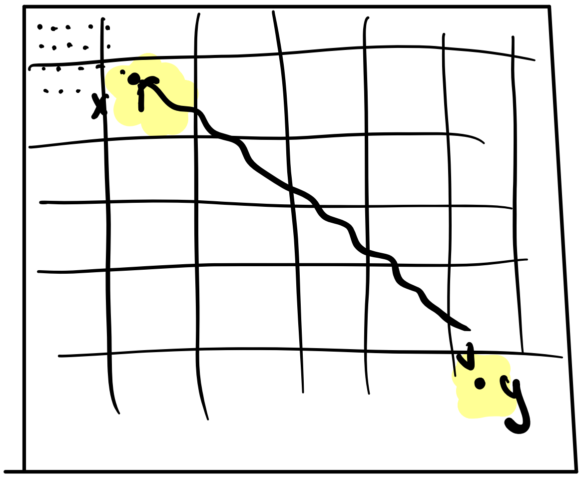
1. The renormalisation group

Kadanoff: Block spin picture (1960s)

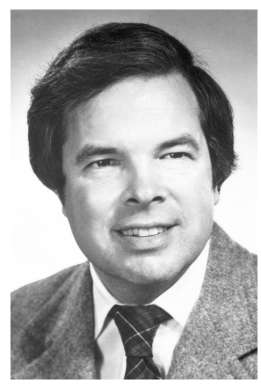


$$\langle \sigma_x \sigma_y \rangle \approx |x-y|^{-\alpha}$$

Divergences and scale dependence of coupling constants in QFT (1950s)

Wilson & others (Fisher, Wegner, ...)

In short the Kadanoff block picture, although absurd, will be the basis for generalizations which are not absurd, and it is helpful to understand the Kadanoff picture in differential form before studying these generalizations.



Computational (and conceptual) picture.

Lectures.

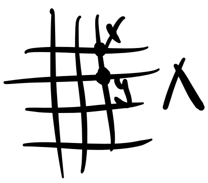
1. Spin models and the renormalised potential
2. Perturbation theory and its three problems
3. Finite-range approach to renormalisation
4. Example of 0-state Potts model and concluding remarks

https://www.dpmms.cam.ac.uk/~rb812/teaching/ihes2022/.

Spin system: $\Lambda \subset \mathbb{Z}^d$ or $\Lambda \subset \varepsilon\mathbb{Z}^d$ $(\varepsilon \to 0)$

Spins: φ_x, $x \in \Lambda$ — could be vector-valued
discrete / continuous /
anticommuting

Expectation: $$\langle F \rangle = \frac{1}{Z} \int F(\varphi)\, e^{-\frac{1}{2}(\varphi, -\Delta\varphi) - V(\varphi)}\, d\varphi$$



$$\frac{1}{2}\sum_{x \sim y} (\varphi_x - \varphi_y)^2 \qquad V(\varphi) = \sum_{x \in \Lambda} V_x(\varphi_x)$$

<u>Critical temperature</u>. Ising model: $\beta > 0$

$$\langle F \rangle_\beta = \frac{1}{Z_\beta} \sum_{\sigma \in \{\pm 1\}^\Lambda} F(\sigma)\, e^{-\frac{\beta}{2}(\sigma, -\Delta\sigma)}$$

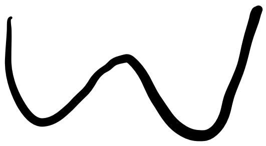
$$\begin{array}{cccc} + & + & - & + \\ + & + & + & - \\ + & + & - & + \\ + & + & - & + \end{array}$$

$\beta < \beta_c$: disordered
$\beta > \beta_c$: ordered
$\beta = \beta_c$: critical $\langle \sigma_x \sigma_y \rangle \approx |x-y|^{-\alpha}$

Ginzburg-Landau φ^4 model: $g > 0$, $\nu \in \mathbb{R}$

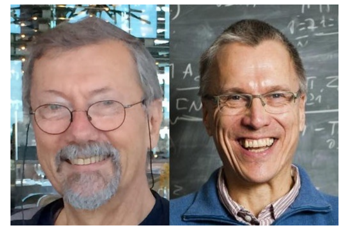
$$\langle F \rangle_{g,\nu} = \frac{1}{Z_{g,\nu}} \int_{\mathbb{R}^\Lambda} F(\varphi)\, e^{-\frac{1}{2}(\varphi, -\Delta\varphi) - \sum_{x \in \Lambda}\left(\frac{1}{4} g\, \varphi_x^4 + \frac{1}{2}\nu\, \varphi_x^2\right)} d\varphi$$

$(d = 4)$



$\nu < \nu_c(g)$: ordered
$\nu > \nu_c(g)$: disordered
$\nu = \nu_c(g)$: critical

$$\langle \varphi_x \varphi_y \rangle \approx |x-y|^{-\alpha}$$



Same universality class: $\mathbb{Z}_2$ symmetry

Many other interesting universality classes:

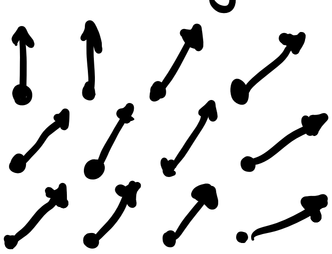
S_n : Potts
$O(n)$: $O(n)$
⋮

<u>Low temperature</u>. Heisenberg model:

Haar on S^{n-1}

$$\langle F \rangle_\beta = \frac{1}{Z_\beta} \int_{(S^{n-1})^\Lambda} e^{-\frac{\beta}{2}(\sigma, -\Delta \sigma)} F(\sigma) \prod_{x \in \Lambda} d\sigma_x$$

$d \geq 3$: Again disordered for $\beta \ll 1$, ordered for $\beta \gg 1$.



$$\langle \sigma_x \cdot \sigma_y \rangle_\beta \approx m(\beta)^2 + \frac{C(\beta)}{|x|^{d-2}}$$

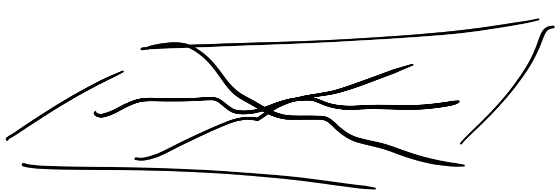
Spin wave or Goldstone picture.

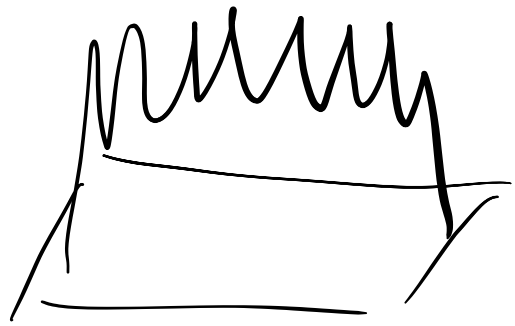
High temperature. Discrete Gaussian model $d=2$:

$$\langle F\rangle_\beta = \frac{1}{Z_\beta} \sum_{\substack{\sigma\in\mathbb{Z}^\Lambda \\ \sigma_0=0}} e^{-\frac{\beta}{2}(\sigma,-\Delta\sigma)} F(\sigma)$$

$\beta>\beta_c$: flat (localised) $\langle(\sigma_0-\sigma_x)^2\rangle \approx 1$

$\beta<\beta_c$: rough (delocalised) $\langle(\sigma_0-\sigma_x)^2\rangle \approx \log|x|$





Lattice sine-Gordon model:

$$\langle F\rangle_{\beta,z} = \frac{1}{Z_{\beta,z}} \int_{\substack{\mathbb{R}^\Lambda \\ \varphi_0=0}} e^{-\frac{1}{2}(\varphi,-\Delta\varphi)-\sum_{x\in\Lambda} 2z\cos(\sqrt{\beta}\varphi_x)} F(\varphi)\,d\varphi$$

$\equiv$ Coulomb gas

Above: infrared problem

Continuum limits $\Lambda_\varepsilon \subset \varepsilon\mathbb{Z}^d$ $(\varepsilon \to 0)$

$$\langle F \rangle_{\lambda,\mu} \propto \int_{\mathbb{R}^{\Lambda_\varepsilon}} e^{-\varepsilon^d \sum_{x\in\Lambda_\varepsilon} \left(\frac{1}{2}\varphi_x(-\Delta^\varepsilon\varphi)_x + \frac{1}{4}\lambda\varphi_x^4 + \frac{1}{2}(\mu + a^\varepsilon(\lambda))\varphi_x^2 \right)} F(\varphi)\, d\varphi$$

$$\Delta^\varepsilon \varphi = \varepsilon^{-2} \sum_{x\sim y} (\varphi_y - \varphi_x)$$

divergent counterterm

$(d=2,3)$

$$\langle \varphi_x \varphi_y \rangle_{\lambda,\mu} \approx |x-y|^{-(d-2)} \qquad (|x-y| \to 0)$$

NLSM $d=2$, ...

Dynamics
Combinatorial models: SAW, dimer, ...
Quantum stat. phys
⋮

solution will look like. Nevertheless, the renormalization group continues to be less important than one might expect. It is at present an approach of last resort, to be used only when all other approaches have been tried and discarded. The reason for this is that it is rather difficult to formulate renormalization group methods for new problems; in fact, the renormalization group approach generally seems as hopeless as any other approach until someone succeeds in solving the problem by the renormalization group approach. Where the renormalization group approach has been successful a lot of ingenuity has been required: one cannot write a renormalization group cookbook. (In contrast, Feynman diagram techniques can be reduced to simple strict rules.) Even if one succeeds in formulating the renormalization group approach for a particular problem, one is likely to have to carry out a complicated computer calculation, which makes most theoretical physicists cringe. Especially in the case of strong interactions of elementary particles, most theorists hope to solve the problem without turning to modern renormalization group methods. It will probably require several years of stagnation in elementary particle theory before theorists will accept the inevitability of the renormalization group approach despite its difficulties.

2. Gaussian integration and renormalised potential

2.1. Gaussian integrals

$$\langle F \rangle_{GFF} \propto \int_{\mathbb{R}^\Lambda} e^{-\frac{1}{2}(\varphi,(-\Delta+m^2)\varphi)} F(\varphi)\, d\varphi$$

$$\langle \varphi_x \varphi_y \rangle = (-\Delta+m^2)^{-1}_{xy}$$

Fact. $\Lambda = \mathbb{Z}^d / L\mathbb{Z}^d$, $x, y \in \Lambda$

$$\lim_{m^2 \downarrow 0} \lim_{L\to\infty} (-\Delta+m^2)^{-1}_{xy} \sim \begin{cases} |x|^{-(d-2)} & (d \geq 3) \\ +\infty & (d \leq 2) \end{cases}$$

$$\lim_{m^2 \downarrow 0} \lim_{L\to\infty} \left[(-\Delta+m^2)^{-1}_{xy} - (-\Delta+m^2)^{-1}_{xx}\right] \sim \begin{cases} -c_2 \log|x| & (d=2) \\ -c_1 |x| & (d=1) \end{cases}$$

Defn. $P_C(\varphi) = (\det(2\pi C))^{-1/2} e^{-\frac{1}{2}(\varphi, C^{-1}\varphi)}$

$$\mathbb{E}_C F = \int F(\varphi)\, P_C(\varphi)\, d\varphi$$

Fact. If $C = C_1 + C_2$ with $C_1, C_2 \geq 0$ then

$$P_C = P_{C_1} * P_{C_2} \iff \mathbb{E}_C F(\varphi) = \mathbb{E}_{C_1} \mathbb{E}_{C_2} F(\varphi_1 + \varphi_2)$$

Continuously: if $C = \int_0^\infty \dot{C}_t\, dt$ with $\dot{C}_t \geq 0$ then

$$\frac{\partial}{\partial t} P_{C_t} = \frac{1}{2} \Delta_{\dot{C}_t} P_{C_t} \quad \text{and} \quad \varphi = \int_0^\infty \sqrt{\dot{C}_t}\, dW_t \sim P_C$$

$$\Delta_{\dot{C}_t} = \sum_{x,y\in\Lambda} \dot{C}_t(x,y) \frac{\partial^2}{\partial\varphi_x \partial\varphi_y}$$

How to decompose the GFF?

- Heat-kernel decomp.

$$(-\Delta+m^2)^{-1} = \int_0^\infty \underbrace{e^{\Delta t - m^2 t}}_{\text{pointw.} \geq 0} \, dt$$

- Pauli-Villars decomp.

$$(-\Delta+m^2)^{-1} = \int_0^\infty \underbrace{\frac{\partial}{\partial t}(-\Delta+m^2+1/t)^{-1}}_{(-t\Delta+tm^2+1)^{-2}\text{ : locality}} \, dt$$

- Block spin decomp.
- Finite-range decomp.

$$(-\Delta+m^2)^{-1} = \int_0^\infty \dot{C}_t \, dt$$

$$\dot{C}_t(x,y) = 0 \quad \text{if} \quad |x-y| > t$$

- Momentum space decomp: local in Fourier space

Technical difficulty: different decompositions have different advantages. How can we make use of all?

2.2. The renormalised potential

Interested in $\langle \cdot \rangle_0 \propto e^{-\frac{1}{2}(\varphi, C^{-1}\varphi) - V_0(\varphi)} d\varphi$

$$C_t = \int_0^t \dot{C}_s \, ds \qquad : \text{ cov. decomp.}$$

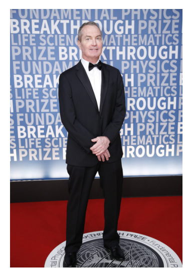
$$e^{-V_t(\varphi)} = (P_{C_t} * e^{-V_0})(\varphi) \qquad : \text{ ren. potential}$$

$$= \mathbb{E}_{C_t}\left(e^{-V_0(\varphi + \zeta)}\right)$$

<u>Fact.</u> $e^{-V_t(\varphi)} = \mathbb{E}_{C_t - C_s}\left(e^{-V_s(\varphi+\zeta)}\right) \quad \forall t > s$

$$\frac{\partial}{\partial t} e^{-V_t} = \frac{1}{2} \Delta_{\dot{C}_t} e^{-V_t}$$

$$= \frac{1}{2} \sum_{x,y \in \Lambda} \dot{C}_t(x,y) \frac{\partial^2}{\partial \varphi_x \partial \varphi_y} e^{-V_t}$$



<u>Polchinski equation:</u>

$$\frac{\partial V_t}{\partial t} = \frac{1}{2} \Delta_{\dot{C}_t} V_t - \frac{1}{2} (\nabla V_t)^2_{\dot{C}_t}$$

$$= \frac{1}{2} \sum_{x,y \in \Lambda} \dot{C}_t(x,y) \left[\frac{\partial^2 V_t}{\partial \varphi_x \partial \varphi_y} - \frac{\partial V_t}{\partial \varphi_x} \frac{\partial V_t}{\partial \varphi_y} \right]$$

Renormalised measure:

$$\langle F \rangle_t \propto \int_{\mathbb{R}^\Lambda} e^{\underbrace{-\frac{1}{2}(\varphi, (C_\infty - C_t)^{-1}\varphi) - V_t(\varphi)}_{H_t(\varphi) \text{ Wilson action}}} F(\varphi)\, d\varphi$$

Given an observable $F: \mathbb{R}^\Lambda \to \mathbb{R}$, find F_t s.t.

$\langle F \rangle_0 = \langle F_t \rangle_t$?

Fact. Suppose

$$\frac{\partial}{\partial t} F_t = L_t F_t, \qquad L_t = \frac{1}{2}\Delta_{\dot{C}_t} - (\nabla V_t, \nabla)_{\dot{C}_t}$$

Then

$$\frac{\partial}{\partial t}\langle F_t \rangle_t = \frac{\partial}{\partial t} \int \underbrace{P_{C_\infty - C_t}(\varphi)}_{\frac{\partial}{\partial t}(\cdot) = -\frac{1}{2}\Delta_{\dot{C}_t} P_{C_\infty - C_t}} \underbrace{e^{-V_t(\varphi)} F_t(\varphi)}_{\frac{\partial}{\partial t}(\cdot) = \frac{1}{2}\Delta_{\dot{C}_t}(e^{-V_t} F_t)}\, d\varphi = 0$$

Explicitly: $F_t(\varphi) = e^{+V_t(\varphi)} \int P_{C_t}(\zeta)\, e^{-V_0(\varphi+\zeta)} F(\varphi+\zeta)\, d\zeta$

<u>Exercise:</u> Derive equation for the Wilson action H_t, for the renormalised measure, ...

$\Rightarrow$ Beautiful structure: How to analyse it?

Rk. (Rescaling). The covariances $\dot{C}_t$ are (approx.) scale invariant: if $\dot{C}_t = e^{\Delta t}$ then

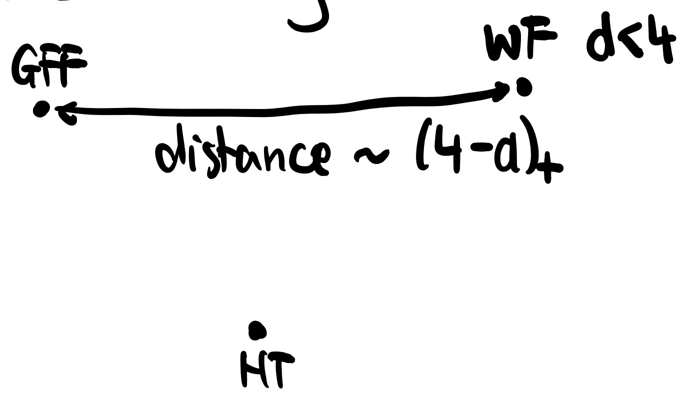
$$\dot{C}_t(x,y) \approx \frac{1}{t^{d/2}} e^{-c|x-y|/t^2} + \text{lattice effects}$$

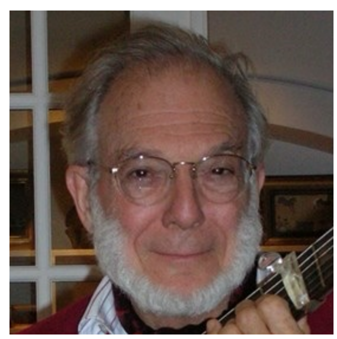
$$\Rightarrow \dot{C}_{L^2 t}(Lx, Ly)\, L^2 dt \approx L^{-(d-2)} \dot{C}_t(x,y)\, dt$$

It is thus natural to also add rescaling to the renormalisation group map – or one can compose V_t with a rescaling map.

Renormalisation group fixed points should be (asymp.) stationary solutions to the Polchinski up to rescaling.

Picture for Ising / φ^4 models:





3. Perturbation theory and its problems

3.1. Perturbation theory

$$V_0(\varphi) = \sum_{x\in\Lambda}\left(\tfrac{1}{4}g_0\varphi^4 + \tfrac{1}{2}\nu_0\varphi^2\right)$$

Local potential approximation:

renormalised coupling constants

$$V_t(\varphi) \approx \sum_{x\in\Lambda}\left(\tfrac{1}{4}g_t\varphi^4 + \tfrac{1}{2}\nu_t\varphi^2\right)$$

First-order perturbation theory:

$$\begin{aligned} V_t(\varphi) &= -\log \mathbb{E}_{C_t}\left(e^{-V_0(\varphi+\zeta)}\right) \\ &\approx \mathbb{E}_{C_t}\left(V_0(\varphi+\zeta)\right) \\ &= \sum_{x\in\Lambda}\left(\tfrac{1}{4}g_0\varphi^4 + \tfrac{1}{2}\underbrace{(\nu_0 + 3g_0\, C_t(0,0))}_{\approx\, \nu_t}\varphi_x^2\right) + \text{const} \end{aligned}$$

Second-order perturbation theory: $\tfrac{1}{4}g_0\varphi_x^4 + \cdots$ $\quad$ $\tfrac{1}{4}g_0\varphi_y^4 + \cdots$

$$V_t(\varphi) \approx \mathbb{E}_{C_t}(V_0(\varphi+\zeta)) \underbrace{- \tfrac{1}{2}\mathbb{E}_{C_t}\left(V_0(\varphi+\zeta);\, V_0(\varphi+\zeta)\right)}$$

$$\begin{aligned} -\sum_{\substack{x\in\Lambda\\ y\in\Lambda}}\Big(&\tfrac{1}{2}g_0^2\, C_t(x,y)\,\varphi_x^3\varphi_y^3 + \tfrac{9}{4}g_0^2\, C_t(x,y)^2\,\varphi_x^2\varphi_y^2 \\ &+ 3g_0^2\, C_t(x,y)^3\varphi_x\varphi_y + \tfrac{1}{2}g_0\nu_0\, C_t(x,y)\,\varphi_x^3\varphi_y \\ &+ \tfrac{3}{2}g_0\nu_0\, C_t(x,y)^2\varphi_x^2 + \tfrac{1}{2}\nu_0^2\, C_t(x,y)\,\varphi_x\varphi_y \Big) \end{aligned}$$

Better to write: $\varphi_x^2 \varphi_y^2 = \varphi_x^4 + \varphi_x^2(\varphi_y^2 - \varphi_x^2)$

$$\varphi_x \varphi_y = \varphi_x^2 + \underbrace{\varphi_x(\varphi_y - \varphi_x)}$$

gradient terms: smaller variance

$\dot{C}_t = e^{t\Delta}$: $\dot{C}_t(x,x) \sim t^{-d/2}$ $\quad (t \gg 1)$

$\nabla^\alpha \dot{C}_t(x,x) \sim t^{-(d+\alpha)/2}$

Local potential approximation: drop gradient terms

Also dropping the φ^6 terms give:

$$(*_0) \quad g_t \approx g_0 - 9 g_0^2 \Big[\sum_{x \in \Lambda} C_{0,t}(x)^2\Big], \quad C_{s,t} = \int_s^t e^{u\Delta}\, du$$

$$\nu_t \approx \nu_0 + 3 g_0 C_{0,t}(0) - 3 g_0 \nu_0 \Big[\sum_x C_{0,t}(x)^2\Big]$$

$$+ 6 g_0^2 \Big[\sum_x C_{0,t}(x)^3\Big]$$

$\dot{C}_t = e^{t\Delta^\varepsilon}$:

$\sim \log \varepsilon^{-1}$ $\quad (d=2)$

$\sim \frac{1}{\varepsilon}$ $\quad (d=3)$

~ 1 $\quad (d=2)$

$\sim \log \varepsilon^{-1}$ $\quad (d=3)$

Applying the same instead from $s \to t$ gives:

$$(*_s) \quad g_t \approx g_s - 9 g_s^2 \Big[\sum_{x \in \Lambda} C_{s,t}(x)^2\Big]$$

$$\nu_t \approx \nu_s + 3 g_s C_{s,t}(0) - 3 g_s \nu_s \Big[\sum_x C_{s,t}(x)^2\Big]$$

$$+6g_s^2\Big[\sum_x C_{s,t}(x)^3\Big]$$

<u>Rk.</u> • For lattice φ^4 in $d=4$ (small $g>0$, conj. Ising):

$$\chi_{g,\nu_c(g)+\varepsilon} = \sum_x \langle \varphi_0 \varphi_x \rangle_{g,\nu_c(g)+\varepsilon} \sim C\varepsilon^{-1}(\log \varepsilon^{-1})^{3/9}$$

• For cont. φ^4 in $d=2,3$, the counterterms $a^\varepsilon(\lambda)$ are exactly the yellow terms.

<u>Lesson 1.</u> Let $d=4$ and $C_t(x) = \int_0^t e^{s\Delta}(0,x)\, ds$, (unit lattice)

$$\Rightarrow \sum_{x\in\Lambda} C_{0,t}(x)^2 \sim c\log t \qquad (t \gg 1).$$

$(*_0)$ gives $\quad g_t \approx g_0 - c\log t\, g_0^2$

$(*_s)$ gives $\quad g_t \approx g_s - c g_s^2 \qquad$ if $t=2s$

Iterating $(*_s)$ with $(s,t) = (2^j, 2^{j+1})$, $j\in\mathbb{N}$ gives

$$g_{j+1} = g_j - c g_j^2 \quad\Rightarrow\quad g_j \approx \frac{1}{g_0^{-1}+cj} \longrightarrow 0$$

$\Rightarrow$ Perturbation theory must be renormalised.

Lesson 2.

$$g \in [0,\infty) \longmapsto \int_{\mathbb{R}} e^{-\frac{1}{2}\varphi^2 - \frac{g}{4}\varphi^4} \, d\varphi$$

is not analytic at $g=0$ (∞ for $g<0$)

Related to this: $\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}\varphi^2} \frac{1}{n!} \frac{g^n}{4^n} \varphi^{4n} \, d\varphi$

$$\sim \frac{1}{n!} \left(\tfrac{1}{4} g\right)^n \underbrace{(4n-1)!!}_{\sim C^n (n!)^2}$$

$\Rightarrow$ Perturbation theory must be truncated carefully.
Large field problem.

Lesson 3.

If $V(\varphi) = \sum_{x \in \Lambda} V_x(\varphi_x)$ as above

$$\left| -\log \mathbb{E}_{C_t}\left(e^{-V_t(\varphi+\zeta)}\right) - \mathbb{E}_{C_t} V(\varphi+\zeta) \right| = O(V^2)$$

is not only nonuniform in φ, but also in $|\Lambda|$.

$\Rightarrow$ Perturbation theory must be localised.

$|\Lambda| \rightsquigarrow t^{d/2}$

Rk. Despite these problems, V_t is perfectly well-defined (and its exp. is integrable) in finite volume.

3.2. Fermionic fields

Defn. Let Ω be the Grassmann algebra with generators $\psi_x, \bar{\psi}_x$ where $x \in \Lambda$ (finite).

$$\psi_x \psi_y = -\psi_y \psi_x, \quad \psi_x \bar{\psi}_y = -\bar{\psi}_y \psi_x, \quad \bar{\psi}_x \bar{\psi}_y = -\bar{\psi}_y \bar{\psi}_x$$

Notation: $\bar{\psi}_x = \psi_{\bar{x}}$ with $\bar{x} \in \bar{\Lambda}$ (second copy).

In particular: $\psi_x^2 = \psi_x \psi_x = -\psi_x \psi_x = 0$.

Every $F \in \Omega$ is a polynomial in $\psi_x, \bar{\psi}_x$.

$$F = F(\psi, \bar{\psi}) \quad \text{(but careful about order).}$$

The Grassmann derivative is defined by

$$\partial_{\psi_x}(\underbrace{\psi_x F}_{\in \Omega}) = F \quad \text{if } F \text{ does not contain } \psi_x$$

The Grassmann integral is defined by

$$\int \partial_\psi \partial_{\bar{\psi}} F = \partial_{\psi_1} \partial_{\bar{\psi}_1} \cdots \partial_{\psi_N} \partial_{\bar{\psi}_N} F \in \mathbb{R}$$
if $\Lambda = \{1, \dots, N\}$

Simply extracts top degree coefficient.

The Grassmann Gaussian integral with covariance C (symmetric, pos.-det.) is

$$\mathbb{E}_C F = \frac{\int \partial_\psi \partial_{\bar\psi}\, e^{-(\psi, C^{-1}\bar\psi)}\, F}{\int \partial_\psi \partial_{\bar\psi}\, e^{-(\psi, C^{-1}\bar\psi)}}.$$

<u>Exercise</u>. $\int \partial_\psi \partial_{\bar\psi}\, e^{-(\psi, C^{-1}\bar\psi)} = \det C^{-1}$.

Fermionic Wick formula:

$$\mathbb{E}_C\Big(\prod_{i=1}^{p} \bar\psi_{x_i} \psi_{y_i}\Big) = \sum_{\sigma \in S_p} (-1)^\sigma \prod_{i=1}^{p} C_{x_i y_{\sigma(i)}}$$

$$= \det\big((C_{x_i y_j})_{i,j=1}^{p}\big).$$

<u>Example</u>. Let Δ be the graph Laplacian on Λ:

$$(\Delta f)_x = \sum_{y\sim x} (f_y - f_x)$$

Denote by Δ^0 its version with the first row and column removed. Then (matrix tree thm)

$$(\det \Delta^0) = \#\text{ spanning trees on } \Lambda$$
(Kirchhoff)

$$\mathbb{E}_C\Big(\prod_{i=1}^{p} (\bar\psi_{x_i} - \bar\psi_{y_i})(\psi_{x_i} - \psi_{y_i})\Big)$$
$$= \mathrm{Prob}\big(\text{edges } x_i y_i \in \mathrm{UST}\big).$$

Fermionic convolution: take Ω^2 two copies of Ω with generators $\psi_x, \bar\psi_x, \xi_x, \bar\xi_x$, $x \in \Lambda$. Define $\mathbb{E}_C$ to act only on $\xi_x, \bar\xi_x$.

$$\mathbb{E}_C F = (\det C) \int \partial_\xi \partial_{\bar\xi}\, e^{-(\xi, C^{-1}\bar\xi)} F \in \Omega(\psi, \bar\psi)$$

if $F \in \Omega(\psi, \bar\psi, \xi, \bar\xi)$.

Convolution $\mathbb{E}_C F(\bar\psi+\bar\xi, \psi+\xi) \in \Omega(\psi,\bar\psi)$
for $F \in \Omega(\psi,\bar\psi)$

Exercise: $\mathbb{E}_C F(\bar\psi+\bar\xi, \psi+\xi) = e^{\mathcal{L}_C} F$

$$\mathcal{L}_C = \sum_{x,y\in\Lambda} C(x,y) \frac{\partial}{\partial\psi_x}\frac{\partial}{\partial\bar\psi_y}.$$

$\Rightarrow$ Again $\mathbb{E}_{C_t} F$ satisfies a heat equation.

$$\mathbb{E}_C F(\bar\psi+\bar\xi, \psi+\xi)$$
$$= \mathbb{E}_{C_2}\mathbb{E}_{C_1} F(\bar\psi+\bar\xi_1+\bar\xi_2, \psi+\xi_1+\xi_2)$$

if $C = C_1 + C_2$

Upshot: Formally everything identical to the 'bosonic' case. Can define renormalised potential and so on.

<u>Lemma</u> (Gram's inequality). Let $(V, \langle\cdot,\cdot\rangle)$ be an inner product space. Then

$$|\det((\langle u_i, v_j\rangle)_{i,j=1}^p)| \le \prod_{i=1}^p \|u_i\| \, \|v_i\|$$

$\|u\| = \langle u,u\rangle^{1/2}$.

Cor. Assume C is pos.-def. Then no $p!$

$$\mathbb{E}_C\Big(\prod_{i=1}^p \bar\psi_{x_i}\psi_{y_i}\Big) \le \prod_{i=1}^p \big(C_{x_ix_i}C_{y_iy_i}\big)^{1/2}.$$

Proof. Set $\langle u,v\rangle = (u, Cv) = \sum_{x,y} u_x v_x C_{xy}$.

$\Rightarrow$ $\mathrm{LHS} = \det(C_{x_iy_j})_{i,j=1}^p$

$$= \det\big((\langle \delta_{x_i}, \delta_{y_j}\rangle)_{i,j=1}^p\big)$$

$$\le \prod_{i=1}^p \|\delta_{x_i}\|\,\|\delta_{y_i}\| = \mathrm{RHS}.$$

Rk. Compare this bound to the naive bound

$$|\det(C_{x_iy_j})| \le \sum_{\sigma\in S_p} \Big|(-1)^\sigma \prod_{i=1}^p C_{x_iy_{\sigma(i)}}\Big|$$

$$\le p!\,\max_{x_i,y_j} |C_{x_iy_j}|^p$$

$\Rightarrow$ Fermions behave like bounded random var.

4. The finite-range renormalisation group

4.1. Finite-range decomposition

Goal: $(-\Delta+m^2)^{-1} = \int_0^\infty \dot{C}_t \, dt$

$\dot{C}_t$ pos.-def.

$\dot{C}_t(x,y) = 0$ if $|x-y| > t$

approximately scale invariant.

Example. For any sufficiently integrable f,

$$\frac{1}{|x|^{d-2}} = C_f \int_0^\infty f(|x|/t)\, t^{-(d-1)} \, dt.$$

For $f(|\cdot|)$ pos. def with comp. support gives FRD.

Example.

$$\frac{1}{-\Delta} = \int_0^\infty f(t\sqrt{-\Delta})\, t \, dt$$

Take $f(s) = \int_{-1}^{1} \hat{f}(u)\, e^{isu} \, du \geq 0$

$\Rightarrow e^{i\sqrt{-\Delta}t}$ is the wave operator

finite prop. speed : $(e^{i\sqrt{-\Delta}t})(x,y) = 0$ if $|x-y| > t$.

Again gives finite range decomposition.

<u>Prop</u>. Let Δ be the Laplacian on $\mathbb{Z}^d$. Then

$$(-\Delta + m^2)^{-1} = \int_0^\infty \dot{C}_t \, dt$$

where the $\dot{C}_t$ are pos.-def matrices with

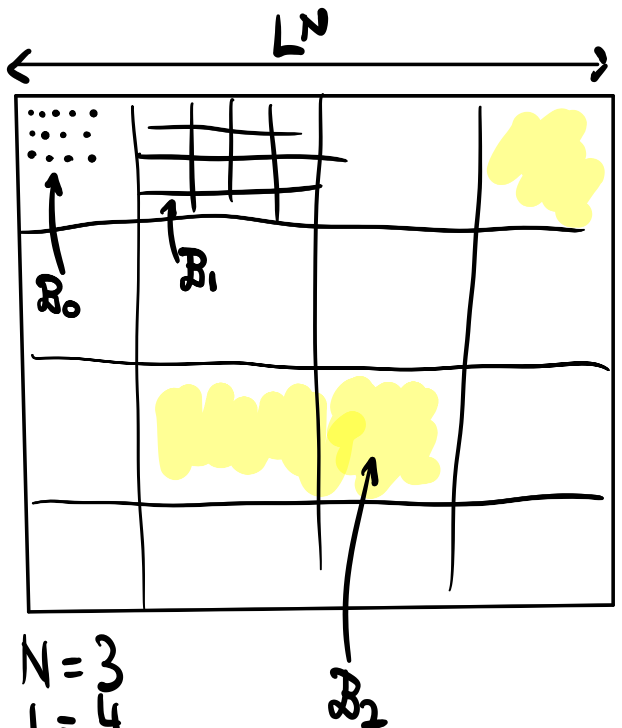
$$\dot{C}_t(x,y) = 0 \quad \text{if} \quad |x-y| > t$$

$$|\nabla^\alpha \dot{C}_t(x,y)| \leq c_\alpha t^{-(d-1)-|\alpha|}$$

Idea: Use discrete time wave equation.

4.2. Blocks and polymers

$\Lambda = \Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d, \quad L \geq 2, \quad N \to \infty$

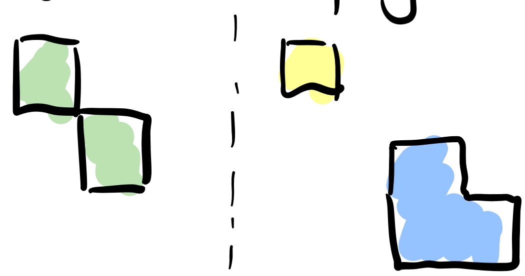


$N = 3$
$L = 4$

$\mathcal{B}_j = \{\text{blocks}\}$

$\mathcal{P}_j = \{\text{unions of blocks}\}$ "polymers"

$\mathcal{P}_j^c = \{\text{connected polymers}\}$



$$C_j(x,y) = \int_{\frac{1}{2}L^{j-1}}^{\frac{1}{2}L^j} \dot{C}_t(x,y)\, dt, \qquad j = 1, \dots, N-1$$

$$C_0(x,y) = \int_0^{\frac{1}{2}L^{-1}} \dot{C}_t(x,y)\, dt$$

$$\Rightarrow \quad (-\Delta + m^2)^{-1} = \sum_{j=0}^{N} C_j$$

$$C_N(x,y) = \int_{\frac{1}{2}L^{N-1}}^{\infty} \dot{C}_t(x,y)\, dt$$

$C_j(x,y) = 0$ if $|x-y| > \frac{1}{2} L^j$

$C_{j+1}(Lx, Ly) \approx L^{-(d-2)} C_j(x,y)$

$m^2 \ll L^{-2j}$
$j \gg 1$

Upshot: if $\varphi \sim \mathcal{N}(0, C_{j+1})$ then

$$\mathrm{Var}(\varphi_x)^{1/2} \approx L^{-\frac{d-2}{2}j}$$

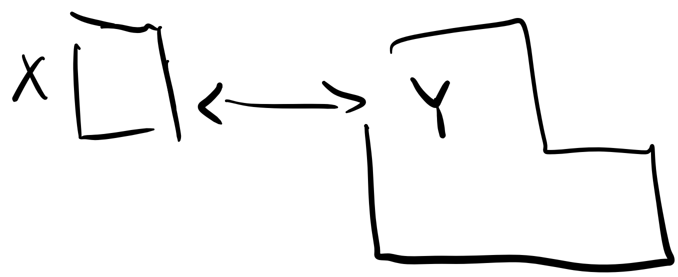
$$\mathrm{Var}((\nabla^\alpha \varphi)_x)^{1/2} \approx L^{-(\frac{d-2}{2}+\alpha)j}$$

<u>Factorisation</u>: if $F(X,\varphi)$ depends on $\varphi|_X$ $(X \in \mathcal{P}_{j+1})$ then

$$\mathbb{E}_{C_{j+1}}\left[F(X,\varphi+\zeta)\, F(Y,\varphi+\zeta)\right]$$

$$= \mathbb{E}_{C_{j+1}}\left[F(X,\varphi+\zeta)\right] \mathbb{E}_{C_{j+1}}\left[F(Y,\varphi+\zeta)\right]$$

if $X, Y \in \mathcal{P}_{j+1}$ that do not touch: $X \not\sim Y$.



<u>Proof</u>. Jointly Gaussian random variables are independent iff they are uncorrelated.

Analogous for Grassmann variables. Follows from

$$\mathbb{E}_C F(\psi+\xi, \bar\psi+\bar\xi) = e^{\mathcal{L}_C} F(\psi, \bar\psi).$$

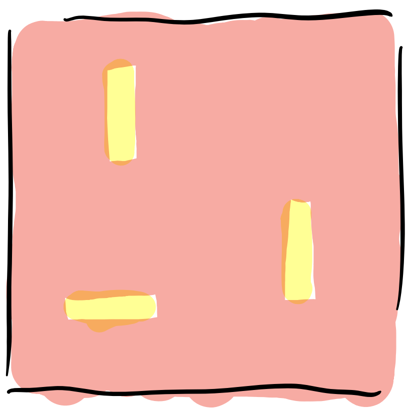
4.3. Coordinates

Instead of parametrising $V_j(\varphi)$ parametrise $e^{-V_j(\varphi)} = Z_j(\varphi)$ satisfying

$$Z_{j+1}(\varphi) = \mathbb{E}_{C_{j+1}} Z_j(\varphi + \zeta).$$

The parametrisation is

$$Z_j(\varphi) = e^{+E_j|\Lambda|}\Big[\sum_{X\in\mathcal{P}_j} e^{-V_j(\Lambda\setminus X,\varphi)} K_j(X,\varphi)\Big]$$



Λ

$$= e^{+E_j|\Lambda|}\Big[e^{-V_j(\Lambda,\varphi)} + \sum_{B\in\mathcal{B}_j} e^{-V_j(\Lambda\setminus B,\varphi)} K_j(B,\varphi) + \cdots$$

- The E_j are constants.
- The $V_j(\Lambda\setminus X,\varphi)$ are approximate local versions of the renormalised potential. E.g.

$$V_j(X,\varphi) = \sum_{x\in X}\left(\tfrac{1}{4} g_j \varphi_x^4 + \tfrac{1}{2}\nu_j \varphi_x^2\right).$$

- The $K_j(X,\varphi)$ are error terms (complicated).

<u>Factorisation properties:</u> Set $I(X) = e^{-V_j(X)}$

$$I(X) = \prod_{B \in \mathcal{B}_j(X)} I(B) = I^X$$: block factorisation

$$K(X) = \prod_{Y \in \mathrm{Comp}(X)} K(Y)$$: component factorisation

conn. comp.

Abstractly the representation is

$$Z_j = e^{+E_j|\Lambda|} \underbrace{\left(\sum_{X \in \mathcal{P}_j} I_j^{\Lambda \setminus X} K_j(X) \right)}_{(I_j \circ K_j)(\Lambda)}$$

<u>Example.</u> If K also factorises over blocks,

$$\sum_{X \in \mathcal{P}} I^{\Lambda \setminus X} \underbrace{K(X)}_{K^X = \prod_{B \in \mathcal{B}(X)} K(B)} = \prod_{B \in \mathcal{B}_j} (I(B) + K(B))$$

<u>Goal:</u> $(V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$ s.t.

$$\mathbb{E}_{C_{j+1}} Z_j(\varphi + \zeta) = Z_{j+1}(\varphi)$$

Example (Simple reblocking map). Assume

$$Z_j(\varphi) = \sum_{X\in P_j} I_j^{\Lambda\setminus X}(\varphi)\, K_j(X,\varphi)$$

$B\in \mathcal{B}_j$

Write $\varphi = \varphi' + \zeta$ and assume $\tilde{I}(B,\varphi')$ is given. Then

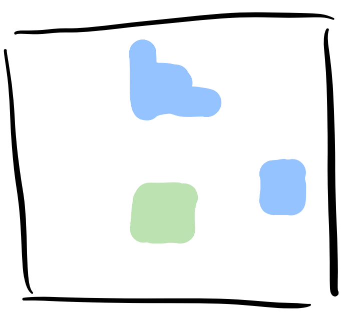
$$\mathbb{E}_{C_{j+1}} Z_j(\varphi'+\zeta) = \sum_{U\in P_{j+1}} \tilde{I}^{\Lambda\setminus U}(\varphi')\, \tilde{K}(U,\varphi')$$

where $\tilde{K}(U,\varphi')$ is given by a formula below.

Rk. $\tilde{I}$ is arbitrary! Take e.g. $\tilde{I}(B,\varphi') = I(B,\varphi')$.

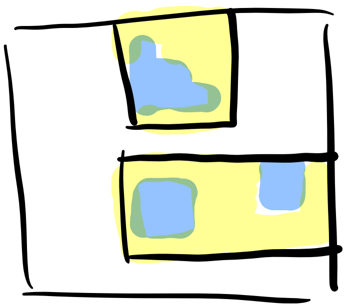
Proof. Start from

$$Z_j = \sum_{X\in P_j} I^{\Lambda\setminus X} K(X) = \sum_{X\in P_j} \underbrace{(\tilde{I} + \overset{I(B)-\tilde{I}(B)}{\delta I})^{\Lambda\setminus X}}_{\sum_{Y\in P_j(\Lambda\setminus X)} \tilde{I}^{\Lambda\setminus(X\cup Y)}(\delta I)^{Y}} K(X)$$



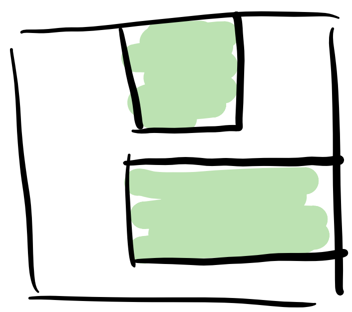
$$= \sum_{X\in P_j} \tilde{I}^{\Lambda\setminus X} \underbrace{\sum_{Y\in P_j(X)} (\delta I)^{X\setminus Y} K(Y)}_{\tilde{K}'(X)}$$

Now reblock:



$$Z_j = \sum_{U \in \mathcal{P}_{j+1}} \tilde{I}^{\Lambda \setminus U} \underbrace{\sum_{\substack{X \in \mathcal{P}_j(U) \\ \bar{X} = U}} \tilde{I}^{U \setminus X} \tilde{K}'(X)}_{\tilde{K}''(U)}$$

$$\Rightarrow \mathbb{E}_{C_{j+1}} Z_j(\varphi' + \zeta) = \sum_{U \in \mathcal{P}_{j+1}} \tilde{I}^{\Lambda \setminus U}(\varphi') \underbrace{\mathbb{E}_{C_{j+1}} \tilde{K}''(U, \varphi', \zeta)}_{\tilde{K}(U, \varphi')}$$



The finite range property implies that $\tilde{K}$ factors over components at scale $j+1$.

Defn. $K = (K(X))_{X \in \mathcal{P}_j^c}$ is called a polymer activity. It extends to $(K(X))_{X \in \mathcal{P}_j}$ by fact.

$$K(X) = \prod_{Y \in \mathrm{Comp}(X)} K(Y).$$

Goal: Define a good map

$$\Phi_{j+1} : (V_j, K_j) \mapsto (V_{j+1}, K_{j+1})$$

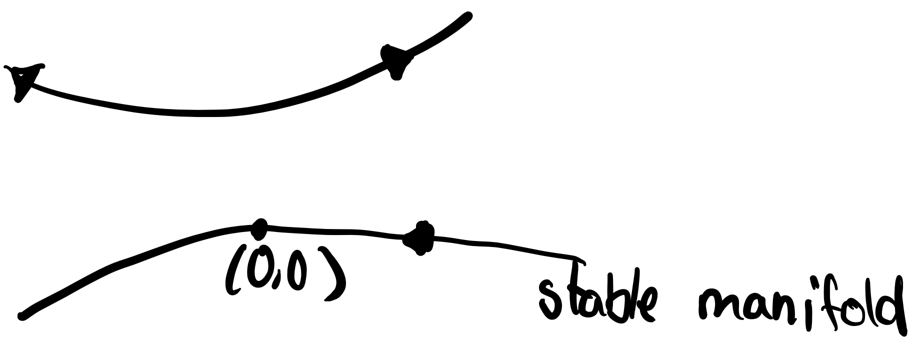
GFF fixed point

such that $\Phi_{j+1}(0,0) = (0,0)$ and

$$V_{j+1} = \text{Explicit}(V_j) + O(K_j)$$

$$K_{j+1} = \underbrace{\mathcal{L}_{j+1}(K_j)}_{\le \varkappa \|K_j\|} + O(V_j + K_j)^2 \qquad \varkappa < 1$$

for suitable spaces and norms.



4.4. Norms

$\mathcal{N}_j(X)$ = space of $K(X) = K(X,\varphi) = K(X, \varphi|_{X^+})$

<u>Example</u>. $\mathcal{N}_j(X) = L^\infty(\mathbb{R}^{X^*})$.

<u>Desired properties</u>:

- Submultiplicativity: $\|F(X)\,G(Y)\|_j \le \|F(X)\|_j \, \|G(Y)\|_j$ for $X \not\sim Y$.

- Contractivity of $\mathbb{E}$: $\|\mathbb{E}_{C_{j+1}} F(X, \cdot + \zeta)\|_{j+1} \le \|F(X)\|_j$

<u>Example</u>. (Fermions). $\mathcal{N}_j(X) = \operatorname{span}\{\psi_x, \bar{\psi}_x : x \in X\}$

$= \operatorname{span}\{\psi_x : x \in X \cup \bar{X}\}$

Every $F \in \mathcal{N}_j(X)$ can be written as

$$F = \sum_p \frac{1}{p!} \sum_{x_1,\dots,x_p \in X \cup \bar{X}} \underbrace{F_{x_1,\dots,x_p}}_{\in \mathbb{R} \text{ antisymmetric}} \psi_{x_1} \cdots \psi_{x_p}$$

$$= \sum_{z} \frac{1}{|z|!} F_z \psi^z$$

$z = (x_1, \dots, x_p) \subset X \cup \bar{X}$ finite sequence

Define $L^{-\frac{d-2}{2}j}$

$$\|F\|_h = \sum_{z} \frac{h^{|z|}}{|z|!} |F_z| \quad , \quad h > 0$$

<u>Exercise</u>: The Gram inequality implies that if $\max C_{xx} \le \ell^2$ then

$$\| \mathbb{E}_C F(\cdot + \xi, \cdot + \bar{\xi}) \|_h \le \|F\|_{h+\ell} .$$

<u>Sketch</u>. if z is a finite sequence,

$$(\psi + \xi)^z = \sum_{z' \subset z} \xi^{z \setminus z'} \psi^{z'} \operatorname{sign}(z, z')$$

$$\Rightarrow \mathbb{E}_C (\psi + \xi)^z = \sum_{z' \subset z} \underbrace{\left(\mathbb{E}_C \xi^{z \setminus z'} \right)}_{\le \ell^{|z \setminus z'|}} \psi^{z'} \operatorname{sign}(z, z')$$

Gram ineq. $\le \ell^{|z \setminus z'|}$

$$\rightarrow \mathbb{E}_C F(\psi + \xi, \bar{\psi} + \bar{\xi}) = \sum_{z} \frac{1}{|z|!} F_z \sum_{z' \subset z} \left(\mathbb{E}_C \xi^{z \setminus z'} \right) \psi^{z'} \operatorname{sign}$$

$$\Rightarrow \left(\mathbb{E}_C F(\psi+\xi, \bar\psi+\bar\xi)\right)_{Z'} = \sum_{Z\supset Z'} \frac{|Z'|!}{|Z|!} F_Z \left(\mathbb{E}_C\, \xi^{Z\setminus Z'}\right) \text{ sign}$$

$$\Rightarrow \|\mathbb{E}_C F(\cdot\,\xi, \cdot+\bar\xi)\|_h = \sum_{Z'} \frac{h^{|Z'|}}{|Z'|!} \left|(\cdots)_{Z'}\right|$$

$$\le \sum_Z \frac{|F_Z|}{|Z|!} \underbrace{\sum_{Z'\subset Z} h^{|Z'|} \ell^{|Z\setminus Z'|}}_{(h+\ell)^{|Z|}}$$

$$= \|F\|_{h+\ell}\,.$$

4.5. Reblocking

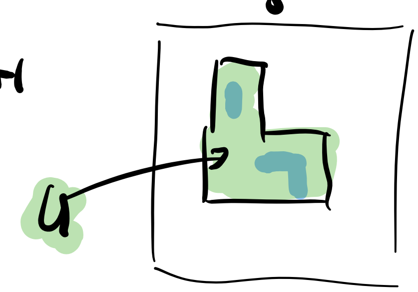
For a polymer activity $F = (F(X))_{X\in\mathcal{P}_j^c}$ define

$$\|F\|_j = \max_{X\in\mathcal{P}_j^c} A^{|X|_j} \|F(X)\|_j$$

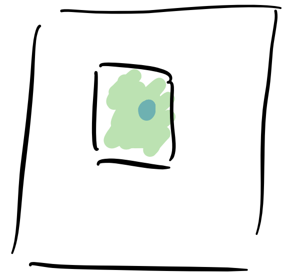
$A > 1$ parameter.

Prototype for reblocking: $\bar F(U) = \sum_{X\in\mathcal{P}_j^c} F(X)$

$U\in\mathcal{P}_{j+1}$



Example. $\mathcal{U} \in \mathcal{B}_{j+1}$

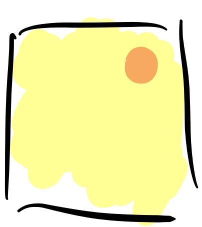


$$\underbrace{A^{|\mathcal{B}_{j+1}(\mathcal{U})|}}_{A} \; \underset{\uparrow E_{c_{j+1}}}{\|\tilde{F}(\mathcal{U})\|_{j+1}}$$

$$\le A \underbrace{\sum_{B \in \mathcal{B}_j(\mathcal{U})}}_{L^d} \underbrace{\|F(B)\|_j}_{\le A^{-1}\|F\|_j} + \cdots$$

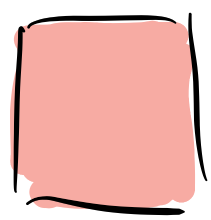
$$\le L^d \|F\|_j + \cdots$$

Upshot: Blocks are obstacle to contraction.



$|\mathcal{B}_j(X)| = 1$

$|\mathcal{B}_{j+1}(\bar{X})| = 1$



$|\mathcal{B}_j(X)| = L^d$

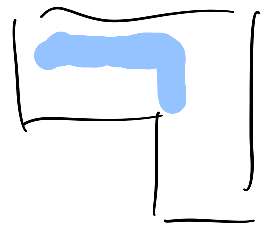
$|\mathcal{B}_{j+1}(\bar{X})| = 1$

Lemma. Let $L \ge 2^d + 1$. There is a geometric constant $\eta = \eta(d) > 0$ s.t. if $X \in \mathcal{P}_j$ and $|\mathcal{B}_j(X)| \ge 2^d + 1$

$$\Rightarrow \quad (1+\eta)\,|\mathcal{B}_{j+1}(\bar{X})| \le |\mathcal{B}_j(X)| \qquad (*)$$

$\underline{\text{Defn}}$. $\mathcal{S}_j = \{ X \in \mathcal{P}_j^c : |\mathcal{B}_j(X)| \leq 2^d \}$: small sets.

$\underline{\text{Example}}$. Suppose F satisfies $F(X) = 0$ if $X \in \mathcal{S}_j$.
Then



$$\|\bar{F}(U)\|_{j+1} \leq \sum_{\substack{X \in \mathcal{P}_j^c \\ \bar{X} = U}} \|F(X)\|_j$$

$$\leq \|F\|_j \sum_{\substack{X \in \mathcal{P}_j^c \\ \bar{X} = U}} A^{-|\mathcal{B}_j(X)|}$$

$$\leq \|F\|_j \, A^{-(1+\eta)|\mathcal{B}_{j+1}(U)|} \underbrace{\sum_{\substack{X \in \mathcal{P}_j^c \\ \bar{X} = U}} 1}_{2^{|\mathcal{B}_j(U)|} = 2^{L^d |\mathcal{B}_{j+1}(U)|}}$$

$$\leq \|F\|_j \, A^{-|\mathcal{B}_{j+1}(U)|} \underbrace{\left(A^{-\eta} 2^{L^d}\right)}_{\leq A^{-\frac{1}{2}\eta} \ll 1}{}^{|\mathcal{B}_{j+1}(U)|}$$

$$\ll \|F\|_j \, A^{-|\mathcal{B}_{j+1}(U)|}$$

$$\Rightarrow \|\bar{F}\|_{j+1} \ll \|F\|_j$$

Upshot: Large sets contract for geometric reasons.

4.6. Extraction of local parts

Suppose: $K(X) = \mathrm{Loc}_X K(X) + (1-\mathrm{Loc}_X) K(X)$

$X \in \mathcal{S}_j$

where $\|(1-\mathrm{Loc}_X) K(X)\|_{j+1} \ll L^{-d} \|K\|_j$

$\|\mathrm{Loc}_X K(X)\|_{j+1} \leq C \|K\|_j$

In practice, $\mathrm{Loc}_X K(X)$ is 'explicit' of the form needed be considered as part of V. Think:

$$\mathrm{Loc}_X K(X) = \sum_{x \in X} \left(\frac{1}{4} g_K \varphi_x^4 + \frac{1}{2} \nu_K \varphi_x^2 + \frac{1}{2} z_K (\nabla\varphi)_x^2 \right)$$

with g_K and ν_K determined by matching Taylor expansions w.r.t. constant φ.

Locality: $\mathrm{Loc}_X K(X) = \sum_{B \in \mathcal{B}_j(X)} \mathrm{Loc}_{X,B} K(X)$

$$\mathrm{Loc}_{X,B} K(X) = \sum_{x \in B} \left(\frac{1}{4} g_K \varphi_x^4 + \cdots \right)$$

Want to change coordinates $(V, K) \to (V', K')$ to move $\mathrm{Loc}_X K(X)$ from K to V'.

Difficulty: $V = V(B)$ indexed only by $B \in \mathcal{B}_j$
but want to move $X \in \mathcal{S}_j \supsetneq \mathcal{B}_j$.
Possible by locality.

Assume first we only need to cancel $X = B \in \mathcal{B}_j$.
Then the simple reblocking map suffices.

<u>Example</u> (simple reblocking cont'd). There is a choice of map $(V_j, K_j) \longrightarrow (V_{j+1}, K_{j+1})$,

$$V_{j+1}(B) = \mathbb{E}_{C_{j+1}}\big(V_j(B, \cdot + \zeta) - \underbrace{Q_j(B, \cdot + \zeta)}_{O(K)}\big)$$

$$K_{j+1}(U) = \mathcal{L}_{j+1}(U) + O(V+K)^2$$

where

$$\mathcal{L}_{j+1}(U) = \sum_{X \in \mathcal{P}_j^c} e^{-V_{j+1}(U \setminus X)} \mathbb{E}_{C_{j+1}}\big(K_j(X) + (\delta I)^X\big)$$

$U \in \mathcal{P}_{j+1}^c$

$$\prod_{B \in \mathcal{B}_j(X)} \underbrace{\left(e^{-V_j(B, \varphi+\zeta)} - e^{-V_{j+1}(B, \varphi)}\right)}$$

$\Rightarrow -\mathbb{E}_{C_{j+1}} Q(B)$ if $X = B$
$+ O(V+K)^2$

Choose: $Q(B) = \mathrm{Loc}_B K_j(B)$.

$\Rightarrow$ $K(B)$ replaced by $(1-\mathrm{Loc}_B)K(B)$.

What about $X \in \mathfrak{Z}_j \setminus \mathcal{B}_j$?

$$K(X) = \underbrace{J(X)}_{\mathrm{Loc}_X K(X)} + \big(K(X) - J(X)\big)$$

Locality: $$J(X) = \sum_{B \in \mathcal{B}_j(X)} \underbrace{J(B,X)}_{\mathrm{Loc}_{B,X} K(X)}$$

Define $J(B,B) = -\sum_{X \in \mathfrak{Z}_j \setminus \mathcal{B}_j :\, X \supset B} J(B,X)$ so that

$$\sum_{\substack{X \in \mathfrak{Z}_j \\ X \supset B}} J(B,X) = 0.$$

<u>Example</u> There is a choice $(V_j, K_j) \to (V_{j+1}, K_{j+1})$,

$$V_{j+1}(B) = \mathbb{E}_{C_{j+1}}\big(V_j(B, \cdot + \zeta) - \underbrace{Q_j(B, \cdot + \zeta)}_{O(K)}\big)$$

$$K_{j+1}(U) = \mathcal{L}_{j+1}(U) + O(V+K)^2$$

where

$$\mathcal{L}_{j+1}(U) = \sum_{X \in \mathcal{P}_j^c} e^{-V_{j+1}(U \setminus X)}\, \mathbb{E}_{C_{j+1}} \Big(K_j(X) - \sum_{B \in \mathcal{B}_j(X)} J(B,X) + (\delta I)^X \Big)$$

$U \in \mathcal{P}_{j+1}^c$

$$(\delta I)^X = \prod_{B \in \mathcal{B}_j(X)} \left(e^{-V_j(B,\varphi+\zeta)} - e^{-V_{j+1}(B,\varphi)} \right)$$

provided $Q(B) = O(K)$ and $J(B,X) = O(K)$ and

$$\sum_X J(B,X) = 0.$$

Upshot:

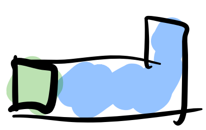
for $X \in \mathcal{S}_j \setminus \mathcal{B}_j$: $\quad K_j(X) - \sum_B J(B,X) = (1 - \mathrm{Loc}_X)\, K(X)$

good!

for $X \in \mathcal{B}_j$: $\quad \mathbb{E}_{C_{j+1}} \Big[K_j(B) - J(B,B) + \underbrace{(\delta I)^B}_{} - Q(B) \Big]$

$$= \mathbb{E}_{C_{j+1}} \big((1 - \mathrm{Loc}_B) K(B) \big)$$

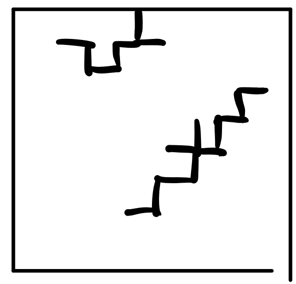
if $Q(B) = -J(B,B) + \mathrm{Loc}_B K(B)$



$$= \sum_{\substack{X \in \mathcal{S}_j \\ X \supset B}} \mathrm{Loc}_{X,B}\, K(X)$$

Upshot: expanding part cancelled from all small sets.

5. Example: 0-state Potts model



$\Lambda_N = \mathbb{Z}^d / L^N \mathbb{Z}^d$ torus, $d \geq 3$

$$\mathbb{P}_\beta(F) = \frac{1}{Z_\beta} \beta^{|E(F)|} 1_{F \text{ is forest}}$$

$(\beta > 0)$

Percolation model of random forests

Thm. If $d \geq 3$, $\beta \geq \beta_0$, $L \geq L_0$, $N \to \infty$,

$$\mathbb{P}_\beta(0 \leftrightarrow x) = \theta(\beta)^2 + \frac{c(\beta)}{\beta |x|^{d-2}} + O(\beta^{-1}|x|^{-(d-2)-\varepsilon}) + O(\beta^{-1} L^{-\kappa N}),$$

$$\theta(\beta) = 1 - O(1/\beta)$$

Prop. (CJSSS)

$$\mathbb{P}_\beta(0 \leftrightarrow x) \propto \int \prod_{y \in \Lambda} \partial_{\psi_y} \partial_{\bar\psi_y} \frac{\bar\psi_0 \psi_x}{\beta} \exp\Big(- \sum_{x \in \Lambda} (\nabla \psi)_x (\nabla \bar\psi)_x - \frac{1}{\beta} \sum_{x \in \Lambda} \psi_x \bar\psi_x - \frac{1}{\beta} \sum_{x \in \Lambda} \psi_x \bar\psi_x (\nabla \psi_x)(\nabla \bar\psi_x) \Big).$$

Looks like fermionic φ^4 model, but careful: coupling $1/\beta$ appears in two places.
There are nonperturbative (all orders) cancellations due to hidden symmetry. ($OSp(1|2)$)

Two options to deal with symmetry in RG:

- Preserve symmetry along RG flow.
- Derive nonperturbative cancellations from Ward identities.

Forest model $\equiv$ $OSp(1|2)$ NLSM

ξ_x, η_x, $x \in \Lambda$ Grassmann

$z_x = \sqrt{1 - 2\xi_x \eta_x} = 1 - \xi_x \eta_x$ even (commuting)

$\Rightarrow u_x = (\xi_x, \eta_x, z_x)$ supervector

$$u_x \cdot u_x = -1$$

$$u_x \cdot u_y = \underbrace{\eta_x \xi_y + \eta_y \xi_x}_{} - z_x z_y$$

$OSp(1|2)$ inner prod.

OSp(1|2) invariant superintegral:

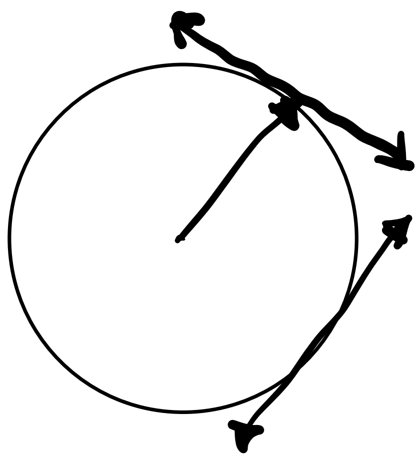
$$\int \prod_{x\in\Lambda} \partial_{\eta_x}\partial_{\xi_x} \frac{1}{z_x}(\cdot) = \int_{H^{0|2}} (\cdot)$$

$$\langle F\rangle_{\beta,h} \propto \int \prod_{x\in\Lambda} \partial_{\eta_x}\partial_{\xi_x} \frac{1}{z_x}\, e^{-\frac{\beta}{2}(u,-\Delta u) - \sum_{x\in\Lambda} h z_x} F$$

all inner prod. w.r.t. "."

$$\sum_{xy} (u_x-u_y)\cdot(u_x-u_y)$$

<u>Prop.</u> $P_\beta(0 \leftrightarrow x) = -\langle u_0\cdot u_x\rangle_{\beta,0}$



$\bar\psi_x = \sqrt{\beta}\,\xi_x$, $\psi_x = \sqrt{\beta}\,\eta_x$

<u>Ward identies:</u>

$$-\langle u_0\cdot u_x\rangle_{\beta,0} = -\langle z_0 z_x\rangle_{\beta,0}$$
$$= \langle \xi_0 \eta_x\rangle_{\beta,0}$$
$$= 1 - \langle \xi_0\eta_0\xi_x\eta_x\rangle_{\beta,0}$$

$$\sum_{x\in\Lambda} \langle \xi_0\eta_x\rangle_{\beta,h} = \frac{\langle z_0\rangle_{\beta,h}}{h}.$$

result from OSp(1|2) symmetry.

In terms of $\psi, \bar\psi$:

$$\langle z_0 \rangle = \langle 1 - \xi_0 \eta_0 \rangle = 1 - \frac{1}{\beta} \langle \bar\psi_0 \psi_0 \rangle$$

$$\langle \xi_x \eta_y \rangle = \frac{1}{\beta} \langle \bar\psi_x \psi_y \rangle$$

<u>General form:</u>

$$\exp\left(- \left((\nabla\psi)(\nabla\bar\psi) + m^2 \psi\bar\psi \right) \right)$$

$$\times \exp\left(- \sum_{x \in \Lambda} \underbrace{\left(z_0 (\nabla\psi)_x (\nabla\bar\psi)_x + a_0 \psi_x \bar\psi_x + b_0 \psi_x \bar\psi_x (\nabla\psi)_x (\nabla\bar\psi)_x \right)}_{V_0(\Lambda)} \right)$$

<u>Thm</u>. Given $b_0 > 0$ small, $m^2 > 0$, there exist

$$z_0^c(b_0, m^2) = O(b_0)$$

$$a_0^c(b_0, m^2) = O(b_0)$$

stable manifold

as well as

$$\lambda(b_0, m^2) = 1 - O(b_0)$$

$$\tilde a_N(b_0, m^2) = \cdots$$

$$\tilde u_N(b_0, m^2) = \cdots$$

s.t.

$$\langle \bar{\psi}_0 \psi_0 \rangle = (-\Delta^{\mathbb{Z}^d} + m^2)^{-1}(0,0)$$
$$+ \frac{\lambda\left(m^{-2} + O(L^{+2N})\right)|\Lambda_N|^{-1}}{1+\tilde{u}_N} + \text{error}$$

$\to 0$ if $N \to \infty$
$m^2 > 0$ fixed

$$\sum_x \langle \psi_0 \bar{\psi}_x \rangle = \frac{1}{m^2} - \frac{1}{m^4} \frac{\tilde{a}_N}{1+\tilde{u}_N}$$

⋮