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1. **Approximation of Ising by φ^4 model.** Show that for any $F : \mathbb{R}^\Lambda \to \mathbb{R}$ bounded and continuous, $\langle F(\varphi)\rangle_H \to \langle F(\sigma)\rangle_{J,g}$ as $\lambda \to \infty$ where the left-hand side denotes the generalised Ising model with

$$H(\varphi) = -\sum_{xy\in E} J_{xy}\varphi_x\varphi_y - \sum_{x\in\Lambda} g_x\varphi_x + \sum_{x\in\Lambda} \lambda(\varphi_x^2 - 1)^2 \tag{0.1}$$

and the right-hand side usual Ising model with coupling constants J and external field g.

2. **FKG inequality.** By either considering the Ising model as a limit of a continuous Ising model and applying the version of the FKG inequality for such models proved in class, or by adapting its proof to the Ising case, show carefully that the Ising model (with $J_{xy} \geqslant 0$ and any $g_x \in \mathbb{R}$) satisfies the FKG inequality.

3. **Susceptibility of Curie-Weiss model.** For the Curie–Weiss model, extend the analysis from class to show that the magnetisation satisfies

$$M(\beta_c, h) \sim (3h)^{1/3} \qquad (h \downarrow 0),$$

and that susceptibility satisfies

$$\chi(\beta, 0) = \frac{1}{\beta_c - \beta} \qquad (\beta < \beta_c), \qquad \chi(\beta, 0+) \sim \frac{1}{2(\beta - \beta_c)} \qquad (\beta \downarrow \beta_c).$$

4. **Mean-field $O(3)$ model.** For the $O(3)$ model on the complete graph, show that the critical inverse temperature is $\beta_c = 3$ and compute the divergence of the susceptiblity.

(*) What is the critical temperature of the $O(n)$ model for general n?

5. **Ginibre inequality for the clock model.** Extend the Ginibre inequality to the clock model which is defined like the XY model except that the spins σ_x take values in the discrete circle $\mathbb{Z}_q$ viewed as a subset of the circle $\mathbb{S}^1 \subset \mathbb{R}^2$.

6. **A correlation inequality for $O(n)$ models.** Consider the $O(n)$ model with $n \geqslant 2$ and assume that the constant external field is in direction $e = (1, 0, \dots, 0)$. Show that

$$\langle ((\sigma_x^1)^2 - (\sigma_x^2)^2)\sigma_y^1\rangle_{\beta,h}^\Lambda \geqslant 0, \quad \langle ((\sigma_x^1)^2 - (\sigma_x^2)^2)\sigma_y^1\sigma_z^1\rangle_{\beta,h}^\Lambda \geqslant 0.$$

7. **Infinite volume limit of XY model.** Show that the infinite volume limit of the XY model with free boundary conditions exists for all local functions, when $\beta \geqslant 0$ and $h \geqslant 0$.

8. **High-temperature expansion for the $O(n)$ model.** By deriving a version of the high temperature expansion for the $O(n)$ model, show that the spin-spin correlation function $\langle \sigma_x \cdot \sigma_y\rangle_{\beta,0}^\Lambda$ decays exponentially when $\beta \leqslant \beta_0(d,n)$ for some $\beta_0(d,n) > 0$.

9. **Convex functions**. Let $f_n : \mathbb{R} \to \mathbb{R}$ be a sequence of convex and differentiable functions. Assume that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each $x \in \mathbb{R}$. Show that f is convex and that for each $x \in \mathbb{R}$ at which f is differentiable,

$$f_n'(x) \to f'(x). \tag{0.2}$$

Give an example of differentiable and convex functions f_n and $x \in \mathbb{R}$ such that f is not differentiable at x.

10. **Aizenman–Simon bound (*).** Let $\beta \geqslant 0$ and $h = 0$. Show that, for all $a, b \in \Lambda$,

$$\langle \sigma_a \cdot \sigma_b\rangle_{2\beta,0}^{\Lambda,\mathrm{XY}} \leqslant \langle \sigma_a\sigma_b\rangle_{\beta,0}^{\Lambda,\mathrm{Ising}}.$$

Consequently, $2\beta_c^{\mathrm{XY}}(d) \geqslant \beta_c^{\mathrm{Ising}}(d)$. *[Hint: Consider the XY model with additional term $\lambda\sum_x \cos(4\theta_x)$ in the exponential. Thus for $\lambda = 0$ this is the usual XY model and as $\lambda \to \infty$ the measure converges to clock model with $q = 4$. Relate the $q = 4$ clock model to two independent copies of the Ising model.]*

11. **Covariance of the Gaussian free field.** Consider the GFF with mass $m > 0$ on the Λ_L with periodic boundary conditions. Show that its covariance is given by the $\Lambda_L \times \Lambda_L$ matrix

$$C_{m^2}^{\Lambda_L,\text{periodic}}(x,y) = ((-\Delta^{\Lambda_L,\text{periodic}} + m^2)^{-1}(x,y))_{x,y\in\Lambda_L}.$$

where $\Delta^{\Lambda_L,\text{periodic}}$ is the discrete Laplacian on Λ_L with periodic boundary conditions, i.e., $(\Delta^{\Lambda_L,\text{periodic}} f)_x = \sum_{x\sim y}(f_y - f_x)$ for $x, y \in \Lambda_L$ and $x \sim y$ denoting that x and y are nearest-neighbours with opposite sides of Λ_L identified. Show that

$$\lim_{m^2\downarrow 0}\lim_{L\to\infty} C_{m^2}^{\Lambda_L,\text{periodic}}(0,0) = \lim_{m^2\downarrow 0} C_{m^2}^{\mathbb{Z}^d}(0,0) \begin{cases} < \infty & (d > 2) \\ = \infty & (d \leqslant 2) \end{cases}$$

where $C_{m^2}^{\mathbb{Z}^d}(x,y)$ is the Green's function of $-\Delta^{\mathbb{Z}^d} + m^2$ and can be written as a Fourier integral over $[-\pi,\pi)^d$. Show that the limit $C_0^{\mathbb{Z}^d}(x,y)$ of $C_{m^2}^{\mathbb{Z}^d}(x,y)$ as $m^2 \downarrow 0$ exists when $d > 2$, while in $d \leqslant 2$ the limit

$$\bar{C}_0^{\mathbb{Z}^d}(x,y) = \lim_{m^2\downarrow 0}\lim_{L\to\infty}[C_{m^2}^{\Lambda_L,\text{periodic}}(x,y) - C_{m^2}^{\Lambda_L,\text{periodic}}(0,0)]$$

exists and can again be written as as an analogous Fourier integral. Show that

$$C_0^{\mathbb{Z}^d}(x,y) \sim C_d|x-y|^{-(d-2)} \qquad (d > 2)$$
$$\bar{C}_0^{\mathbb{Z}^d}(x,y) \sim -\frac{1}{2\pi}\log|x| \qquad (d = 2).$$

(*) means harder