Roland Bauerschmidt ⟨rb812@cam.ac.uk⟩ *Lent 2021*

Renewal theory and applications.

1. A cinema shows a single movie at a time, and changes the programme after a number of months that is uniformly distributed on the interval $[0, 3]$. Purchasing the rights to a new movie costs £10,000. How much have they spent after 5 years?

Clara goes to the movies once every two months, but only enters the cinema if the movie is less than one month old. In 5 years, how many movies has she seen? What if each programme lasts for an exponentially distributed period with mean 1.5 months?

2. Suppose that the lifetime of a car is a random variable with density function f. Piet buys a new car as soon as either the old one breaks down or it reaches the age of T years. A new car costs c, while an additional a is incurred if the car breaks down before T. In the long-run, how much does Piet spend on his cars per year?

3. Let $(X_t : t \geq 0)$ be a renewal process with interarrival times having the Gamma $(2, \lambda)$ distribution. Determine the limiting excess-life distribution. Determine also the expected number of renewals up to time t.

4. A barber takes an exponentially distributed amount of time, with mean 20 minutes, to complete a haircut. Customers arrive at rate 2 per hour, but leave if both chairs in the waiting room are full. Devise a Markov chain model on the state space $\{0, 1, 2, 3\}$. Using Little's formula, find the average waiting time in the system (including service time) of a customer.

5. A PC keyboard has 100 different keys and a monkey is tapping them (uniformly) at random. Assuming no power failure, use the elementary renewal theorem to find the expected number of keys tapped until the first appearance of the sequence of fourteen characters 'W. Shakespeare'.

Answer the same question for the sequence 'omo'.

Spatial Poisson processes.

6. In a certain town at time $t = 0$ there are no bears. Brown bears and grizzly bears arrive as independent Poisson processes B and G with respective intensities β and γ.

(a) Show that the first bear is brown with probability $\beta/(\beta + \gamma)$.

(b) Find the probability that between two consecutive brown bears, there arrive exactly r grizzly bears.

(c) Given that $B(1) = 1$, find the expected value of the time at which the first bear arrived.

7. **Campbell–Hardy theorem.** Let Π be the points of a non-homogeneous Poisson process on $\mathbb{R}^d$ with intensity function λ. Let $S = \sum_{x \in \Pi} g(x)$ where g is a smooth function which we assume for convenience to be non-negative. Show that $\mathbb{E}(S) = \int_{\mathbb{R}^d} g(u)\lambda(u)\, du$ and $\operatorname{var}(S) = \int_{\mathbb{R}^d} g(u)^2 \lambda(u)\, du$, provided these integrals converge.

8. Let Π be a Poisson process with constant intensity λ on the surface of the sphere of $\mathbb{R}^3$ with radius 1. Let P be the process given by the (X, Y) coordinates of the points projected on a plane passing through the centre of the sphere. Show that P is a Poisson process, and find its intensity function.

9. Repeat the previous exercise, when Π is a homogeneous Poisson process on the ball $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq 1\}$.

Note for supervisors: The example sheets are essentially the same as those from last year and solutions are available.