Lambda,\xi} \leq \langle F\rangle_{\beta,h}^{\Lambda,+}$$

$$\langle F\rangle_{\beta,h}^{\Lambda,-} \leq \langle F\rangle_{\beta,h}^{\Lambda} \leq \langle F\rangle_{\beta,h}^{\Lambda,+}.$$

Proof. For any $F:\{\pm1\}^\Lambda\to\mathbb{R}$,

$$\langle F\rangle_{\beta,h}^{\Lambda,+} = \frac{\langle FG\rangle_{\beta,h}^{\Lambda,\xi}}{\langle G\rangle_{\beta,h}^{\Lambda,\xi}}, \quad \text{where}$$

$$G(\sigma) = e^{\beta\sum_{x\in\Lambda}\sum_{y\notin\Lambda}\sigma_x}\, e^{-\beta\sum_{x\in\Lambda}\sum_{y\notin\Lambda}\sigma_x\xi_x}.$$

Since G is increasing and F is increasing (by assumption), the FKG inequality implies

$$\langle F\rangle^{\Lambda,+}_{\beta,h} \geq \langle F\rangle^{\Lambda,\xi}_{\beta,h}.$$

The argument is the same for free boundary cond.

In fact, F does not need to be supported in Λ, i.e. can see the boundary condition as well (the first equality then becomes $\geq$ using that $F\uparrow$).

<u>Exercise</u>. Consider periodic boundary conditions for Λ_L, i.e., opposite sides are identified. Show

$$\langle F\rangle^{\Lambda_{L-1},-}_{\beta,h} \leq \langle F\rangle^{\Lambda_L,\text{periodic}}_{\beta,h} \leq \langle F\rangle^{\Lambda_{L-1},+}_{\beta,h}$$

for F increasing. (Last remark is useful.)

<u>Exercise</u>. Assume $\xi \leq \xi'$. For F increasing, show

$$\langle F\rangle^{\Lambda,\xi}_{\beta,h} \leq \langle F\rangle^{\Lambda,\xi'}_{\beta,h}.$$

1.7. Infinite volume limit

Let $n_x = \frac{1}{2}(1+\sigma_x) \in \{0,1\}$ and recall the notation

$$\sigma^S = \prod_{x\in S} \sigma_x, \qquad n^S = \prod_{x\in S} n_x.$$

<u>Lemma</u>. Let $F: \{\pm 1\}^\Lambda \to \mathbb{R}$. For all $S \subset \Lambda$, there are $\hat{F}(S) \in \mathbb{R}$ and $\tilde{F}(S) \in \mathbb{R}$ s.t.

$$F = \sum_{S\subset\Lambda} \hat{F}(S)\,\sigma^S = \sum_{S\subset\Lambda} \tilde{F}(S)\, n^S.$$

<u>Rk</u>. These are Fourier representations of F. The functions $\sigma \mapsto \sigma^S$ (indexed by $S \subset \Lambda$) form an orthonormal basis for the space of functions on $\{\pm 1\}^\Lambda$ with inner product $2^{-|\Lambda|}\sum_\sigma F(\sigma)G(\sigma)$.

<u>Claim</u> (completeness). $2^{-|\Lambda|}\sum_{S\subset\Lambda} \sigma^S \tilde{\sigma}^S = 1_{\sigma=\tilde{\sigma}}$.

<u>Proof</u>. By the claim, $F(\sigma) = \sum_{\tilde{\sigma}} F(\tilde{\sigma})\, 1_{\sigma=\tilde{\sigma}}$

$$= \sum_{\tilde{\sigma}} F(\tilde{\sigma})\, 2^{-|\Lambda|} \sum_S \sigma^S \tilde{\sigma}^S$$

$$= \sum_S \sigma^S \Big[\underbrace{2^{-|\Lambda|} \sum_{\tilde{\sigma}} F(\tilde{\sigma})\tilde{\sigma}^S}_{\hat{F}(S)} \Big].$$

The second identity follows from the first.

Proof of claim. If $\sigma = \tilde\sigma$ then $\sigma^S \tilde\sigma^S = 1$ for all S and the claim holds since $2^{|\Lambda|}$ is the number of subsets $S \subset \Lambda$. If $\sigma \neq \tilde\sigma$, i.e. $\sigma_x \neq \tilde\sigma_x$ for some $x \in \Lambda$,

$$\sum_{S \subset \Lambda} \sigma^S \tilde\sigma^S = \sum_{S \subset \Lambda \setminus \{x\}} \underbrace{\left(\sigma^S \tilde\sigma^S + \sigma^{S \cup \{x\}} \tilde\sigma^{S \cup \{x\}} \right)}_{\sigma^S \tilde\sigma^S (1 + \sigma_x \tilde\sigma_x) = 0}.$$

Defn. $F : \{\pm 1\}^{\mathbb{Z}^d} \to \mathbb{R}$ is called local (or a cylinder function) if $F(\sigma)$ only depends only on $\sigma|_\Lambda$ for some $\Lambda \subset \mathbb{Z}^d$ finite. The smallest Λ is the support of F (denoted $\operatorname{supp} F$).

Thm. For any $\Lambda_n \uparrow \mathbb{Z}^d$ and F local, the limits

$$\langle F \rangle_{\beta,h}^{\mathbb{Z}^d,+} = \lim_{n\to\infty} \langle F \rangle_{\beta,h}^{\Lambda_n,+}, \quad \langle F \rangle_{\beta,h}^{\mathbb{Z}^d,-} = \lim_{n\to\infty} \langle F \rangle_{\beta,h}^{\Lambda_n,-}$$

exist, do not depend on the sequence Λ_n, and are translation invariant, i.e.,

$$\langle \theta_x F \rangle_{\beta,h}^{\mathbb{Z}^d,\pm} = \langle F \rangle_{\beta,h}^{\mathbb{Z}^d,\pm} \quad \forall x \in \mathbb{Z}^d$$

where θ_x is the action of a translation by x:

$$\theta_x F(\sigma) = F(\theta_x \sigma), \quad (\theta_x \sigma)_y = \sigma_{y-x}.$$

Proof. Since $F = \sum_{S \subset \operatorname{supp} F} \tilde{F}(S) n^S$, it suffices to assume $F = n^S$ for some finite $S \subset \Lambda$. Since n^S is increasing

$\langle n^S \rangle_{\beta,h}^{+,\Lambda_n}$ is decreasing in n.

Since also $\langle n^S \rangle_{\beta,h}^{+,\Lambda_n} \in [0,1]$, the limit of $\langle n^S \rangle_{\beta,h}^{\Lambda_n,+}$ exists.

To see it is independent of Λ_n, consider another sequence Λ_n'. Since $\Lambda_n \uparrow \mathbb{Z}^d$ and $\Lambda_n' \uparrow \mathbb{Z}^d$ we can find a third increasing sequence $\Gamma_n \uparrow \mathbb{Z}^d$ s.t.

$(\Gamma_k)_{k \text{ odd}}$ is a subsequence of (Λ_n)
$(\Gamma_k)_{k \text{ even}}$ is a subsequence of (Λ_n').

By the above applied to $\Lambda_n, \Lambda_n', \Gamma_n$, we find

$$\lim_{n\to\infty} \langle F \rangle^{\Lambda_n,+} = \lim_{n\to\infty} \langle F \rangle^{\Gamma_n,+} = \lim_{n\to\infty} \langle F \rangle^{\Lambda_n',+},$$

so the limit is independent of the sequence.

For translation invariance, note

$$\langle F \rangle^{\mathbb{Z}^d,+} \xleftarrow{n\to\infty} \langle F \rangle^{\Lambda_n,+} = \langle \theta_x F \rangle^{\theta_x \Lambda_n,+} \xrightarrow{n\to\infty} \langle \theta_x F \rangle^{\mathbb{Z}^d,+}$$

by defn. limit is indep. of (Λ_n).

Notation. $\lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \cdot \rangle^\Lambda = \langle \cdot \rangle$ means the limit exists for any local F and increasing sequence $\Lambda_n \uparrow \mathbb{Z}^d$.

Cor. If $\langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,+} = \langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,-}$ then for any $\xi \in \{\pm 1\}^{\mathbb{Z}^d}$:

$$\langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,\xi} = \lim_{n\to\infty} \langle \cdot \rangle_{\beta,h}^{\Lambda_n,\xi} = \langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,+}$$

and likewise for free and periodic boundary cond.

Exercise. Show that $\langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,\text{free}} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \cdot \rangle_{\beta,h}^{\Lambda}$ exists and is translation invariant, for all $\beta \geq 0$, $h \in \mathbb{R}$ (without the assumption $\langle \cdot \rangle^{\mathbb{Z}^d,+} = \langle \cdot \rangle^{\mathbb{Z}^d,-}$)

Thm. Let $\beta \geq 0$, $h \in \mathbb{R}$. The following are equivalent:

(i) $\langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,+} = \langle \cdot \rangle_{\beta,h}^{\mathbb{Z}^d,-}$

(ii) $\langle \sigma_0 \rangle_{\beta,h}^{\mathbb{Z}^d,+} = \langle \sigma_0 \rangle_{\beta,h}^{\mathbb{Z}^d,-}$

Proof. (i)$\Rightarrow$(ii) is trivial, so assume (ii). For any $S \subset \mathbb{Z}^d$ finite, it suffices to show that

$$\langle n^S \rangle_{\beta,h}^{\mathbb{Z}^d,+} = \langle n^S \rangle_{\beta,h}^{\mathbb{Z}^d,-}$$

Note that $\sum_{x\in S} n_x - \underbrace{\prod_{x\in S} n_x}_{n^S}$ is increasing.

$$\Rightarrow \underbrace{\langle \sum_{x\in S} n_x - n^S \rangle^{\mathbb{Z}^d,+}}_{|S|\langle n_0\rangle^{\mathbb{Z}^d,+} - \langle n^S\rangle^{\mathbb{Z}^d,+}} \geq \underbrace{\langle \sum_{x\in S} n_x - n^S \rangle^{\mathbb{Z}^d,-}}_{\underbrace{|S|\langle n_0\rangle^{\mathbb{Z}^d,-}}_{\overset{\text{ass.}}{=} |S|\langle n_0\rangle^{\mathbb{Z}^d,+}} - \langle n^S\rangle^{\mathbb{Z}^d,-}}$$

$$\Rightarrow \langle n^S\rangle^{\mathbb{Z}^d,+} \leq \langle n^S\rangle^{\mathbb{Z}^d,-}$$

But since n^S is increasing, also

$$\langle n^S\rangle^{\mathbb{Z}^d,+} \geq \langle n^S\rangle^{\mathbb{Z}^d,-}.$$

<u>Exercise.</u> By HT expansion, for β small enough,

$$\langle \sigma_0\rangle^{\mathbb{Z}^d,+}_{\beta,0} = \langle \sigma_0\rangle^{\mathbb{Z}^d,-}_{\beta,0} = 0$$

Thus $\langle \cdot \rangle^{\mathbb{Z}^d}_{\beta,0}$ is 'unique': independent of b.c.

By LT expansion, for β large enough,

$$\langle \sigma_0\rangle^{\mathbb{Z}^d,+}_{\beta,0} = -\langle \sigma_0\rangle^{\mathbb{Z}^d,-}_{\beta,0} > 0.$$

Thus $\langle \cdot \rangle^{\mathbb{Z}^d,+}_{\beta,0} \neq \langle \cdot \rangle^{\mathbb{Z}^d,-}_{\beta,0}$: coexistence of two phases.

2. Mean-field theory

We understand $\beta \ll 1$ and $\beta \gg 1$, but what about $\beta \approx \beta_c$? To get intuition, consider the Ising model on the complete graph (Curie-Weiss model):

$$\mathbb{P}^N_{\beta,h}(\sigma) = \frac{1}{Z^N_{\beta,h}} e^{\frac{\beta}{2N}\sum_{x,y=1}^{N} \sigma_x \sigma_y + h\sum_{x=1}^{N}\sigma_x}, \quad \sigma \in \{\pm 1\}^N$$

$$\sum_x \sigma_x \underbrace{\frac{1}{N}\sum_y \sigma_y}_{\text{'mean field' } m(\sigma)}$$

$$= \frac{1}{Z^N_{\beta,h}} e^{+\frac{\beta N}{2} m(\sigma)^2 + hN m(\sigma)}$$

The CW model is the simplest example of an exactly solvable model.

Example sheet: 'Solve' CW model combinatorially.

Let $M_N(\beta,h) = \langle \sigma_1 \rangle^N_{\beta,h}$ be the magnetisation.

$$\chi_N(\beta,h) = \frac{\partial}{\partial h} M_N(\beta,h) = \sum_{x=1}^{N} \langle \sigma_1 ; \sigma_x \rangle^N_{\beta,h}$$

be the susceptibility.

Thm. For all $\beta \geq 0$, $h \in \mathbb{R}$, the following limits exist:

$$M(\beta,h) = \lim_{N\to\infty} M_N(\beta,h)$$
$$M_+(\beta) = \lim_{h\downarrow 0} M(\beta,h)$$

and satisfy, with $\beta_c = 1$,

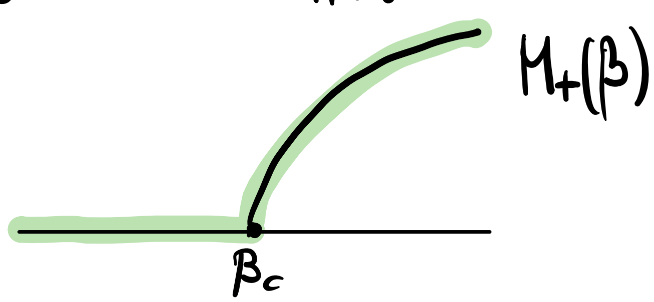
$$M_+(\beta) \begin{cases} >0 & \text{if } \beta > \beta_c \\ =0 & \text{if } \beta \leq \beta_c \end{cases}$$

$$M_+(\beta) \sim (3(\beta-\beta_c))^{1/2} \qquad (\beta \downarrow \beta_c)$$

$$\lim_{\beta\downarrow\beta_c} \frac{M_+(\beta)}{3(\beta-\beta_c)^{1/2}} = 1$$

Rk. $M_+(\beta)$ is called spontaneous magnetisation.
Note that when $\beta > \beta_c$:

$$\lim_{h\downarrow 0} \lim_{N\to\infty} M_N(\beta,h) = M_+(\beta) > 0$$
$$\lim_{N\to\infty} \lim_{h\downarrow 0} M_N(\beta,h) = \lim_{N\to\infty} M_N(\beta,0) = 0.$$



Exercise: $M(\beta_c,h) \sim (3h)^{1/3} \qquad (h\downarrow 0).$

Lemma For $\beta \geq 0$, $h \in \mathbb{R}$,

$$Z^N_{\beta,h} \propto \int_{\mathbb{R}} d\varphi\, e^{-N S(\varphi)}, \quad \text{with}$$

$$S(\varphi) = \frac{\beta}{2}\varphi^2 - \log\cosh(\beta\varphi + h)$$

implied constant does not depend on β, h (but does depend on N)

This is an instance of the "Hubbard-Stratonovich transform" (Ex. Sheet 1)

Proof.

$$\sqrt{2\pi} = \int_{\mathbb{R}} e^{-\frac{1}{2}\varphi^2} d\varphi = \int_{\mathbb{R}} e^{-\frac{1}{2}(\varphi - t)^2} d\varphi \quad (\text{any } t \in \mathbb{R})$$

$$\Rightarrow \quad e^{+\frac{1}{2}t^2} = (2\pi)^{-1/2} \int e^{-\frac{1}{2}\varphi^2 + \varphi t}\, d\varphi$$

$$\Rightarrow \quad e^{+\frac{\beta N}{2} m^2} \propto \int e^{-\frac{\beta N}{2}\varphi^2 + N\varphi m}\, d\varphi$$

$$\Rightarrow \quad Z^N = \sum_\sigma e^{+\frac{\beta N}{2}\left(\frac{1}{N}\sum \sigma_x\right)^2 + h\sum_x \sigma_x}$$

$$\propto \int d\varphi\, e^{-\frac{\beta}{2}N\varphi^2} \underbrace{\sum_\sigma \underbrace{e^{\beta\varphi\sum_x \sigma_x + h\sum_x \sigma_x}}_{\prod_x e^{(\beta\varphi + h)\sigma_x}}}_{\prod_x \left(e^{\beta\varphi + h} + e^{-(\beta\varphi + h)}\right) = \left(2\cosh(\beta\varphi + h)\right)^N}$$

<u>Laplace Principle</u> (Example Sheet 1). Let $S: \mathbb{R} \to \mathbb{R}$ be continuous, bounded below, and assume $\{S(\varphi) \le \min S + 1\}$ is compact, $\int e^{-S(\varphi)} d\varphi < \infty$. Then

$$-\frac{1}{N} \log \int e^{-NS(\varphi)} d\varphi \xrightarrow{N\to\infty} \min S.$$

<u>Cor.</u> $F(\beta,h) = \lim_{N\to\infty} \Big(\underbrace{-\frac{1}{N} \log Z^N_{\beta,h}}_{F_N(\beta,h)}\Big) = \min S.$

<u>Fact.</u> $$M_N(\beta,h) = \frac{\partial}{\partial h} \frac{1}{N} \log Z^N_{\beta,h} = -\frac{\partial F_N}{\partial h}$$

$$\chi_N(\beta,h) = \frac{\partial}{\partial h} M_N(\beta,h) = -\frac{\partial^2 F_N}{\partial h}$$

Since $\chi_N(\beta,h) = \mathrm{Var}_{\beta,h}\left(\frac{1}{N}\sum \sigma_x\right) > 0$, in particular

$$h \mapsto -F_N(\beta,h) \text{ is convex.}$$

<u>Exercise:</u> Let $f_n: \mathbb{R} \to \mathbb{R}$ be a sequence of differentiable and convex functions. Assume that $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$. Show that f is convex and that for every x at which f is differentiable,

$$f_n'(x) \longrightarrow f'(x).$$

What are the minima of $S(\varphi) = \frac{\beta}{2}\varphi^2 - \log\cosh(\beta\varphi + h)$?

$$S'(\varphi) = \beta\varphi - \beta\tanh(\beta\varphi + h) = 0$$

$$\Leftrightarrow \varphi = \tanh(\beta\varphi + h)$$

This sometimes called the self-consistent eqn.

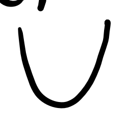
The second derivative is of course also useful:

$$S''(\varphi) = \beta - \frac{\beta^2}{\cosh(\beta\varphi + h)^2}$$

<u>Exercise</u>. This is a calculus exercise.

(i) For any β and $h \neq 0$, S has a unique global minimum φ_0 of the same sign as h.

(ii) For $\beta \underset{(-)}{\leq} 1$ and $h \in \mathbb{R}$, S is (strictly) convex and thus has a unique global minimum which tends to 0 when $h \to 0$.



(iii) For $\beta > 1$ and $h = 0$, S has two global minima $\pm\varphi_0 \neq 0$ and as $h \to \pm 0$ the unique global minimum tends to $\pm\varphi_0 \neq 0$.

(iv) The global minimum is differentiable in h when $h \neq 0$ or $\beta < 1$.

<u>Proof of theorem.</u> It is immediate from the two exercises that

$$\begin{aligned} M_+(\beta) &= -\frac{\partial F}{\partial h}(\beta, 0+) \\ &= -\lim_{h\downarrow 0} \frac{\partial}{\partial h}\left[\frac{\beta}{2}\varphi_0^2 - \log\cosh(\varphi_0\beta + h)\right] \\ &= \lim_{h\downarrow 0} \tanh(\varphi_0\beta) = \begin{cases} =0 & (\beta \le 1) \\ >0 & (\beta > 1) \end{cases} \end{aligned}$$

where $\varphi_0 = \varphi_0(\beta, h)$ is the unique $(h>0)$ solution to the self-consistent equation.

To see how $M_+(\beta)$ tends to 0 as $\beta \downarrow 1$, note

$$\tanh(\varphi_0\beta) = \varphi_0\beta - \frac{1}{3}(\varphi_0\beta)^3 + O(\varphi_0\beta)^5 \quad (\varphi_0\beta \to 0).$$

Thus the self-consistent equation becomes

$$\varphi_0(1-\beta) = \frac{1}{3}(\varphi_0\beta)^3 + o(\varphi_0\beta)^3 \quad (\varphi_0\beta \to 0).$$

Since $\varphi_0(\beta, 0+) > 0$ for $\beta > 1$, we can divide by φ_0:

$$3(1-\beta) = \varphi_0^2\beta^3(1+o(1)) \text{ as } \varphi_0 \to 0 \text{ (i.e. } \beta \downarrow 1)$$

$$\Rightarrow \varphi_0(\beta, 0+) \sim \sqrt{3(1-\beta)} \quad \text{as } \beta \downarrow 1.$$

Thus

$$\begin{aligned} M_+(\beta) &= \tanh(\varphi_0(\beta, 0+)\beta) \\ &\sim \varphi_0(\beta, 0+) \sim \sqrt{3(1-\beta)} \quad (\beta \downarrow 1). \end{aligned}$$

Exercise (susceptibility). The susceptibility is finite for all $h \in \mathbb{R}$ if $\beta < 1$ and for $h \neq 0$ if $\beta > 1$. Moreover,

$$\chi(\beta, 0) = \frac{1}{\beta_c - \beta} \qquad (\beta \uparrow \beta_c)$$

$$\chi(\beta, 0+) \sim \frac{1}{2(\beta - \beta_c)} \qquad (\beta \downarrow \beta_c).$$

Rk. The powers in the behaviour of the magnetisation and susceptibility are called critical exponents. Very generally, it is expected that

$$M_+(\beta) \approx A_1 (\beta - \beta_c)^{\beta} \qquad (\beta \downarrow \beta_c)$$
$$M(\beta_c, h) \approx A_2 h^{1/\delta} \qquad (h \downarrow 0)$$
$$\chi(\beta, 0) \approx A_3 (\beta_c - \beta)^{-\gamma} \qquad (\beta \uparrow \beta_c)$$
$$\vdots$$

for critical exponents $\beta, \delta, \gamma, \ldots$

We have seen that for the Curie-Weiss model:

$$\beta = \frac{1}{2}, \quad \delta = 3, \quad \gamma = 1$$

<u>Thm</u>. The Ising model on $\mathbb{Z}^d$ with $d \geq 5$ has the same critical exponents as the CW model.

<u>Thm</u>. The Ising model on $\mathbb{Z}^2$ has critical exponents

$$\beta = \frac{1}{8}, \quad \delta = 15, \quad \gamma = \frac{7}{4}.$$

3. O(n) models

3.1. Overview

<u>Defn</u>. Let $G=(\Lambda,E)$ be a finite graph. For $n=1,2,3,\dots,$ the (vector) O(n) model (with free boundary conditions) is the probability measure

$$\mathbb{P}^{\Lambda}_{\beta,h}(d\sigma) = \frac{1}{Z^{\Lambda}_{\beta,h}} e^{+\beta\sum_{xy\in E}\sigma_x\cdot\sigma_y + h\sum_{x\in\Lambda}\sigma_x\cdot e} \prod_{x\in\Lambda} d\sigma_x$$

where $e=(1,0,\dots,0)\in\mathbb{R}^n$ (say) and $d\sigma_x$ is the uniform (Haar) measure on $\mathbb{S}^{n-1}=\{x\in\mathbb{R}^n : |x|^2=1\}$.

- $n=1$ is the Ising model.
- $n=2$ is known as XY or rotator model.
- $n=3$ is the classical Heisenberg model.

If $n\geq 2$, these models have continuous symmetry:
Let $R\in O(n)$ and $(R\sigma)_x = R\sigma_x$ for all $x\in\Lambda$. Then:

$$\langle F(\sigma)\rangle^{\Lambda}_{\beta,0} = \langle F(R\sigma)\rangle^{\Lambda}_{\beta,0}$$

The external field is said to break this symmetry explicitly (as opposed to spontaneously).

Defn. Spontaneous symmetry breaking (SSB) holds if:

$$\lim_{h \downarrow 0} \limsup_{\Lambda} \langle \sigma_0 \cdot e \rangle^{\Lambda}_{\beta,h} > 0 \quad \text{(Note: } \langle \sigma_0 \cdot e \rangle^{\Lambda}_{\beta,0} = 0. \text{)}$$

An alternative definition would involve boundary cond:

$$\limsup_{\Lambda} \langle \sigma_0 \cdot e \rangle^{\Lambda, e}_{\beta,h} > 0$$

where "e" refers to constant boundary conditions $\xi_x = e$ for all $x \in \mathbb{Z}^d$ (analogous to + b.c.)

Rk. Both are equal for the Ising model.

3.2. Correlation inequalities

Exercise (Example Sheet 1). The "1st Griffiths inequality" holds for the $O(n)$ model, all $n \geq 1$:

For all $\beta \geq 0$, $h \geq 0$, for all $x_i \in \Lambda$, $a_i \in \{1, \dots, n\}$:

$$\langle \sigma_{x_1}^{a_1} \cdots \sigma_{x_n}^{a_n} \rangle_{\beta,h} \geq 0.$$

The "2nd Griffiths inequality" ($\Rightarrow$ monotonicity in β)

$$\langle \sigma_a \cdot \sigma_b \, ; \, \sigma_c \cdot \sigma_d \rangle_{\beta,h}, \quad (a,b,c,d \in \Lambda)$$

holds also when $n=2$, but remains a conjecture for any $n \geq 3$. For $n=2$, it is a special case of the Ginibre inequality.

For $n=2$, write $\sigma_x = (\cos\theta_x, \sin\theta_x)$, $\theta_x \in [0, 2\pi)$:

$$\langle F(\sigma) \rangle_{\beta,h}^{\Lambda} = \langle F(\cos\theta, \sin\theta) \rangle_{\beta,h}^{\Lambda}$$

$$\propto \int_{[0,2\pi)^\Lambda} e^{\beta \cos(\theta_x - \theta_y) + h\cos\theta_x} F(\cos\theta, \sin\theta) \, d\theta$$

$$d\theta = \prod_{x \in \Lambda} d\theta_x$$

Then:

$$\langle \sigma_x \cdot \sigma_y \rangle_{\beta,h}^{\Lambda} = \langle \cos(\theta_x - \theta_y) \rangle_{\beta,h}^{\Lambda}$$

$$\langle \sigma_x \cdot e \rangle_{\beta,h}^{\Lambda} = \langle \cos\theta_x \rangle_{\beta,h}^{\Lambda}$$

Thm (Ginibre inequalities). Let $n=2$. For any $\beta \geq 0$, $h \geq 0$ and $a, b \in \mathbb{Z}^\Lambda$,

$$\langle \cos(a\theta) \rangle^\Lambda_{\beta,h} \geq 0$$

$$\langle \cos(a\theta) ; \cos(b\theta) \rangle^\Lambda_{\beta,h} \geq 0$$

where $a\theta = \sum_{x \in \Lambda} a_x \theta_x$ and $b\theta = \sum_{x \in \Lambda} b_x \theta_x$.

Cor. $\langle \sigma_x \cdot \sigma_y \rangle^\Lambda_{\beta,h}$ is increasing in $\beta \geq 0$, $h \in \mathbb{R}$, Λ.

$\vdots$

Proof. The idea is the same as for the Griffiths inequalities. We only consider the second inequality because its proof is the more difficult one.

Denote by $\tilde{\theta}$ an independent copy of θ so that

$$\langle \cos a\theta ; \cos b\theta \rangle = \tfrac{1}{2} \langle\langle (\cos a\theta - \cos a\tilde{\theta})(\cos b\theta - \cos b\tilde{\theta}) \rangle\rangle$$

where on the RHS

$$\langle\langle F(\theta, \tilde{\theta}) \rangle\rangle \propto \int e^{-[H(\theta) + H(\tilde{\theta})]} F(\theta, \tilde{\theta}) \, d\theta \, d\tilde{\theta}$$

with $H(\theta) = \beta \sum_{xy} \cos(\theta_x - \theta_y) + h \sum_x \cos \theta_x$, $\beta, h \geq 0$.

Expanding the exponential, it suffices to show

$$\int d\theta\, d\tilde{\theta} \prod_i \left(\cos(m_i\theta) \pm \cos(m_i\tilde{\theta})\right) \geq 0$$

for $m_i \in \mathbb{Z}^\Lambda$ and an i-dependent sign $\pm$. Let

$$\alpha_x^\pm = \tfrac{1}{2}(\tilde{\theta}_x \pm \theta_x)$$

so that

$$\cos m\theta + \cos m\tilde{\theta} = 2\cos(m\alpha^+)\cos(m\alpha^-)$$
$$\cos m\theta - \cos m\tilde{\theta} = 2\sin(m\alpha^+)\sin(m\alpha^-)$$

Therefore:

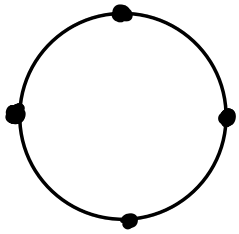
$$\int d\theta\, d\tilde{\theta} \prod_{i=1}^{k} \left(\cos(m_i\theta) + \cos(m_i\tilde{\theta})\right) \times \prod_{i=k+1}^{k+\ell} \left(\cos(m_i\theta) - \cos(m_i\tilde{\theta})\right)$$

$$= 2^{k+\ell} \underbrace{\int d\theta\, d\tilde{\theta} \prod_i \cos(m_i\alpha^+)\cos(m_i\alpha^-) \times \prod_j \sin(m_j\alpha^+)\sin(m_j\alpha^-)}_{\int d\theta\, d\tilde{\theta}\, F(\alpha^+)F(\alpha^-)}$$

$$\propto \int d\alpha^+ d\alpha^-\, F(\alpha^+)F(\alpha^-)$$

$$= \left(\int d\alpha^+\, F(\alpha^+)\right)^2 \geq 0$$

Exercise. Extend the Ginibre inequalities to the case that the σ_x take values in the discrete circle $\mathbb{Z}_q \subset S^1$.



$q=4$

Rk. Note that what facilitated the proof is that $O(2) = U(1)$ is abelian (trigonometric identities).

Rk. Using improvements of the Ginibre inequalities, it can also be shown that the infinite volume limit of the $O(2)$ model with Dirchlet boundary condition ($\xi_x = e \ \forall x \in \mathbb{Z}^d$) exists.

Existence of the infinite volume limit is not known for the $O(n)$ model with $n \geq 3$, but one can always construct subsequential limits:

For any boundary condition ξ (or free or periodic boundary conditions) and $\Lambda_n \uparrow \mathbb{Z}^d$ (hypercube for periodic b.c.) there is a subsequence $(n_k)_k$ s.t.

$\forall F : (S^{n-1})^{\mathbb{Z}^d} \to \mathbb{R}$ local

$$\lim_{k \to \infty} \langle F \rangle_{\beta,h}^{\Lambda_{n_k}, \xi} \text{ exists.}$$

Eg., this a consequence of the (weak-*) compactness of the space of probability measures on the compact space $(S^{n-1})^{\mathbb{Z}^d}$.

<u>Defn.</u> We then say $\langle\cdot\rangle_{\beta,h}^{\mathbb{Z}^d,\xi} = \lim_{k\to\infty} \langle\cdot\rangle_{\beta,h}^{\Lambda_k,\xi} = \lim_{\Lambda\uparrow\mathbb{Z}^d} \langle\cdot\rangle_{\beta,h}^{\Lambda,\xi}$ along a subsequence is an infinite volume limit of the $O(n)$ model.

We can now ask the same questions as for the Ising model:

- Is the magnetisation $\langle\sigma_0\cdot e\rangle_{\beta,h}^{\mathbb{Z}^d}$ nonzero?
- For which (β,h), does $\langle\sigma_0\cdot\sigma_x\rangle_{\beta,h}^{\mathbb{Z}^d}\to 0$ as $|x|\to\infty$?
- If $\langle\sigma_0\cdot\sigma_x\rangle\to 0$ as $|x|\to\infty$, is it exponentially?

<u>Exercise</u> (Example Sheet 2). In mean field theory (complete graph), the situation is essentially the same as for the Ising model.

<u>Exercise</u> (Example Sheet 2). High temperature expansion also works very similarly. Thus for $\beta<\beta_0$:

$$\langle\sigma_0\cdot e\rangle_{\beta,0}^{\mathbb{Z}^d}=0, \quad \langle\sigma_0\cdot\sigma_x\rangle_{\beta,0}^{\mathbb{Z}^d}\le C_\beta e^{-c_\beta|x|}.$$

3.3. Aside: the Gaussian free field (GFF)

Defn. Let $\Lambda \subset \mathbb{Z}^d$ or let Λ be a torus (periodic boundary conditions). The Gaussian free field is the probability measure on $\mathbb{R}^\Lambda$ given by

$$\mathbb{P}^{\Lambda}_{GFF,m^2}(d\sigma) \propto e^{-\frac{1}{2}\sum_{xy \in E}(\varphi_x-\varphi_y)^2 - \frac{m^2}{2}\sum_{x\in\Lambda}\varphi_x^2} \underbrace{d\varphi}_{\text{Lebesgue on } \mathbb{R}^\Lambda}$$

The parameter $m>0$ is called the mass. For $m^2=0$ the same defn. but with $\varphi_0=0$ applies and we say φ is pinned at $x=0$.

Similarly, the GFF with boundary cond. $\xi \in \mathbb{R}^\Lambda$ is

$$\mathbb{P}^{\Lambda,0}_{GFF,m^2}(d\sigma) \propto e^{-\frac{1}{2}\sum_{xy\in \bar{E}}(\varphi_x-\varphi_y)^2 - \frac{m^2}{2}\sum_{x\in\Lambda}\varphi_x^2} d\varphi$$

where $\bar{E}$ includes the boundary edges and $\varphi_x = \xi_x$ for $x \notin \Lambda$. Now $m=0$ is allowed.

Fact. $\sum_{xy} J_{xy}(f_x-f_y)^2 = \frac{1}{2}\sum_{x,y} J_{xy}(f_x-f_y)^2$

$$= \sum_x f_x \underbrace{\sum_y J_{xy}(f_x-f_y)}_{\text{Defn. } -(\Delta_J f)_x} = -\sum f_x(\Delta_J f)_x$$

Cor. The covariance of the GFF with mass $m^2>0$ is:

$$C^\Lambda_{m^2}(x,y) = (-\Delta^\Lambda + m^2)^{-1}(x,y), \quad x,y\in\Lambda$$
$$= \int (e^{\Delta^\Lambda t})(x,y)\, e^{-m^2 t}\, dt$$

where $\Delta^\Lambda = \Delta_J$ with standard weights $J_{xy} = 1_{x\sim y} 1_{x,y\in\Lambda}$.

The covariance of the GFF with Dirichlet b.c. is:

$$C^{\Lambda,0}(x,y) = (-\Delta^{\Lambda,0})^{-1}(x,y), \quad x,y\in\Lambda$$

where $(\Delta^{\Lambda,0} f)_x = (\Delta^\Lambda f)_x - \sum_{y\notin\Lambda} J_{xy} f_x$.

Rk. The GFF with b.c. ξ has the distribution of a GFF with b.c. 0 + a deterministic function satisfying $(\Delta h)_x = 0$ $(x\in\Lambda)$ and $h_x = \xi_x$ $(x\notin\Lambda)$.

Defn. The continuous-time SRW on Λ with jump rates $J_{xy} = J_{yx} \geq 0$ for $x,y\in\Lambda$ can be defined as follows: Let $(Y_n)_{n\in\mathbb{N}}$ be a discrete-time SRW with jump probabilities $p_{xy} = J_{xy}/J_x$ where $J_x = \sum_y J_{xy}$. Let (τ_n) be an independent sequence of $\mathrm{Exp}(J_{Y_n})$ random variables, i.e., with density $J_{Y_n} e^{-J_{Y_n} t}\, dt$ and thus mean $1/J_{Y_n}$. Then $(X_t)_{t\geq 0}$ is defined by

$$X_t = Y_{N_t}, \quad N_t = \max\Big\{n : \sum_{i=1}^n \tau_i \leq t\Big\}.$$

__Thm.__ (AP) X is a continuous-time (right-continuous) Markov process with generator (Q matrix) Δ_J:

$$\forall f:\Lambda\to\mathbb{R}, \quad \frac{\partial}{\partial t}\mathbb{E} f(X_t) = \mathbb{E}(\Delta_J f(X_t)).$$

__Ex.__ The covariance of the GFF with mass $m^2>0$ is

$$C^{\Lambda}_{m^2}(x,y) = \int_0^\infty \mathbb{P}^{\Lambda}_x(X_t=y)\, e^{-m^2 t}\, dt$$

The covariance of the GFF with Dirichlet b.c. is

$$C^{\Lambda,0}_{m^2}(x,y) = \int_0^\infty \mathbb{E}^{\mathbb{Z}^d}_x\left(1_{X_t=y}\, 1_{\tau_{\Lambda^c}>t}\right) e^{-m^2 t}\, dt$$

where $\tau_{\Lambda^c} = \inf\{t\geq 0 : X_t\notin\Lambda\}$. Also

$$\frac{C^{\Lambda,0}_0(0,x)}{C^{\Lambda,0}_0(0,0)} = \mathbb{P}_x(\tau_0<\tau_{\Lambda^c})$$

Indeed, by symmetry & the strong Markov property,

$$C^{\Lambda,0}_0(0,x) = C^{\Lambda,0}_0(x,0) = \mathbb{E}_x\left(\int_0^\infty 1_{X_t=0}\, 1_{t<\tau_{\Lambda^c}}\right)$$

$$= \mathbb{E}_x\left(\int_{\tau_0}^\infty 1_{X_t=0}\, 1_{t<\tau_{\Lambda^c}}\right) = \mathbb{P}_x(\tau_0<\tau_{\Lambda^c})\, C^{\Lambda,0}_0(0,0).$$

Exercise. Since SRW on $\mathbb{Z}^d$ is transient iff $d>2$:

$$\lim_{m^2\downarrow 0}\lim_{L\to\infty}\langle \varphi_0^2\rangle^{\Lambda_L}_{GFF,m^2} = \lim_{L\to\infty}\langle \varphi_0^2\rangle^{\Lambda_L,0}_{GFF,0} \begin{cases} =\infty & (d\le 2) \\ <\infty & (d>2) \end{cases}$$

Moreover, in $d\ge 3$,

$$C_0(x,y) = \lim_{m^2\downarrow 0}\lim_{L\to\infty}\langle \varphi_x\varphi_y\rangle^{\Lambda_L}_{GFF,m^2} \text{ exists } \in(0,\infty)$$

$$= \lim_{L\to\infty} \underbrace{\langle \varphi_x\varphi_y\rangle^{\Lambda_L,0}_{GFF,0}}_{C_0^{\Lambda_L}(x,y)} = \langle \varphi_x\varphi_y\rangle^{\mathbb{Z}^d}_{GFF,0}$$

$$= \frac{1}{(2\pi)^d}\int_{[0,2\pi]^d} dp \, \frac{1}{\sum_{i=1}^d (2-2\cos(p_i))} e^{ip\cdot(x-y)}$$

$$\overset{|x|\to\infty}{\approx} \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} dp \, \frac{1}{|p|^2} e^{ip\cdot(x-y)} = C_d |x-y|^{-(d-2)}$$

whereas, in $d\le 2$,

$$\bar{C}_0(x,y) = \lim_{m^2\downarrow 0}\lim_{L\to\infty}\left[\langle \varphi_x\varphi_y\rangle^{\Lambda_L}_{GFF,m^2} - \langle \varphi_0^2\rangle^{\Lambda_L}_{GFF,m^2}\right]$$

$$= \lim_{L\to\infty}\left[\langle \varphi_x\varphi_y\rangle^{\Lambda_L,0}_{GFF,0} - \langle \varphi_0^2\rangle^{\Lambda_L,0}_{GFF,0}\right] \text{ exists.}$$

$$= (\text{formula for } C_0(x,y) \text{ in } d\ge 3).$$

In $d=2$,

$$\bar{C}_0(x,y) \sim \frac{-1}{2\pi} \log|x-y|$$

$$|\bar{C}_0(x,y) - \bar{C}_0(x,y+e)| \leq 2$$

$$\frac{C_0^{\Lambda_L,0}(0,x)}{C_0^{\Lambda_L,0}(0,0)} \xrightarrow{L\to\infty} 1 \qquad \forall x \in \mathbb{Z}^d$$

$$\| P_x(\tau_0 < \tau_{\Lambda_L^c})$$

3.4. Mermin-Wagner theorem

The next theorems show that a Peierls argument cannot work for $n \geq 2$. Intuitively, continuous spins can be deformed slowly without affecting their energy much.

Thm Let $d=2$, $n \geq 2$, $\beta \geq 0$, $h=0$, $\xi \in \mathbb{Z}^d$. Then for every infinite volume limit $\langle \cdot \rangle$ and every cont. local $F: (S^{n-1})^{\mathbb{Z}^d} \to \mathbb{R}$

$$\forall R \in SO(n): \langle F(R\sigma) \rangle = \langle F(\sigma) \rangle.$$

i.e., all infinite volume limits are $SO(n)$ invariant for all $\beta \geq 0$! In particular, the magnetisation vanishes:

$$\langle \sigma_0 \cdot e \rangle = -\langle \sigma_0 \cdot e \rangle = 0$$

Proof. Fix some axis of rotation in $\mathbb{R}^n$ and denote by $R(\varphi)$ the rotation by angle $\varphi \in [0, 2\pi)$ about this axis, e.g., $R(\varphi)$ can be rotation in xy plane.

Let $\varphi_x \in [0, 2\pi)$ be angles that can depend on $x \in \Lambda' \subset\subset \Lambda$ and set $\varphi_x = 0$, $x \notin \Lambda'$.

Here: $\Lambda' \subset\subset \Lambda$ means $J_{xy} = 0$ if $x \in \Lambda'$, $y \notin \Lambda$.

For $\sigma \in (S^{n-1})^{\mathbb{Z}^d}$, define

$$(R(\varphi)\sigma)_x = R(\varphi_x)\sigma_x .$$

$$\Rightarrow \langle F(R(\varphi)\sigma)\rangle^{\Lambda,\xi} = \frac{1}{Z^{\Lambda,\xi}} \int d\sigma \, F(R(\varphi)\sigma)\, e^{-\beta H_\Lambda(\sigma)}$$

$\Lambda' \subset\subset \Lambda$ $\Rightarrow$ does not affect b.c.

$$= \frac{1}{Z^{\Lambda,\xi}} \int d\sigma \, F(\sigma)\, e^{-\beta H_\Lambda(R(-\varphi)\sigma)}$$

$$= \langle F(\sigma)\, e^{-\beta \delta_\varphi H(\sigma)}\rangle^{\Lambda,\xi}$$

where $H_\Lambda(\sigma) = -\sum_{xy \in E} \sigma_x \cdot \sigma_y$ and

$$\delta_\varphi H(\sigma) = H_{\bar\Lambda'}(R(-\varphi)\sigma) - H_{\bar\Lambda'}(\sigma)$$

$\varphi_x = 0$ for $x \notin \Lambda'$

Since

$$R(\varphi_x)^{-1}\sigma_x \cdot R(\varphi_y)^{-1}\sigma_y = \sigma_x \cdot R(\varphi_x) R(\varphi_y)^{-1} \sigma_y$$
$$= \sigma_x \cdot \sigma_y + \sigma_x \cdot (1 - R(\varphi_x - \varphi_y))\sigma_y$$

we have

$$\delta_\varphi H(\sigma) = \sum_{xy \subset \bar\Lambda'} \sigma_x \cdot (1 - R(\varphi_x - \varphi_y))\sigma_y .$$

Since $\delta_\varphi H$ is a local function, for every infinite volume limit we obtain that

$$\langle F(R(\varphi)\sigma)\rangle = \langle F(\sigma)\, e^{-\beta\delta_\varphi H(\sigma)}\rangle$$

Goal: Choose φ s.t. $\varphi_x \to \alpha$ for all $x \in \operatorname{supp} F$

$$\delta_\varphi H(\sigma) \longrightarrow 0$$

Claim: $|\delta_\varphi H(\sigma)| \leq C \sum_{xy \subset \overline{\Lambda}} (\varphi_x - \varphi_y)^2$.

Proof of claim: WLOG $n=2$. Then can view $S^1 \subset \mathbb{C}$:

$$\sigma_x = (\operatorname{Re} e^{i\theta_x}, \operatorname{Im} e^{i\theta_x}), \quad \theta_x \in [0, 2\pi)$$

$$\sigma_x \cdot \sigma_y = \cos(\theta_x - \theta_y) = \operatorname{Re}\left(e^{i\theta_x}\, \overline{e^{i\theta_y}}\right)$$

$$R(\varphi)\sigma_x = \left(\operatorname{Re}\left(e^{i(\theta+\varphi)}\right), \operatorname{Im}\left(e^{i(\theta+\varphi)}\right)\right)$$

$$\Rightarrow \Big|\sum_{xy \subset \Lambda'} \sigma_x \cdot (1 - R(\varphi_x - \varphi_y))\sigma_y\Big|$$

$$= \Big|\sum_{xy \subset \Lambda'} \operatorname{Re}\Big(e^{i\theta_x}\, \overline{\underbrace{\left(1 - e^{i(\varphi_x - \varphi_y)}\right)}_{i(\varphi_x - \varphi_y) + O(\varphi_x - \varphi_y)^2} e^{i\theta_y}}\Big)\Big|$$

not correct!

$$\leq C \sum_{xy \subset \Lambda'} (\varphi_x - \varphi_y)^2 \text{ provided } |\varphi_x - \varphi_y| \leq 2 \text{ (say)}$$

Fix of the problem:

Define $\delta_\varphi H = \frac{1}{2}\,\delta_\varphi^+ H(\sigma) + \frac{1}{2}\,\delta_\varphi^- H(\sigma)$

with $\delta_\varphi^\pm H(\sigma) = H_{\Lambda'}(R(\mp\varphi)\sigma) - H_{\Lambda'}(\sigma)$

Then the claim is correct, but the final part of the proof must be modified: As before,

$$\left|\langle F(R(\varphi)\sigma)\rangle - \langle F(\sigma)\rangle\right| = \left|\left\langle F\left(e^{-\beta\delta_\varphi^+ H} - 1\right)\right\rangle\right|$$

This can be bounded by

$$\|F\|_\infty \underbrace{\left\langle \left|e^{-\beta\delta_\varphi^+ H} - 1\right| \right\rangle}$$

$$\overset{(+)}{\leq} \sqrt{2\langle \beta\,\delta_\varphi^+ H\rangle} \quad \text{(see Friedlin-Velenik (9.11))}$$

$$\overset{(-)}{\leq} \sqrt{2\langle \beta\,\delta_\varphi^+ H\rangle + 2\langle \beta\,\delta_\varphi^- H\rangle}$$

$$= 2\sqrt{\beta\langle \delta_\varphi H\rangle}$$

and the RHS goes to 0 as before.

(+) is Pinsker's ineq. which is not trivial
→ Friedlin - Velenik (9.11)

(−) is $e^{-\beta\langle\delta_\varphi^\pm H\rangle} \overset{\text{Jensen}}{\leq} \langle e^{-\beta\delta_\varphi^\pm H}\rangle = 1 \Rightarrow \langle \delta_\varphi^\pm H\rangle \geq 0$

<u>End of fix.</u> [The fix is not examinable.]

Now choose $\varphi_x = \alpha \dfrac{C^{\Lambda',0}(0,x)}{C^{\Lambda',0}(0,0)}$ for some $\alpha \in \mathbb{R}$

where $C^{\Lambda',0}(0,x) = (-\Delta^{\Lambda',0})^{-1}(0,x)$.

Then:

$(\Delta^{\Lambda',0}\varphi)_x = 0$ for $x \in \Lambda' \setminus \{0\}$

$\varphi_x = 0$ for $x \notin \Lambda'$

$\varphi_0 = \alpha$

$\forall x \in \mathbb{Z}^2$: $\varphi_x \to \alpha$ as $\Lambda' \uparrow \mathbb{Z}^2$

Therefore

$$|\delta_\varphi H| \leq C \sum_{xy} (\varphi_x - \varphi_y)^2$$

$$= C \sum_x \varphi_x \underbrace{(\Delta^{\Lambda',0}\varphi)_x}_{=0 \text{ unless } x=0},$$

$$= C \varphi_0 \sum_{y \sim 0} (\varphi_y - \varphi_0)$$

$\leftarrow$ finitely many terms!

As $\Lambda' \uparrow \mathbb{Z}^2$, $\varphi_y \longrightarrow \alpha = \varphi_0$. Therefore $\delta_\varphi H \to 0$ uniformly in σ.

In conclusion, since F is continuous and local,

$$\langle F(R(\alpha)\sigma)\rangle \overset{\varphi_x \to \alpha\ \forall x}{=} \lim_{\Lambda'} \langle F(R(\varphi)\sigma)\rangle$$

$$= \lim_{\Lambda'} \langle F(\sigma)\, e^{-\beta\delta_\varphi H}\rangle$$

$$\overset{\delta_\varphi H \to 0}{=} \langle F(\sigma)\rangle .$$

This is superceded by the fix above

correction: order of quantifiers

Thm (McBryan–Spencer). Let $d=2$ and $n\geq 2$. For any $\varepsilon>0$ and $\beta\geq\beta_0(\varepsilon)$, there are $C_{\beta,\varepsilon}$ s.t.

$$\limsup_{\Lambda\uparrow\mathbb{Z}^2} \langle \sigma_0\cdot\sigma_x\rangle^{\Lambda,e}_{\beta,0} \leq C_{\beta,\varepsilon}(1+|x|)^{-(1-\varepsilon)/2\pi\beta}$$

Proof. WLOG $n=2$. Write $\sigma_x=(\cos\theta_x,\sin\theta_x)$ so that

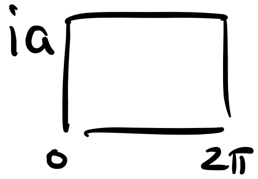
$$\langle\sigma_0\cdot\sigma_x\rangle^{\Lambda,e} = \langle e^{i(\theta_0-\theta_x)}\rangle^{\Lambda,e} \quad (\text{because } \langle\sin(\theta_0-\theta_x)\rangle=0)$$

$$\propto \int \exp\Big(\beta\sum_{uv}\cos(\theta_u-\theta_v)+i(\theta_0-\theta_x)\Big)\, d\theta$$

Now do the following complex translation:

$$\theta_u \mapsto \theta_u + i a_u \quad \forall u\in\Lambda, \quad a_u=0 \quad \forall u\notin\Lambda$$

Note that the integrand is holomorphic and the vertical pieces of the contour vanish by periodicity.



$$\langle \sigma_0 \cdot \sigma_x \rangle = \frac{1}{Z} e^{-(a_0-a_x)} \int \exp\Big(\beta \sum_{uv} \cos(\theta_u-\theta_v)\cosh(a_u-a_v)\Big) \times e^{i(\theta_0-\theta_x)}\, d\theta$$

$$\leq e^{-(a_0-a_x)} \Big\langle \exp\Big(\beta \sum_{uv} \cos(\theta_u-\theta_v) \underbrace{\big(\cosh(\theta_u-\theta_v)-1\big)}_{=O(a_u-a_v)^2}\Big)\Big\rangle$$

Assume $\max_{u,v} |a_u-a_v| < \delta$ is small enough that

$$\cosh(a_u-a_v)-1 \leq \tfrac{1}{2}(1+\varepsilon)(a_u-a_v)^2$$

$$\Rightarrow \langle \sigma_0 \cdot \sigma_x \rangle^{\Lambda} \leq e^{-(a_0-a_x) + \frac{\beta}{2}(1+\varepsilon) \underbrace{\sum (a_u-a_v)^2}_{(a,-\Delta^{\Lambda,0} a)}}$$

Choose $a = \frac{1}{\beta}\underbrace{(-\Delta^{\Lambda,0})^{-1}(\delta_0-\delta_x)}_{C_0^{\Lambda,0}(0,\cdot)-C_0^{\Lambda,0}(x,\cdot)}$. Then $|a_u-a_v| \leq \frac{4}{\beta} \leq \delta$ for $\beta \geq \beta_0(\varepsilon)$

and $\frac{\beta}{2}(a,-\Delta^{\Lambda,0} a) = \frac{1}{2}(a_0-a_x)$

$$\Rightarrow \langle \sigma_0 \cdot \sigma_x \rangle^{\Lambda,0} \leq e^{-\frac{1}{2}(1-\varepsilon)(a_0-a_x)}$$

$$= e^{-(1-\varepsilon)\frac{1}{\beta}\frac{1}{2}\underbrace{\left[C_0^{\Lambda,0}(0,0)-C_0^{\Lambda,0}(x,0)-C_0^{\Lambda,0}(0,x)+C_0^{\Lambda,0}(x,x)\right]}_{\sim \frac{1}{2\pi}\log|x|}}$$

3.5. Infrared bound

Let $\Lambda = \Lambda_L$ be a discrete torus of side length L even.

Let μ be a measure on $\mathbb{R}^n$ (e.g., the uniform measure on S^{n-1}) and consider the spin model

$$\langle F \rangle \propto \int_{(\mathbb{R}^n)^\Lambda} \prod_{x\in\Lambda} \mu(d\varphi_x) \underbrace{e^{-\frac{1}{2}\sum_{xy\in E} |\varphi_x - \varphi_y|^2}}_{e^{-\frac{1}{2}(\varphi, -\Delta\varphi)}}$$

$(\Delta\varphi)^a_x = (\Delta\varphi^a)_x$, $x \in \Lambda$, $a \in [n]$

$(f,\varphi) = \sum_{x\in\Lambda} f_x \cdot \varphi_x$

Always assume $\int \mu(d\varphi)\, e^{t|\varphi|^2} < \infty$ for all $t \in \mathbb{R}$.
(technical assumption)

Rk. Inverse temperature β and external field h can be added by suitable choice of μ. For the $O(n)$ model, take μ to be the uniform measure on $\sqrt{\beta}\, S^{n-1}$ and set $\sigma = \varphi/\sqrt{\beta}$. To add $h \neq 0$ then similarly set

$$\mu_h(d\varphi) = \mu(d\varphi)\, e^{+h\sigma\cdot e} = \mu(d\varphi)\, e^{(h/\sqrt{\beta})\sigma\cdot e}.$$

Thm. (Fröhlich-Simon-Spencer). For any $f: \Lambda \to \mathbb{R}^n$ with $\sum_{x\in\Lambda} f_x = 0$,

$$\langle e^{(f,\varphi)} \rangle \leq e^{+\frac{1}{2}(f,(-\Delta)^{-1}f)}.$$ Gaussian domination

(Proof next time)

Here note that $\ker(-\Delta) = \{\text{constant functions on } \Lambda\}$.
Thus $-\Delta$ can be inverted when restricted to $\{\text{constant}\}^{\perp}$.

The condition $\sum_{x\in\Lambda} f_x = 0$ is exactly $f \in \{\text{constants}\}^{\perp}$:

$$(f, 1^a) = \sum_x f_x \underbrace{1^a_x}_{\text{constant function } (1^a_x)^b_y = \delta_{ab} \ (x\in\Lambda,\, a,b\in[n])} = \sum f^a_x = 0.$$

Note also that $\langle (f,\varphi)\rangle = \sum_x f_x \langle \varphi_x\rangle = \langle \varphi_0\rangle \cdot \sum_{x\in\Lambda} f_x = 0.$

<u>Cor.</u> $\underbrace{\langle (f,\varphi)^2\rangle - \langle (f,\varphi)\rangle^2}_{\text{variance of } (f,\varphi)} = \langle (f,\varphi)^2\rangle \leq (f, (-\Delta)^{-1} f)$.

independent of μ!

<u>Proof.</u> By the theorem with f replaced by tf,

$$\underbrace{\langle e^{t(f,\varphi)}\rangle}_{1 + t(f,\varphi) + \frac{t^2}{2}(f,\varphi)^2 + O_f(t^3 e^{C\|\varphi\|_2^2})} \leq \underbrace{e^{\frac{1}{2}t^2 (f,(-\Delta)^{-1} f)}}_{1 + \frac{t^2}{2}(f,(-\Delta)^{-1}f) + O_f(t^3)}$$

Since $\langle (f,\varphi)\rangle = 0$ and $\langle e^{C\|\varphi\|_2^2}\rangle < \infty$, thus

$$t^2 \langle (f,\varphi)^2\rangle \leq t^2 (f,(-\Delta)^{-1} f) + O(t^3)$$

$$\Rightarrow \langle (f,\varphi)^2\rangle \leq (f,(-\Delta)^{-1} f).$$

<u>Aside</u>: Fourier analysis on Λ_L. Let

$$\Lambda_L^* = \{ \tfrac{2\pi}{L} n : n \in \Lambda_L \} \subset [-\pi, \pi)^d$$

$$e_p(x) = e^{ip\cdot x} \text{ for } p \in \Lambda_L^*.$$

Then e_p, $p \in \Lambda_L^*$ is an orthogonal basis for $\mathbb{C}^{\Lambda_L}$ with inner product

$$(f,g) = \sum_{x \in \Lambda_L} \bar{f}_x g_x .$$

Normalisation: $(e_p, e_p) = |\Lambda_L|$ (e_p is not normalised!)

Fourier transform: $\hat{f}(p) = (e_p, f) = \sum_x f_x e^{-ip\cdot x}$.

$$\Rightarrow f_x = \frac{1}{|\Lambda_L|} \sum_{p \in \Lambda_L^*} \hat{f}(p) e^{ipx}$$

$$(\Delta e_p)_x = \sum_{y \sim x} (e^{ip\cdot y} - e^{ip\cdot x})$$

$$= \underbrace{\sum_{e \sim 0} (e^{ip\cdot e} - 1)}\, e^{ip\cdot x} = \hat{\Delta}(p)\, e_p(x)$$

$$\hat{\Delta}(p) = \sum_{i=1}^{d} (2\cos(p_i) - 2)$$

$$\Rightarrow \widehat{\Delta f}(p) = \hat{\Delta}(p) \hat{f}(p)$$

$\sum f_x = 0 \Leftrightarrow \hat{f}(0) = 0$

$\Rightarrow \widehat{(-\Delta)^{-1} f}(p) = -\hat{\Delta}(p)^{-1} \hat{f}(p)$

Parseval identity: $\sum_{x \in \Lambda_L} \bar{f}_x g_x = \frac{1}{|\Lambda_L|} \sum_{p \in \Lambda_L^*} \bar{\hat{f}}(p) \hat{g}(p)$

(both $(\delta_x)_{x \in \Lambda}$ and $\left(\frac{1}{|\Lambda_L|^{1/2}} e_p\right)_{p \in \Lambda_L^*}$ are ONBs)

Infinite volume: $\Lambda_L \rightsquigarrow \mathbb{Z}^d$

$$\sum_{x \in \Lambda_L} \rightsquigarrow \sum_{x \in \mathbb{Z}^d}$$

$$\frac{1}{|\Lambda_L^*|} \sum_{p \in \Lambda_L^*} \rightsquigarrow \frac{1}{(2\pi)^d} \int_{[0,2\pi)^d} dp$$

<u>Cor.</u> For $p \neq 0$, $\langle |\hat{\varphi}(p)|^2 \rangle \leq \langle (e_p, (-\Delta)^{-1} e_p) \rangle$

$$\Rightarrow \frac{1}{|\Lambda_L|} \langle |\hat{\varphi}(p)|^2 \rangle \leq \frac{1}{-\hat{\Delta}(p)} \overset{p \to 0}{\sim} \frac{1}{|p|^2}$$

What about $p = 0$?

$$\frac{1}{|\Lambda_L|} \langle |\hat{\varphi}(0)|^2 \rangle = \sum_{x \in \Lambda} \langle \varphi_0 \varphi_x \rangle.$$

High temperature: $\sum_{x\in\Lambda} \langle \varphi_0 \varphi_x \rangle \longrightarrow 0$

Long-range order: $\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda} \langle \varphi_0 \varphi_x \rangle \geq c$

<u>Cor.</u> For the $O(n)$ model,

$$\frac{1}{|\Lambda_L|}\sum_x \langle \sigma_0 \sigma_x \rangle^{\Lambda_L}_{\beta,0} \geq 1 - \frac{1}{\beta}\frac{1}{|\Lambda_L|}\sum_{p\neq 0} (\widehat{-\Delta})(p)^{-1}$$

In particular, in $d\geq 3$, there is long-range order for $\beta > \beta_0 = \beta_0(d) > 0$:

$$\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} \langle \sigma_0 \cdot \sigma_x \rangle^{\Lambda_L}_{\beta,0} \geq 1 - \frac{\beta_0}{\beta} > 0 \quad \text{(uniformly in } L\text{)}$$

<u>Proof.</u> Since $\sigma_x \cdot \sigma_x = 1 \ \forall x \in \Lambda_L$, the Parseval identity implies

$$1 = \frac{1}{|\Lambda_L|}\sum_x |\sigma_x|^2 = \frac{1}{|\Lambda_L|^2}\sum_{p\in\Lambda_L^*} |\hat{\sigma}(p)|^2$$

$$\Rightarrow \frac{1}{|\Lambda_L|^2}\langle |\hat{\sigma}(0)|^2 \rangle = 1 - \frac{1}{|\Lambda_L|^2}\sum_{p\neq 0} \langle |\hat{\sigma}(p)|^2 \rangle$$

$$\geq 1 - \frac{1}{\beta}\frac{1}{|\Lambda_L|}\sum_{p\neq 0} (\widehat{-\Delta})(p)^{-1}$$

infrared bound

In particular, in $d \geq 3$, $\frac{1}{|\Lambda_L|}\sum_{p\neq 0} \widehat{(-\Delta)}(p)^{-1}$

$$= \int_{[0,1]^d} \frac{1}{|p|^2}\, dp + O(1) \leq \beta_0 < \infty.$$

<u>Cor.</u> In $d \geq 3$, for $\beta > \beta_0$, there is also spontanaous magnetisation:

$$\lim_{h\downarrow 0} \lim_{L\to\infty} \langle \sigma_0 \cdot e \rangle^{\Lambda_L}_{\beta,h} = 1 - O\left(\tfrac{1}{\beta}\right)$$

<u>Proof</u>. First consider $h=0$. Then the distribution of

$$M = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sigma_x$$

is rotationally invariant, and thus $M/|M|$ is uniform on S^{n-1} and independent of $|M|$ (check!). Define

$$p = \mathbb{P}(|M| > 1-\delta), \quad q = \mathbb{P}\left(\frac{M}{|M|}\cdot e \geq 1-\varepsilon\right).$$

Since $M/|M|$ is uniform, $q > 0$ for any $\varepsilon > 0$.

Since $|M| \leq 1$, $\mathbb{1}_{|M|>1-\delta} \geq |M|\,\mathbb{1}_{|M|>1-\delta}$

$$= |M|\left(1 - \mathbb{1}_{|M|\leq 1-\delta}\right)$$

$$\geq |M|^2 - (1-\delta).$$

$$\Rightarrow p \geq \langle |M|^2 \rangle - (1-\delta) \geq 1 - \frac{c}{\beta} - (1-\delta) \geq \delta - \frac{c}{\beta}$$

Together,

$$\Phi(h) = \frac{1}{|\Lambda|} \log \langle e^{\overbrace{|\Lambda| h M\cdot e}^{h \sum_x \sigma_x \cdot e}} \rangle \geq \frac{1}{|\Lambda|} \log\left(pq\, e^{(1-\delta)(1-\varepsilon) h |\Lambda|}\right)$$

$$= (1-\delta)(1-\varepsilon)h + \frac{1}{|\Lambda|}\log(pq) \geq (1-2\delta)h - O_\delta\left(\frac{1}{|\Lambda|}\right).$$

Finally, $\Phi(h)$ is convex, so

$$\langle \sigma_0 \cdot e \rangle^\Lambda_{\beta,h} = \langle M \cdot e \rangle^\Lambda_{\beta,h} = \Phi'(h) \geq \frac{\Phi(h) - \Phi(0)}{h}$$

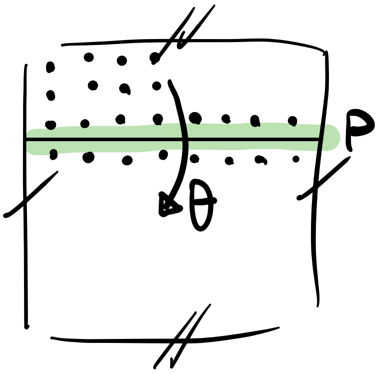
$$\geq 1 - 2\delta - O_\delta\left(\frac{1}{h|\Lambda|}\right).$$

Taking first $L \to \infty$ then $h \downarrow 0$ finishes the proof.

Exercise. If $e' \perp e$ then $\langle \sigma_0 \cdot e' \rangle^\Lambda_{\beta,h} = 0$.

3.6. Reflection positivity and proof of IRB

Consider a plane P going through the midpoints of edges (an edge plane) splitting the torus (of even side length) into two halves:



$\Lambda = \Lambda_+ \cup \Lambda_-$

Let $\theta = \theta_P : \Lambda_\pm \longrightarrow \Lambda_\mp$ be the reflection about P.

For $\varphi : \Lambda \to \mathbb{R}^n$, let

$$(\theta\varphi)_x = \varphi_{\theta x}$$

For $F : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$, let

$$(\theta F)(\varphi) = F(\theta\varphi).$$

<u>Defn</u>. A measure $\langle\cdot\rangle$ on $(\mathbb{R}^n)^\Lambda$ is reflection positive for the reflection θ if

$$\langle F(\theta G)\rangle = \langle(\theta F)G\rangle$$
$$\langle F(\theta F)\rangle \geq 0$$

for all $F, G : (\mathbb{R}^n)^\Lambda \to \mathbb{R}$ (for which this is defined) and which only depend on $\varphi|_{\Lambda_+}$.

Moreover, $\langle\cdot\rangle$ is reflection positive (RP) (for edge planes) if it is RP for θ_P for any edge plane P.

<u>Lemma</u>. Any product measure $\mu^{\otimes\Lambda}$ is RP.

<u>Proof</u>. Since $\varphi|_{\Lambda_+}$ and $\varphi|_{\Lambda_-}$ are independent,

$$\langle F\,\theta G\rangle = \langle F\rangle\langle\theta G\rangle = \langle F\rangle\langle G\rangle = \langle(\theta F)G\rangle$$

$$\langle F(\theta F)\rangle = \langle F\rangle\langle F\rangle \geq 0.$$

By definition, $(F,G)\mapsto\langle F(\theta G)\rangle$ defines a positive semi-definite quadratic form on the algebra $\mathcal{A}_+$ of bounded $F:\underbrace{(\operatorname{supp}\mu)}_{\subseteq\mathbb{R}^n}{}^{\Lambda}\to\mathbb{R}$ only depending on Λ_+.

<u>Cor</u>. (Cauchy-Schwarz inequality). For $F,G\in\mathcal{A}_+$

$$\langle F(\theta G)\rangle^2 \leq \langle F\theta F\rangle\langle G\,\theta G\rangle.$$

<u>Lemma</u>. Let θ be a reflection and $\langle\cdot\rangle$ reflection positive with respect to θ. Then for all $A,B,C,D\in\mathcal{A}_+$,

$$\langle e^{A+\theta B+C\theta D}\rangle^2 \leq \langle e^{A+\theta A+C\theta C}\rangle\langle e^{B+\theta B+D\theta D}\rangle$$

and the measure $\langle(\cdot)\,e^{A+\theta A+C\theta C}\rangle$ is also RP.

The same holds with $C\Theta D$ replaced by $\sum_\alpha C_\alpha \Theta D_\alpha$ and likewise on the RHS.

<u>Proof</u>. Expand the exponential as

$$e^{A+\Theta B+C\Theta D} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{(e^A C^k)}_{X_k} \Theta \underbrace{(e^B D^k)}_{Y_k}.$$

By Cauchy-Schwarz twice,

$$\langle e^{A+\Theta B+C\Theta D}\rangle^2 = \Big[\sum_k \frac{1}{k!} \underbrace{\langle X_k \Theta Y_k\rangle}_{\leq \langle X_k\Theta X_k\rangle^{1/2}\langle Y_k\Theta Y_k\rangle^{1/2}}\Big]^2$$

$$\leq \Big[\sum_k \sqrt{\tfrac{1}{k!}\langle X_k\Theta X_k\rangle}\sqrt{\tfrac{1}{k!}\langle Y_k\Theta Y_k\rangle}\Big]^2$$

$$\leq \Big(\underbrace{\sum_k \frac{1}{k!}\langle X_k\Theta X_k\rangle}\Big)\Big(\sum_k \frac{1}{k!}\langle Y_k\Theta Y_k\rangle\Big)$$

$$= \text{RHS}.$$

To see that $\langle(\cdot)e^{A+\Theta A+C\Theta C}\rangle$ is RP note

$$\langle F(\Theta F)\, e^{A+\Theta A+C\Theta C}\rangle = \sum_k \frac{1}{k!}\langle FX_k\, \Theta(FX_k)\rangle \geq 0.$$

We will now prove the infrared bound. Let

$$H(\varphi) = \frac{1}{2}\sum_{xy} |\varphi_x - \varphi_y|^2 = \frac{1}{2}(\varphi, -\Delta\varphi)$$

$$\Rightarrow H(\varphi) - (f,\varphi) = H(\varphi - (-\Delta)^{-1}f) + \tfrac{1}{2}(f,(-\Delta)^{-1}f)$$

Thus to show that

$$\langle e^{(f,\varphi)} \rangle = \frac{\int e^{-H(\varphi)+(f,\varphi)} \prod_x \mu(d\varphi_x)}{\int e^{-H(\varphi)} \prod_x \mu(d\varphi_x)} \leq e^{+\frac{1}{2}(f,(-\Delta)^{-1}f)}$$

it suffices to show $Z(g) \leq Z(0)$ for all $g \in (\mathbb{R}^n)^\Lambda$, where

$$Z(g) = \int \prod_{x\in\Lambda} \mu(d\varphi_x)\, e^{-H(\varphi+g)}$$

<u>Lemma</u>. Let θ be a reflection (about an edge plane). For $g: \Lambda \to \mathbb{R}^n$ define g_+ by

$$g_+|_{\Lambda_+} = g|_{\Lambda_+}, \quad g_+|_{\Lambda_-} = \theta(g|_{\Lambda_+}).$$

and analogously for g_-. Then:

$$Z(g)^2 \leq Z(g_+)Z(g_-).$$

For simplicity, assume $\operatorname{supp}\mu$ is bounded (e.g. $O(n)$).

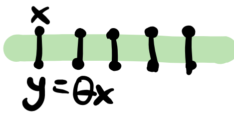
Proof. Denote by $E_\pm$ the edges with both endpoints in $\Lambda_\pm$ and by E_0 the remaining ones.

$$\Rightarrow H(\varphi) = \underbrace{\sum_{e\in E_+} \tfrac12|\varphi_x-\varphi_y|^2}_{H_+(\varphi)} + \underbrace{\sum_{e\in E_-} \tfrac12|\varphi_x-\varphi_y|^2}_{H_-(\varphi)} + \underbrace{\sum_{e\in E_0} \tfrac12|\varphi_x-\varphi_y|^2}_{H_0(\varphi)}$$

Note that $\theta H_+ = H_-$ and that

$$H_0(\varphi) = \sum_{\substack{x\in\Lambda_+:\\ \exists xy\in E_0}} \left(\tfrac12|\varphi_x|^2 + \tfrac12\theta|\varphi_x|^2 + \varphi_x\cdot\theta\varphi_x\right)$$

then $y=\theta x$



Thus H is of the form

$$H = A + \theta A + \sum_\alpha C_\alpha(\theta C_\alpha)$$

and $H(\cdot+g)$ is of the form

$$H(\cdot+g) = A_{g_+} + \theta A_{g_-} + \sum_\alpha C_{\alpha,g_+}\,\theta C_{\alpha,g_-}.$$

The lemma gives

$$\left(\int e^{-H(\varphi+g)}\prod_x \mu(d\varphi_x)\right)^2 \le \left(\int e^{-H(\varphi+g_+)}\prod_x \mu(d\varphi_x)\right) \times \left(\int e^{-H(\varphi+g_-)}\prod_x \mu(d\varphi_x)\right)$$

i.e., $Z(g)^2 \le Z(g_+)\,Z(g_-)$.

Cor. For any g with $\sum g = 0$, $Z(g) \leq Z(0)$

Proof. For $e = xy$, write $|\nabla_e f| = |f_x - f_y|$. Note that
$$|\nabla_e f_\pm| = 0 \quad \text{for} \quad e \in E_0$$
and that $|\nabla_e f| = 0$ implies $|\nabla_e f_\pm| = 0$.

Therefore iterating the previous lemma over all planes (every edge is contained in some plane):
$$Z(g) \leq \sup_{f : |\nabla f| = 0} Z(f).$$

Since $|\nabla f| = 0$ implies that f is constant, we have $Z(f) = Z(0)$ as claimed.

Cor. (Infinite volume IRB). Let $d \geq 3$. Let $\langle \cdot \rangle$ be any infinite volume limit of torus measures $\langle \cdot \rangle^{\Lambda_L}$. Assume also ergodicity: there is m^* s.t.
$$\left\langle \left| \frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} (\sigma_x - m^*) \right|^2 \right\rangle \longrightarrow 0$$

not assuming $\sum f = 0$

Then for all $f : \mathbb{Z}^d \to \mathbb{R}$ with finite support:
$$\sum_{x,y \in \mathbb{Z}^d} f_x f_y \langle (\sigma_x - m^*) \cdot (\sigma_y - m^*) \rangle \leq \frac{n}{2\beta} (f, (-\Delta)^{-1} f).$$

Rk. $(f,(-\Delta)^{-1}f) = \sum_{x,y} (-\Delta)^{-1}(x,y) f_x f_y$ for f with finite supp.
if we define
$$(-\Delta)^{-1}(x,y) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} dp \, (\hat{-\Delta})^{-1}(p) e^{ip\cdot(x-y)}$$

__Proof.__ Suppose $g: \mathbb{Z}^d \to \mathbb{R}$ has finite support and satisfies $\sum_x g_x = 0$. Then the IRB on the torus gives
$$\sum_{x,y} g_x g_y \langle \sigma_x \cdot \sigma_y \rangle^\Lambda \leq \frac{n}{\beta} (g,(-\Delta)^{-1}g)$$
Taking $\Lambda \uparrow \mathbb{Z}^d$,
$$\sum_{x,y} g_x g_y \langle \sigma_x \cdot \sigma_y \rangle^\Lambda \leq \frac{n}{\beta} \underbrace{(g,(-\Delta)^{-1}g)}_{\sum_{x,y} g_x g_y (-\Delta)^{-1}(x,y)}$$
Since $\sum_x g_x = 0$, the LHS equals
$$\sum_{x,y} g_x g_y \langle (\sigma_x - m^*) \cdot (\sigma_y - m^*) \rangle$$
Now given f with (possibly) $\sum f \neq 0$ set
$$g_x = f_x - \underbrace{\frac{1}{|\Lambda_{L'}|} \sum_{x \in \Lambda_{L'}} f_x}_{\bar{f}_{L'} = O_f\left(\frac{1}{|\Lambda_{L'}|}\right)} 1_{\Lambda_{L'}} \quad \text{where } \Lambda_{L'} \supset \operatorname{supp} f.$$
$\Rightarrow \sum g_x = 0$

$$\Rightarrow \sum g_x g_y \langle (\sigma_x - m^*)\cdot(\sigma_y - m^*)\rangle$$
$$= \sum f_x f_y \langle (\sigma_x - m^*)\cdot(\sigma_y - m^*)\rangle + \langle |\bar{f}_{L'} \sum_{x\in\Lambda_{L'}} (\sigma_x - m^*)|^2\rangle$$
$$- 2\langle \bar{f}_{L'} \sum_{x\in\Lambda_{L'}} (\sigma_x - m^*) \cdot \sum_y f_y(\sigma_y - m^*)\rangle$$

By Cauchy-Schwarz and the assumption, the last two terms tend to 0 as $L' \to \infty$.

Similarly, the terms involving $\bar{f}_{L'}$ in

$$(g,(-\Delta)^{-1}g) = (f,(-\Delta)^{-1}f) + \sum_{x,y\in\Lambda_{L'}} \bar{f}_{L'}^2 (-\Delta)^{-1}(x,y) + \cdots$$

tend to 0.

not requiring $\sum f = 0$

<u>Cor.</u> If $\langle \sigma_0 \cdot \sigma_x\rangle \xrightarrow{|x|\to\infty} 0$ then for all $f: \mathbb{Z}^d \to \mathbb{R}$, $d\geq 3$,

$$\sum_{x,y} f_x f_y \langle \sigma_x \cdot \sigma_y\rangle \leq \frac{n}{\beta}(f,(-\Delta)^{-1}f)$$

provided f has sufficient decay (e.g. finite support – but this can of course be relaxed now).

Lemma (Schrader – Messager – Miracle-Solé). Let $n=1,2$. For the $O(n)$ model with homogeneous n.n. coupling on $\mathbb{Z}^d$:

$$\langle \sigma_0 \cdot \sigma_x \rangle^{\mathbb{Z}^d}_{\beta,0} \leq \langle \sigma_0 \cdot \sigma_y \rangle^{\mathbb{Z}^d}_{\beta,0} \quad \text{whenever } |y|_1 \leq |x|_\infty .$$

Proof perhaps later. Assuming this SMMS inequality, we obtain the real space IRB.

Prop. Let $n=1,2$ and $d \geq 3$. Let $\langle \cdot \rangle^{\mathbb{Z}^d}$ be an infinite volume torus limit with $\langle \sigma_0 \sigma_x \rangle^{\mathbb{Z}^d} \to 0$ as $|x| \to \infty$. Then there is $C>0$ s.t.

$$\langle \sigma_0 \cdot \sigma_x \rangle^{\mathbb{Z}^d} \leq \frac{C}{\beta |x|^{d-2}} . \qquad \text{(real space IRB)}$$

Proof. Let $\chi(x) = 1_{|x| \leq L}$ and set $S(x) = \langle \sigma_0 \cdot \sigma_x \rangle$. By the usual infrared bound,

$$\underbrace{(\chi, S\chi)}_{\sum_{x,y} \chi(x)\chi(y) S(x-y)} \leq \frac{n}{\beta} (\chi, (-\Delta)^{-1} \chi) \leq \frac{C}{\beta} L^{d+2}$$

where we used that

$$\sum_{|x|\le L}\sum_{|y|\le L} \underbrace{(-\Delta)^{-1}(x,y)}_{\le C(1+|x-y|)^{-(d-2)}} \le \sum_{j=0}^{1+\log_2 L} O(2^{-(d-2)j}) \underbrace{\sum_{|x|\le L}}_{L^d} \underbrace{\sum_{y} 1_{2^j \le |x-y| \le 2^{j+1}}}_{O(2^{dj})}$$

$$\le \sum_{j=0}^{1+\log_2 L} O(2^{2j})\, L^d \le O(L^{2+d}).$$

Thus

$$\min_{|x|\le 2L} S(x) \le (2L+1)^{-2d} \underbrace{\sum_{|x|\le L}\sum_{|y|\le L} S(x-y)}_{(\mathcal{X}, S\mathcal{X})}$$

$$\le \frac{C}{\beta} L^{-2d} L^{2+d} = \frac{C}{\beta} L^{2-d}$$

By the SMMS inequality, for $|x|_\infty \le 2L$,

$$S(x) \le \min_{|y|_1 \le |x|_\infty} S(y) \le \min_{|y|_1 \le 2L} S(y) \le \frac{C}{\beta} L^{2-d}$$

For any $x \in \mathbb{Z}^d$, choose L s.t. $L \le |x|_\infty \le 2L$. Then

$$S(x) \le \frac{C'}{\beta} |x|^{2-d}.$$

4. Random walk and current representations

4.1. Random walk representation

For the random walk representation, we consider the continuous spin model:

$$\langle \cdot \rangle^{\Lambda}_{g,\nu} = \int_{(\mathbb{R}^n)^\Lambda} (\cdot) \underbrace{e^{-\frac{1}{2}\sum_{xy\in E}|\varphi_x-\varphi_y|^2}}_{e^{\frac{1}{2}(\varphi,\Delta\varphi)}} e^{-\sum_{x\in\Lambda}\left(\frac{1}{4}g|\varphi_x|^4+\frac{1}{2}\nu|\varphi_x|^2\right)} d\varphi$$

(Example Sheet 2: For $n=1$, the Ising model can be obtained as a limit. This is analogous for $n\geq 2$.)

<u>Prop.</u> For any sufficiently nice $f: \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\int_{(\mathbb{R}^n)^\Lambda} e^{\frac{1}{2}(\varphi,\Delta\varphi)} f(\tfrac{1}{2}|\varphi|^2)\, \varphi_x^1 \varphi_y^1$$

$$= \int_{(\mathbb{R}^n)^\Lambda} e^{\frac{1}{2}(\varphi,\Delta\varphi)} \int_0^\infty \mathbb{E}_x\Big(f\big(\underbrace{\tfrac{1}{2}|\varphi|^2+L(t)}_{(\frac{1}{2}|\varphi_x|^2+L_x(t))_{x\in\Lambda}}\big)\mathbb{1}_{X_t=y}\Big)\,dt$$

where (X_t) is a SRW and (L_t) its local-time:

$$L(t)_x = \int_0^t \mathbb{1}_{X_s=x}\, ds.$$

Cor.

$$\langle \varphi_x^1 \varphi_y^1 \rangle_{g,\nu}^{\Lambda} = \int_0^\infty \mathbb{E}_x\Big(\underbrace{\frac{Z(L_t)}{Z(0)}}_{\mathcal{Z}(L_t)} 1_{X_t = y} \Big)\, dt$$

where for $\ell \in \mathbb{R}^\Lambda$,

$$Z(\ell) = \int_{\mathbb{R}^\Lambda} e^{+\frac{1}{2}(\varphi,\Delta\varphi)} f\big(\tfrac{1}{2}|\varphi|^2 + \ell\big)\, d\varphi,$$

$$f(\ell) = \exp\Big(-\sum_{x\in\Lambda}\big(g\,\ell_x^2 + \nu \ell_x\big)\Big)$$

$$\mathcal{Z}(\ell) = \frac{Z(\ell)}{Z(0)}$$

Example Sheet 3: For any nice $F : \Lambda \times \mathbb{R}^\Lambda \to \mathbb{R}$,

$$\frac{\partial}{\partial t} \mathbb{E}\big(F(X_t, L_t)\big) = \mathcal{L}\, \mathbb{E}_x\big(F(X_t, L_t + \ell)\big), \quad \mathcal{L} = \Delta_x + \frac{d}{d\ell_x}$$

Thus (X_t, L_t) is a Markov process with generator $\mathcal{L}$. (Kolmogorov backward equation).

Lemma.

$$-\sum_{z\in\Lambda} \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \varphi_x^1 \varphi_z^1\, \mathcal{L} F\big(z, \tfrac{1}{2}|\varphi|^2\big)\, d\varphi$$

$$= \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} F\big(x, \tfrac{1}{2}|\varphi|^2\big)\, d\varphi$$

Proof. Recall $\mathcal{L} = \Delta + \frac{\partial}{\partial \ell_x}$ and notice that

$$\varphi'_z \frac{\partial}{\partial \ell_z} F(z, \underbrace{\tfrac{1}{2}|\varphi|^2}_{\ell}) = \frac{\partial}{\partial \varphi'_z} F(z, \tfrac{1}{2}|\varphi|^2).$$

$$\Rightarrow \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \varphi'_x \sum_z \varphi'_z \mathcal{L} F(z, \tfrac{1}{2}|\varphi|^2) \, d\varphi$$

$$= \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \varphi'_x \sum_z \Big[\underbrace{\varphi'_z \sum_y \Delta_{zy} F(y, \tfrac{1}{2}|\varphi|^2)}_{\sum_y (\Delta\varphi')_y F(y, \frac{1}{2}|\varphi|^2)} + \frac{\partial}{\partial \varphi'_z} F(z, \tfrac{1}{2}|\varphi|^2) \Big]$$

Integrate the second term by parts:

$$= -\int e^{\frac{1}{2}(\varphi,\Delta\varphi)} F(x, \tfrac{1}{2}|\varphi|^2) \, d\varphi$$

Proof of theorem. Fix $y \in \Lambda$ and, for $z \in \Lambda$, $\ell \in \mathbb{R}^\Lambda$, let

$$F(z,\ell) = \mathbb{E}_z(1_{X_t = y} f(L_t + \ell)).$$

$$\rightarrow \mathcal{L} F(z,\ell) = \frac{\partial}{\partial t} \mathbb{E}_z(1_{X_t = y} f(L_t + \ell))$$

by the exercise on Example Sheet 3.

Apply last lemma:

$$\text{LHS} = -\sum_z \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \varphi'_x \varphi_z \frac{\partial}{\partial t} \mathbb{E}_z \left(1_{X_t=y} f(L_t + \tfrac{1}{2}|\varphi|^2)\right) d\varphi$$

$$\text{RHS} = \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \mathbb{E}_x (1_{X_t=y} f(L_t)) \; d\varphi$$

Integrate from $t=0$ to $t\to\infty$. Since $f\to 0$ as $|\ell|\to\infty$, the boundary term at $+\infty$ from the LHS vanishes:

$$\sum_x \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \varphi'_x \varphi'_y f(\tfrac{1}{2}|\varphi|^2) \, d\varphi$$

$$= \int_0^\infty \int e^{\frac{1}{2}(\varphi,\Delta\varphi)} \mathbb{E}_x \left(1_{X_t=y} f(L_t + \tfrac{1}{2}|\varphi|^2)\right) d\varphi \, dt$$

<u>Ex.</u> Generalise the above to the IBP formula:

$$\langle \varphi'_x F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x \left(\mathcal{I}(L_t) \left\langle \frac{\partial F}{\partial \varphi'_y} \right\rangle_{L_t} 1_{X_t=y} \right) dt$$

where

$$\langle \cdot \rangle_\ell = \frac{\int e^{\frac{1}{2}(\varphi,\Delta\varphi)} f(\frac{1}{2}|\varphi|^2 + \ell)\, (\cdot)}{\int e^{\frac{1}{2}(\varphi,\Delta\varphi)} f(\frac{1}{2}|\varphi|^2 + \ell)}$$

<u>Ex.</u> Derive an expression for $\langle \varphi_{x_1} \varphi_{x_2} \varphi_{x_3} \varphi_{x_4} \rangle$ using two random walks.

4.2. Lebowitz inequality

Lemma. Let $n=1$ and $F(\varphi) = \varphi_{x_1} \cdots \varphi_{x_k}$. Then $\langle F \rangle_\ell$ is decreasing in each ℓ_x. (*)

Proof.

$$\frac{\partial}{\partial \ell_x} \langle F \rangle_\ell = - \langle F ; \lambda g (\tfrac{1}{2}|\varphi|^2 + \ell) + \nu \rangle_\ell$$
$$= - \lambda g \langle F ; \tfrac{1}{2}|\varphi|^2 \rangle_\ell \leq 0$$

by the second Griffiths inequality (extended from Ising to φ^4 models — same proof).

Cor. Let F be as in the previous lemma. Then:

$$\langle \varphi_x^1 F(\varphi) \rangle \leq \sum_y \langle \varphi_x^1 \varphi_y^1 \rangle \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle$$

Proof. By the IBP formula,

$$\langle \varphi_x^1 F(\varphi) \rangle = \sum_y \int_0^\infty \mathbb{E}_x \Big(\mathcal{Z}(L_t) \underbrace{\left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_{L_t}}_{\leq \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle_0 \text{ by } (*)} 1_{X_t = y} \Big) \, dt$$

$$\leq \sum_y \left\langle \frac{\partial F}{\partial \varphi_y^1} \right\rangle \underbrace{\int_0^\infty \mathbb{E}_x \left(\mathcal{Z}(L_t) \, 1_{X_t = y} \right) dt}_{\langle \varphi_x^1 \varphi_y^1 \rangle}$$

Rk. Similar statement for $n=2$ follow from the Ginibre inequality.

Cor (Lebowitz inequality). For $n=1$,

$$\langle \varphi_{x_1} \cdots \varphi_{x_4} \rangle^{\Lambda}_{g,\nu} - \sum_{\pi} \langle \varphi_{x_{\pi(1)}} \varphi_{x_{\pi(2)}} \rangle^{\Lambda}_{g,\nu} \langle \varphi_{x_{\pi(3)}} \varphi_{x_{\pi(4)}} \rangle^{\Lambda}_{g,\nu} \leq 0$$

where the sum is over all pairings (matchings) of $\{1,2,3,4\}$.

Cor. Same for Ising model.

Rk. For a Gaussian measure one has equality.

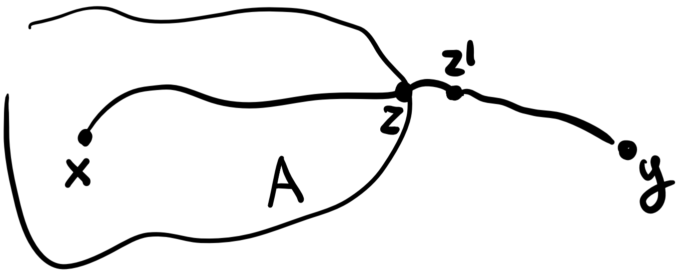
Exercise. Let $\langle \cdot \rangle$ be a Gaussian measure on $\mathbb{R}^{\Lambda}$. Express $\langle \varphi_{x_1} \cdots \varphi_{x_n} \rangle$ in terms of $\langle \varphi_x \varphi_y \rangle$, $x, y \in \Lambda$. [Hint: this is Wick's formula and can be derived from the above IBP inequality which becomes an equality.]

4.3. Simon–Lieb inequality

Let $X^1, \dots, X^p$ be independent simple random walks with local times $L^1, \dots, L^p$ and joint local time $L = L^1 + \dots + L^p$. Denote by $\mathbb{E}_{x_1,\dots,x_p}$ the expectation for these walks with $X_0^i = x_i$.

Lemma. Let $A \subset \Lambda$ and let $T_A = \inf\{t \geq 0 : X_t \notin A\}$. Then for $x \in A$,

$$\int_0^\infty \mathbb{E}_x\big(\mathcal{Z}(L(t))\, 1_{X_t=y}\, 1_{T_A<t}\big)\, dt$$
$$= \sum_{z \in A} \sum_{z' \notin A} J_{zz'} \int_0^\infty \int_0^\infty \mathbb{E}_{x,z'}\big(\mathcal{Z}(L(t))\, 1_{X^1_{t_1}=z}\, 1_{X^1_{[0,t_1]} \subset A}\, 1_{X^2_{t_2}=y}\big)\, dt_1\, dt_2$$



Proof. Fix $t>0$. Then

$$\mathbb{E}_x\big(\mathcal{Z}(L(t))\, 1_{X_t=y}\, 1_{T_A<t}\big)$$
$$= \sum_{i=0}^{2^n-1} \mathbb{E}_x\big(\mathcal{Z}(L(t))\, 1_{X_t=y}\, 1_{T_A \in I_{n,i}}\big)$$

where $I_{n,i} = [t2^{-n} i,\ t 2^{-n}(i+1))$.

Let $I_n(s) = I_{n,i}$ if $s \in I_{n,i}$. Then the RHS is

$$\int_0^t ds \underbrace{|I_n(s)|^{-1} \mathbb{E}_x\big(\mathcal{I}(L(t))\, \mathbb{1}_{X_t = y}\, \mathbb{1}_{T_A \in I_n(s)}\big)}_{(*)}.$$

Claim: For each s,

$$\lim_{n\to\infty} (*) = \sum_{z\in A} \sum_{z'\notin A} J_{zz'}\, \mathbb{E}_{x,z'}\big(\mathcal{I}(L^1(s) + L^2(t-s))$$
$$\mathbb{1}_{X^1_s = z}\, \mathbb{1}_{X^1_{[0,s]} \subset A}\, \mathbb{1}_{X^2_{t-s} = y}\big).$$

Assuming the claim and noticing that

$$(*) \le \underbrace{\max_{|\ell|_1 \le t} \mathcal{I}(\ell)}_{\le C(t)}\ \underbrace{|I_n(s)|^{-1}\, \mathbb{P}_x(T_A \in I_n(s))}_{\le |I_n(s)|^{-1} \mathbb{P}_x(N_{I_n(s)} \ge 1) \le C}.$$

dominated convergence can the following change of variables implies the claim:

$$\int_0^\infty dt \int_0^t ds\, f(s)\, g(t-s) = \int_0^\infty dt_1 \int_0^\infty dt_2\, f(t_1) g(t_2).$$

Proof of claim: Fix $I = [a,b)$. Then

$$\mathbb{E}_x\big(\mathcal{I}(L(t))\, \mathbb{1}_{X_t = y}\, \mathbb{1}_{T_A \in I}\big) - \mathbb{E}_x\big(\mathcal{I}(L(t))\, \mathbb{1}_{X_t = y}\, \mathbb{1}_{T_A > a}$$
$$\mathbb{1}_{X_a \in A}\, \mathbb{1}_{X_b \notin A}\big)$$

$$\le C_t \, \mathbb{E}\left(\left| 1_{T_A \in I} - 1_{T_A > a} 1_{X_b \notin A} \right| \right) \le C_t \, P(N_I \ge 2)$$
$$\le C_t \, |I|^2$$

Thus (if $I_n(s) = I$)

$$II(*) = \mathbb{E}_x \left(\mathcal{I}(L(t)) \, 1_{X_t = y} \, 1_{T_A > a} \, 1_{X_a \in A} 1_{X_b \notin A} \right) + O(|I|^2)$$

$$= \sum_{z \in A} \sum_{z' \notin A} \mathbb{E}_x \left(\mathcal{I}(L(t)) \, 1_{X_t = y} \, 1_{T_A > a} \, 1_{X_a = z} \, 1_{X_b = z'} \right) + O(|I|^2)$$

By the Markov property, this is

$$= \sum_{z \in A} \sum_{z' \notin A} \mathbb{E}_{x,z'} \left(\mathcal{I}(L^1(a) + L^2(t-a)) \, 1_{X^1_a = z} \, 1_{X^1_{[0,a]} \subseteq A} \, 1_{X^2_{t-a} = y} \right)$$
$$\times \underbrace{\mathbb{P}_z (X_{b-a} = z')}_{J_{zz'} \underbrace{|b-a|}_{|I|} + O(|I|^2)} + O(|I|^2)$$

$$\Rightarrow (*) = \sum_{z \in A} \sum_{z' \notin A} J_{zz'} \, \mathbb{E}_{x,z'} \left(\mathcal{I}(L^1(a) + L^2(t-a)) \, 1_{X^1_a = z} \, 1_{X^1_{[0,a]} \subset A} \, 1_{X^2_{t-a} = y} \right) + O(|I|)$$

Using right-continuity of $t \mapsto X^i_t$ the claim follows.

Cor (Simon-Lieb inequality). Let $n=1,2$ and $A\subset\Lambda$. Then for $x\in A$, $y\notin A$,

$$\langle \varphi_x \varphi_y \rangle^\Lambda \le \sum_{z\in A} \sum_{z'\notin A} J_{zz'} \langle \varphi_x \varphi_z \rangle^A \langle \varphi_{z'} \varphi_y \rangle^\Lambda.$$

Proof.

$$\langle \varphi_x \varphi_y \rangle = \int_0^\infty \mathbb{E}_x\left(\mathcal{I}(L^t)\, 1_{X_t=y}\, 1_{T_A<t}\right) dt$$

$$= \sum_{z\in A} \sum_{z'\notin A} J_{zz'} \int_0^\infty \int_0^\infty \mathbb{E}_{x,z'}\left(\mathcal{I}(L(t))\, 1_{X^1_{t_1}=z}\, 1_{X^1_{[0,t_1]}\subset A}\, 1_{X^2_{t_2}=y}\right) dt_1\, dt_2$$

The claim follows if on the RHS $\mathcal{I}(L(t))$ can be replaced by $\mathcal{I}(L^1(t))\,\mathcal{I}(L^2(t))$.

To see this,

$$\int_0^\infty \int_0^\infty \mathbb{E}_{x,z'}\left(\left(\mathcal{I}(L(t)) - \mathcal{I}(L^1(t))\,\mathcal{I}(L^2(t))\right) 1_{X^1_{t_1}=z}\, 1_{X^1_{[0,t]}\subset A}\, 1_{X^2_{t_2}=y}\right)$$

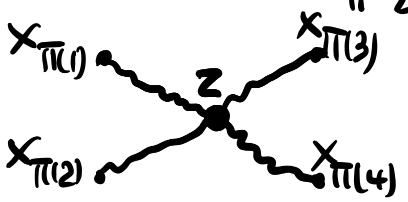
$$= \int_0^\infty \mathbb{E}_x\left(\mathcal{I}(L(t))\, 1_{X_t=z}\, 1_{X_{[0,t]}\subset A}\left(\underbrace{\langle \varphi_{z'} \varphi_y \rangle_{L(t)} - \langle \varphi_{z'} \varphi_y \rangle_0}_{\le 0}\right)\right)$$

4.4. Aizenman-Fröhlich inequality

Thm. (Aizenman, Fröhlich). Let $n=1$. Then

correction

$$U_4(x_1,\dots,x_4) \geq -\sum_{\pi}\sum_{z} \langle \varphi_{x_{\pi(1)}}\varphi_z\rangle\langle\varphi_{x_{\pi(3)}}\varphi_z\rangle \times$$



$$\times \Big[\delta_{z\,x_{\pi(2)}} + \sum_{z'} J_{zz'}\langle\varphi_{z'}\varphi_{x_{\pi(2)}}\rangle\Big]$$

$$\times \Big[\delta_{z'\,x_{\pi(4)}} + \sum_{z''} J_{z'z''}\langle\varphi_{z''}\varphi_{x_{\pi(4)}}\rangle\Big]$$

where

$$U_4(x_1,\dots,x_4) = \langle\varphi_{x_1}\cdots\varphi_{x_4}\rangle - \sum_{\pi}\langle\varphi_{x_{\pi(1)}}\varphi_{x_{\pi(2)}}\rangle\langle\varphi_{x_{\pi(3)}}\varphi_{x_{\pi(4)}}\rangle$$

Rk. Recall the Lebowitz inequality $U_4 \leq 0$.

Lemma. $\mathcal{Z}(\ell+\ell') \geq \mathcal{Z}(\ell)\,\mathcal{Z}(\ell')\, e^{-2g\sum_x \ell_x\ell'_x}$

Proof.

$$\log\mathcal{Z}(\ell+\ell') = \log\mathcal{Z}(\ell) + \int_0^1 \frac{\partial}{\partial u}\log\mathcal{Z}(\ell+u\ell')\,du$$

where

$$\frac{\partial}{\partial u}\log\mathcal{Z}(\ell+u\ell') = -\sum_x \ell'_x\Big\langle 2g\big(\tfrac{1}{2}|\varphi_x|^2+\ell_x+u\ell'_x\big)+\nu\Big\rangle_{\ell+u\ell'}$$

$$\geq -\sum_x \ell'_x\Big\langle 2g\big(\tfrac{1}{2}|\varphi_x|^2+u\ell'_x\big)+\nu\Big\rangle_{u\ell'}$$

$$-2g\sum_x \ell'_x\ell_x$$

$$= \frac{\partial}{\partial u} \log \mathcal{I}(u\ell') - 2g \sum_x \ell'_x \ell_x .$$

$$\Rightarrow \log \mathcal{I}(\ell+\ell') \geq \log \mathcal{I}(\ell) + \log \mathcal{I}(\ell') - 2g \sum_x \ell_x \ell'_x .$$

Rk. Note that if two walks intersect much, then

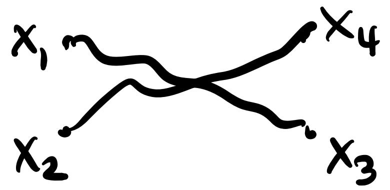
$$e^{-g \sum L(t)_x L'(t)_x}$$

becomes small: the weight suppresses intersections.

Proof of Theorem

$$u(x_1,\dots,x_4)$$

$$= \sum_\pi \int_0^\infty \int_0^\infty \mathbb{E}_{x_{\pi(1)} x_{\pi(3)}} \Big(\big(\mathcal{I}(L^1(t_1)+L^2(t_2)) - \mathcal{I}(L^1(t_1))\, \mathcal{I}(L^2(t_2)) \big) \times 1_{X^1_{t_1}=x_{\pi(2)}} 1_{X^2_{t_2}=x_{\pi(4)}} \Big) dt_1\, dt_2$$



$$\underset{\uparrow \text{ Lemma}}{\geq} \sum_\pi \int_0^\infty \int_0^\infty \mathbb{E}_{x_{\pi(1)} x_{\pi(3)}} \Big(\mathcal{I}(L^1(t_1))\, \mathcal{I}(L^2(t_2))\, 1_{X^1_{t_1}=x_{\pi(2)}} 1_{X^2_{t_2}=x_{\pi(4)}} \times \underbrace{\big(e^{-2g \sum L^1(t_1)_x L^2(t_2)_x} - 1 \big)}_{\substack{=0 \text{ unless } X^1 \text{ and } X^2 \\ \text{intersect, at say } z \in \Lambda.}} \Big) dt_1\, dt_2$$

$$\geq -\sum_{\pi}\sum_{z}\int_0^\infty\!\!\int_0^\infty \mathbb{E}_{x_{\pi(1)}\, x_{\pi(3)}}\Big(\mathcal{I}(L^1(t_1))\,\mathcal{I}(L^2(t_2))\,1_{X^1_{t_1}=x_{\pi(2)}}\,1_{X^2_{t_2}=x_{\pi(4)}}$$
$$1_{z\in X^1_{[0,t_1]}}\,1_{z\in X^2_{[0,t_2]}}\Big)\,dt_1\,dt_2$$

$$= -\sum_{\pi}\sum_{z}\left(\int_0^\infty \mathbb{E}_{x_{\pi(1)}}\big(\mathcal{I}(L(t))\,1_{X_t=x_{\pi(2)}}\,1_{z\in X_{[0,t]}}\big)\,dt\right)$$
$$\times\left(\int_0^\infty \mathbb{E}_{x_{\pi(3)}}\big(\mathcal{I}(L(t))\,1_{X_t=x_{\pi(4)}}\,1_{z\in X_{[0,t]}}\big)\,dt\right)$$

As in the proof of the Simon-Lieb inequality, each integral is bounded by

$$\langle \varphi_{x_{\pi(1)}}\varphi_{x_{\pi(2)}}\rangle\,\delta_{z\,x_{\pi(2)}} + \sum_{z'} J_{zz'}\,\langle \varphi_{x_{\pi(1)}}\varphi_z\rangle\langle \varphi_{z'}\varphi_{x_{\pi(2)}}\rangle\,,$$

<u>Cor.</u>
$$\sum_{x_1,x_2,x_3}|u_4(0,x_1,x_2,x_3)| \leq 3\,|J|^2\,\chi^4\left(1+\frac{1}{|J|\chi}\right)^2$$

where $|J| = \sup_x \sum_y J_{xy} = 2d\beta$, and
$$\chi = \sum_x \langle \varphi_0\varphi_x\rangle$$

Proof. By the Lebowitz inequality $U_4 \le 0$, it suffices to notice that

$$-\sum_{x_1,x_2,x_3} U_4(0,x_1,x_2,x_3) \le 3 \sum_{x_1,x_2,x_3} \sum_z \langle \varphi_0 \varphi_z \rangle \langle \varphi_{x_1} \varphi_z \rangle \times$$

$$\times \Big[\delta_{z x_2} + \sum_{z'} J_{zz'} \langle \varphi_{z'} \varphi_{x_2} \rangle \Big]$$

$$\times \Big[\delta_{z x_3} + \sum_{z''} J_{zz''} \langle \varphi_{z''} \varphi_{x_3} \rangle \Big]$$

$$\le 3 \chi^2 (1 + |J| \chi)^2.$$

Exercise. Consider two independent SRW on $\mathbb{Z}^d$. Show

$$\mathbb{E}_{0,0}\Big[\int_0^\infty \mathbb{1}_{X_t^1 = X_t^2} \, dt \Big] = \begin{cases} < \infty & \text{if } d > 4 \\ = \infty & \text{if } d \le 4. \end{cases}$$

Idea: $\text{LHS} = \sum_z \mathbb{E}_{0,0}\big[\mathbb{1}_{X_t^1 = z} \mathbb{1}_{X_t^2 = z} \big]$

$$= \sum_z C(0,z)^2 \approx \sum_z \frac{1}{(1+|z|)^{2(d-2)}}$$

$$\approx \int_1^\infty dr \, \frac{r^{d-1}}{r^{2d-4}} = \int_1^\infty dr \, r^{-d+3}$$

Intuition: If walks in AF inequality were SRW, expect $U_4(x_1, \dots, x_4) \approx 0$ in $d > 4$ if $x_1, \dots, x_4$ separated.

4.5. Gaussianity of continuum limits in $d \geq 5$.

For $f: \mathbb{R}^d \to \mathbb{R}$ smooth with compact support, define

$$(f_\varepsilon, \varphi) = \sum_x f_\varepsilon(x)\varphi_x = \varepsilon^{d-\alpha} \sum_x f(x\varepsilon)\varphi_x .$$

What can be said about the distribution of this random variable?

<u>Defn.</u> A sequence of random fields φ^ε on $\mathbb{Z}^d$ has a continuum limit φ^0 with scaling exponent $d-\alpha$ if for every $f \in C_c^\infty(\mathbb{R}^d)$,

$$(f_\varepsilon, \varphi^\varepsilon) \xrightarrow{(d)} (f, \varphi) \qquad (\varepsilon \to 0)$$

for a random field φ on $\mathbb{R}^d$ (which is usually a generalised function, a Schwartz distribution).

<u>Exercise</u> The GFF on $\mathbb{Z}^d$, $d \geq 3$ has a continuum limit with $\alpha = \frac{1}{2}(d-2)$, the continuum GFF. Since the field is Gaussian, this amounts to convergence of the covariances.

<u>Lemma</u>. For the Ising model on $\mathbb{Z}^d$, assume β_ε is s.t. $\langle \varphi_0 \varphi_x \rangle_{\beta_\varepsilon} \longrightarrow 0$ as $|x| \to \infty$ (ε fixed) and that $\beta_\varepsilon \geq c > 0$. Then:

$$\langle (f_\varepsilon, \varphi)^2 \rangle_{\beta_\varepsilon} \longrightarrow 0 \quad \text{if} \quad \alpha < \tfrac{1}{2}(d-2).$$

<u>Proof</u>. By the (inf. vol.) infrared bound,

$$\langle (f_\varepsilon, \varphi)^2 \rangle_{\beta_\varepsilon} = \sum_{x,y} f_\varepsilon(x) f_\varepsilon(y) \langle \varphi_x \varphi_y \rangle_{\beta_\varepsilon}$$

$$\leq \varepsilon^{2d} \varepsilon^{-2\alpha} \sum_{x,y} f(x\varepsilon) f(y\varepsilon) \frac{C\varepsilon^{d-2}}{\beta_\varepsilon (\varepsilon |x-y|)^{d-2}}$$

$$= \underbrace{\varepsilon^{-2\alpha+d-2}}_{\substack{\to 0 \\ \text{if } \alpha < \frac{d-2}{2}}} \underbrace{\varepsilon^{2d} \sum_{x,y} f(x\varepsilon) f(y\varepsilon) \frac{C}{(\varepsilon|x-y|)^{d-2}}}_{\approx \int f(x) f(y) \frac{1}{|x-y|^{d-2}} dx\, dy}$$

<u>Upshot</u>: for a nontrivial limit, need $\alpha \geq \frac{d-2}{2}$.

This will be interesting if $\beta_\varepsilon \approx \beta_c$.

Thm. Let $d \geq 5$, $\alpha \geq \frac{1}{2}(d-2)$, $\chi_\varepsilon < \infty$. Then:
for each $f \in C_c^\infty(\mathbb{R}^d)$,

$$\left| \left\langle \frac{(f_\varepsilon, \varphi)^4}{\bar{\chi}_\varepsilon^2} \right\rangle_{\beta_\varepsilon} - 3 \left\langle \frac{(f_\varepsilon, \varphi)^2}{\bar{\chi}_\varepsilon} \right\rangle_{\beta_\varepsilon}^2 \right| \lesssim C_f \, \varepsilon^{d-4} \to 0.$$

where $\bar{\chi}_\varepsilon = \varepsilon^{d-2\alpha} \chi_\varepsilon = \varepsilon^{d-2\alpha} \sum_x \langle \varphi_0 \varphi_x \rangle_{\beta_\varepsilon}$.

Rk. If $(f_\varepsilon, \varphi)/\bar{\chi}_\varepsilon$ was Gaussian, then LHS $= 0$.

Rk. Consider $f(x) = 1_{|x| \leq 1}$. Then, if $\chi_\varepsilon < \infty$,

$$\langle (f_\varepsilon, \varphi)^2 \rangle = \sum_{x,y} f_\varepsilon(x) f_\varepsilon(y) \langle \varphi_x \varphi_y \rangle_{\beta_\varepsilon} \qquad \text{(without proof)}$$

$$= \varepsilon^{2d-2\alpha} \sum_{|x| \leq \varepsilon^{-1}} \sum_{|y| \leq \varepsilon^{-1}} \langle \varphi_x \varphi_y \rangle_{\beta_\varepsilon} \approx \varepsilon^{d-2\alpha} \chi_\varepsilon = \bar{\chi}_\varepsilon .$$

More correctly, we should have defined the normalisation $\bar{\chi}_\varepsilon$ by the LHS. Then the statement can be extended to to critical case $\chi = \infty$, which the current statement only applies in the near-critical case ($\chi_\varepsilon \to \infty$ but $\bar{\chi}_\varepsilon \approx 1$). This would have made the proof more technical.

Proof.

$$\langle (f_\varepsilon,\varphi)^4\rangle = \sum_{x_1,\dots,x_4} f_\varepsilon(x_1)\cdots f_\varepsilon(x_4)\,\langle \varphi_{x_1}\cdots\varphi_{x_4}\rangle$$

$$\langle (f_\varepsilon,\varphi)^2\rangle^2 = \sum_{x_1,\dots,x_4} f_\varepsilon(x_1)\cdots f_\varepsilon(x_4)\,\langle \varphi_{x_1}\varphi_{x_2}\rangle\langle \varphi_{x_3}\varphi_{x_4}\rangle$$

$$\Rightarrow \langle (f_\varepsilon,\varphi)^4\rangle - 3\langle (f_\varepsilon,\varphi)^2\rangle^2$$

$$= \sum_{x_1,\dots,x_4} f_\varepsilon(x_1)\cdots f_\varepsilon(x_4)\, U_4(x_1,\dots,x_4)$$

$$\le \|f\|_\infty^3 \sum_{x_1} |f_\varepsilon(x_1)| \underbrace{\sum_{x_2,\dots,x_4} |U_4(x_1,\dots,x_4)|}_{\overset{\text{transl. inv.}}{=} \sum_{x_2,\dots,x_4} |U_4(0,x_2,\dots,x_4)|}$$

$$= \|f_\varepsilon\|_\infty^3 \|f_\varepsilon\|_1 \sum_{x_1,\dots,x_3} |U_4(0,x_1,\dots,x_3)|$$

$$\overset{\text{supp } f\subset[-\frac{R_f}{2},\frac{R_f}{2}]^d}{\le} \|f\|_\infty^4 R_f^d\, \varepsilon^{3d-4\alpha} \underbrace{\sum_{x_1,\dots,x_3} |U_4(0,x_1,\dots,x_3)|}_{\le 3|J|^2\chi^4\left(1+\frac{1}{|J|^2\chi^2}\right)}$$

$$= R_f^d \|f\|_\infty^4\, \varepsilon^{3d-4\alpha}\left(3|J|^2\chi^4\left(1+\frac{1}{|J|^2\chi^2}\right)\right).$$

$$\Rightarrow \left\langle \frac{(f_\varepsilon,\varphi)^4}{\chi_\varepsilon^2} \right\rangle - 3\left\langle \frac{(f_\varepsilon,\varphi)^2}{\chi_\varepsilon} \right\rangle^2 \leq 3|J|^2 R_f^d \|f\|_\infty^4 \varepsilon^{-d+4\alpha} \times (1+o(1))$$

Since $\alpha \geq \frac{1}{2}(d-2)$, the RHS is

$$\leq C_f\, \varepsilon^{-d+2(d-2)} = C_f\, \varepsilon^{d-4} \longrightarrow 0.$$

<u>Rk</u> It can also be shown that

$$\left\langle \frac{(f_\varepsilon,\varphi)^{2k}}{\chi_\varepsilon^k} \right\rangle - C_k \left\langle \frac{(f_\varepsilon,\varphi)^2}{\chi_\varepsilon} \right\rangle^k \longrightarrow 0$$

where C_k is the number of pairings of $2k$ points. (Again equality in Gaussian case).

<u>Rk.</u> The statement is also true in $d=4$, but much harder — Fields Medal 2022.

4.6. Random current representation

See Example Sheet 4.

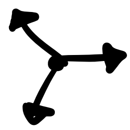
5. FK representation and percolation

<u>Defn</u>. Given a finite graph $G=(\Lambda,E)$ and $\beta\geq 0$, the q-state Potts model is defined on spin configurations $\sigma:\Lambda\longrightarrow\{1,\dots,q\}$ by

$$\mathbb{P}^{\Lambda}_{q,\beta}(\sigma) = \frac{1}{Z^{\Lambda}_{q,\beta}}\, e^{\beta\sum_{xy\in\Lambda}\mathbb{1}_{\sigma_x=\sigma_y}}$$

<u>Ex</u>. Prove there exists $T_q=\{u_1,\dots,u_q\}\subset\mathbb{R}^{q-1}$ s.t.

$$u_i\cdot u_j = \begin{cases} q-1 & (i=j)\\ -1 & (i\neq j)\end{cases}$$

(T_q is a tetrahedron:  $q=1$,  $q=2$, ...)

Identify $\{1,\dots,q\}$ with $\{u_1,\dots,u_q\}$: $u_x=u_{\sigma_x}$. Then

$$u_x\cdot u_y = q\,\mathbb{1}_{\sigma_x=\sigma_y}-1$$

$$\Rightarrow e^{\beta\sum_{xy\in\Lambda}\mathbb{1}_{\sigma_x=\sigma_y}} \propto e^{\frac{\beta}{q}\sum_{xy\in E}u_x\cdot u_y}$$

This is called the tetrahedral representation of the Potts model.

Defn. The random cluster model with parameters $p \in [0,1]$ and $q > 0$ is a probability measure on subgraphs $\omega \subset E$ defined by

$$\mathbb{P}^{\Lambda}_{p,q}(\omega) = \frac{1}{Z^{\Lambda}_{p,q}} p^{|\omega|}(1-p)^{|E \setminus \omega|} q^{k(\omega)}$$

where $k(\omega)$ is the number of connected components (called clusters) of ω.

The special case $q=1$ is called Bernoulli bond percolation.

Thm. (Edwards–Sokal coupling). Define a probability measure on $\{1,\dots,q\}^{\Lambda} \times \{0,1\}^{E}$ by

$$\mathbb{P}(\sigma,\omega) \propto p^{|\omega|}(1-p)^{|E \setminus \omega|} \mathbb{1}\{\sigma \text{ is constant on each cluster of } \omega\}.$$

Then, for $q \in \{2,3,\dots\}$,

(i) The ω marginal is the random cluster model with parameters p and q.

(ii) The σ marginal is the q-state Potts model with β given by $p = 1 - e^{-\beta}$.

(iii) Given ω, the conditional measure on σ is given by choosing σ_x constant on each cluster and uniform from $\{1, \dots, q\}$, independently for different clusters.

<u>Proof</u>. (i) Given ω, there are $q^{k(\omega)}$ choices for σ satisfying the constraint that σ is constant on the clusters of ω. Thus the σ marginal is

$$p^{|\omega|}(1-p)^{|E \setminus \omega|} q^{k(\omega)}.$$

(ii) Example Sheet.

(iii) Similar to (i).

<u>Exercise</u>. If $p = 1 - e^{-\beta}$, then

$$\langle u_x \cdot u_y \rangle_\beta = (q-1)\, \mathbb{P}_{p,q}(x \leftrightarrow y)$$

where the LHS refers to the q-state Potts model and the RHS to the random cluster model, and $x \leftrightarrow y$ is the event that x and y are in the same cluster.

Note that percolation $\mathbb{P}_{p,1} = \mathbb{P}_p$ can immediately be defined on the infinite graph $\mathbb{Z}^d$ (without limit).

Exercise. Let $d=1$. Compute $\mathbb{P}_p(x \leftrightarrow y)$.

Exercise. Let $d \geq 1$. Show there is $p_0 > 0$ such that

$$\mathbb{P}_p(0 \leftrightarrow x) \leq Ce^{-c|x|} \quad \text{for } p < p_0.$$

Exercise. Let $d \geq 2$. Show there is $p_1 < 1$ such that

$$\mathbb{P}_p(0 \leftrightarrow x) \geq c > 0 \quad \text{for } p > p_1.$$