Lent 2026 **Analysis of Functions – Example Sheet 4** P.P. Varjú

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A supervisor sheet is available on Moodle or by email.

1. Let $G(x) = (-4\pi|x|)^{-1}$ for $x \in \mathbf{R}^3$.

(1) Prove that $G \in L^1_{loc}(\mathbf{R}^d)$.

(2) Prove $\Delta G(x) = 0$ for $x \neq 0$. Let $h \in \mathcal{D}(\mathbf{R}^3)$ and $r > 0$. Prove that

$$\int_{|x|>r} G\Delta h dx = \int_{|x|=r} G\frac{\partial h}{\partial \mathbf{n}} dx - \int_{|x|=r} \frac{\partial G}{\partial \mathbf{n}} h dx,$$

where $\mathbf{n}$ is the unit normal vector on the surface $|x| = r$ pointing towards the origin, that is outward from the domain $|x| > r$.

(3) Prove $\Delta T_G = \delta_0$.

2. Suppose that $\Omega \subset \mathbf{R}^n$ is open and bounded, let $f \in C_c^\infty(\Omega)$, and suppose $0 < \varepsilon < 1$.

(1) Show that $\int_\Omega (|f|^2 + \varepsilon)^{\frac{p}{2}} dx \to \|f\|_p^p$ as $\varepsilon \to 0$.

(2) By considering $\int_\Omega (|f|^2 + \varepsilon)^{\frac{p}{2}} dx = \int_\Omega \left(\frac{1}{n}\text{div } x\right) (|f|^2 + \varepsilon)^{\frac{p}{2}} dx$, or otherwise, show that there exists a constant C, depending on Ω, p but not on f, such that $\|f\|_p \leq C\|\nabla f\|_p$.

3. Suppose $u : \mathcal{S}(\mathbf{R}^n) \to \mathbf{C}$ is a linear map. Show that u is continuous if and only if there exist $N, k \in \mathbf{Z}_{>0}$ and $C > 0$ such that:

$$|u(h)| \leq C \sup_{x\in\mathbf{R}^n;|\alpha|\leq k} |(1 + |x|)^N D^\alpha h(x)|, \quad \text{for all } h \in \mathcal{S}(\mathbf{R}^n).$$

In particular, all tempered distributions are of finite order.

4. Let $s \in \mathbf{R}$.

(1) Show that $\mathcal{S}$ is a dense subset of $H^s(\mathbf{R}^n)$.

(2) Find a condition on s such that $\delta_x \in H^s(\mathbf{R}^n)$.

(3) Show that $H^t(\mathbf{R}^n)$ is continuously embedded in $H^s(\mathbf{R}^n)$ for $s < t$.

(4) Show that the derivative D^α is a bounded linear map from $H^{s+k}(\mathbf{R}^n)$ into $H^s(\mathbf{R}^n)$, where $k = |\alpha|$.

(5*) Show that the pairing $\langle , \rangle : H^{-s}(\mathbf{R}^n) \times H^s(\mathbf{R}^n) \to \mathbf{C}$, which acts on $f \in H^{-s}(\mathbf{R}^n), g \in H^s(\mathbf{R}^n)$ by

$$\langle f, g\rangle = \int_{\mathbf{R}^n} \widehat{f}(\xi)\widehat{g}(\xi) d\xi$$

is well defined, and show that the map $g \mapsto \langle f, g\rangle$ is a bounded linear functional on $H^s(\mathbf{R}^n)$. Deduce that $H^s(\mathbf{R}^n)'$ may be identified with $H^{-s}(\mathbf{R}^n)$.

5.

(1) Let $f \in \mathcal{S}(\mathbf{R}^d)$ and let $h \in C_c^\infty(\mathbf{R}^d)$ with $h(0) = 1$. Let $h_n(x) = h(x/n)$ for $n \in \mathbf{Z}_{\geq 1}$. Prove that $fh_n \to f$ in the topology of $H^s(\mathbf{R}^d)$ for all $s \in \mathbf{R}$.

(2) Conclude that $\mathcal{D}(\mathbf{R}^d)$ is dense in $H^s(\mathbf{R}^d)$ and $H^1_0(\mathbf{R}^d) = H^1(\mathbf{R}^d)$.

6. Fix some $f \in \mathcal{S}(\mathbf{R})$ and let $f_r(x) = f(x/r)$.

(1) Prove that $\|f_r\|_{H^{1/2}}$ is bounded for $r \in (0, 1]$.

(2) Prove that $f_r \rightharpoonup 0$ as $r \to 0$ in the Hilbert space $H^{1/2}(\mathbf{R})$.

(3) By considering functions of the form $\sum_{j=1}^{n} a_j f_{r_j}$ for suitable a_j and r_j, or otherwise, show that for all $\varepsilon > 0$, there is a function $g \in H^{1/2}(\mathbf{R})$ with $\|g\|_\infty = 1$ and $\|g\|_{H^{1/2}} < \varepsilon$.

7.

(1) Suppose $s = \frac{n}{2} + \gamma$ for some $0 < \gamma < 1$. Show that there exists a constant $C_{n,\gamma} > 0$ such that for all $x, y \in \mathbf{R}^n$, we have

$$\int_{\mathbf{R}^n} \frac{|e^{2\pi i\langle x,\xi\rangle} - e^{2\pi i\langle y,\xi\rangle}|^2}{|\xi|^{2s}} d\xi \leq C_{n,\gamma}|x-y|^{2\gamma}.$$

(2) Show that if $s = \frac{n}{2} + k + \gamma$ for some $k \in \mathbf{Z}_{\geq 0}$, $0 < \gamma < 1$, then

$$H^s(\mathbf{R}^n) \subset C^{k,\gamma}(\mathbf{R}^n).$$

8. Fix $s \in \mathbf{R}$, and suppose that $f \in H^s(\mathbf{R}^n)$.

(1) Show that there exists a unique $u \in \mathcal{S}'(\mathbf{R}^n)$ which solves:

$$\Delta^2 u + u = f,$$

and express $\widehat{u}$ in terms of $\widehat{f}$.

(2) Show further that $u \in H^{s+4}(\mathbf{R}^n)$ and there exists $C > 0$ such that $\|u\|_{H^{s+4}} \leq C\|f\|_{H^s}$.

(3) Give a condition on s that implies the equation holds in the sense of classical derivatives (possibly after redefining u, f on a set of measure zero)?

9. [The trace theorem] Assume $s > \frac{1}{2}$ and suppose $u \in \mathcal{S}(\mathbf{R}^n)$. Define $Tu \in \mathcal{S}(\mathbf{R}^{n-1})$ by:

$$Tu(x') = u(x', 0), \qquad x' \in \mathbf{R}^{n-1}.$$

(1) Show that if $\xi' \in \mathbf{R}^{n-1}$:

$$\widehat{Tu}(\xi') = \int_{\mathbf{R}} \widehat{u}(\xi', \xi_n) d\xi_n.$$

(2) Deduce that

$$|\widehat{Tu}(\xi')|^2 \leq \left(\int_{\mathbf{R}} (1+|\xi|^2)^s |\widehat{u}(\xi', \xi_n)|^2 d\xi_n\right)\left(\int_{\mathbf{R}} \frac{d\xi_n}{(1+|\xi|^2)^s}\right),$$

where $\xi = (\xi', \xi_n)$.

(3) Show that

$$\int (1+|\xi|^2)^{-s} d\xi_n \leq C(s)(1+|\xi'|^2)^{-s+1/2}$$

for some constant $C(s)$ depending only on s, and hence

$$\|Tu\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{n-1})} \leq C(s)\|u\|_{H^s(\mathbf{R}^n)}.$$

(4) Conclude that T extends to a bounded linear operator $T : H^s(\mathbf{R}^n) \to H^{s-\frac{1}{2}}(\mathbf{R}^{n-1})$.

(5*) Suppose $v \in \mathcal{S}(\mathbf{R}^{n-1})$ and let $\phi \in C_c^\infty(\mathbf{R})$ satisfy $\int_{\mathbf{R}} \phi(t)dt = 1$. Define u through its Fourier transform by:

$$\widehat{u}(\xi', \xi_n) = \frac{\widehat{v}(\xi')}{\sqrt{1+|\xi'|^2}} \phi\left(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}\right).$$

Show that there exists a constant $C > 0$ such that:

$$\|u\|_{H^s(\mathbf{R}^n)} \leq C\|v\|_{H^{s-\frac{1}{2}}(\mathbf{R}^{n-1})}$$

and that $Tu = v$. Conclude that $T : H^s(\mathbf{R}^n) \to H^{s-\frac{1}{2}}(\mathbf{R}^{n-1})$ is surjective.

Remark: This question asks for the proof of a different version of the trace theorem compared to what was stated in the lectures. In the lectures, the trace operator maps $H^s(\Omega) \to H^{s-1/2}(\partial\Omega)$ where $\Omega = \mathbf{R}_{>0} \times \mathbf{R}^{n-1}$. Given the definition of $H^s(\Omega)$ and the norm on it, it is enough to show that the operator constructed in this question satisfies $Tu = 0$ for all $u \in H^s(\mathbf{R}^d)$ such that $u|_\Omega \equiv 0$. To this end, it is enough to show that such functions u may be approximated by Schwartz functions that vanish on Ω. The details of this is an optional exercise.

10.* For two probability measures μ, ν on $\mathbf{R}$ their convolution $\mu * \nu$ is defined via

$$\mu * \nu(\varphi) = \int_{\mathbf{R}} \varphi(x+y) d\mu(x) d\nu(y), \varphi \in \mathcal{S}(\mathbf{R}).$$

(1) Show that $\mu * \nu$ defines an element $T_{\mu*\nu}$ of $\mathcal{S}'(\mathbf{R}^n)$ and find $\widehat{T_{\mu*\nu}}$.

(2) Let μ be a probability measure on $\mathbf{R}$ such that $\int_{\mathbf{R}} x d\mu(x) = 0, \int_{\mathbf{R}} x^2 d\mu(x) = 1$. Denote by $\mu^{*k} = \mu * \cdots * \mu$ the k-fold convolution (with $k \in \mathbf{Z}_{\geq 1}$ factors) and define a Borel measure on $\mathbf{R}$

$$\lambda_k(A) = \mu^{*k}\big(\sqrt{k}A\big), \quad \text{where } \sqrt{k}A = \{\sqrt{k}x : x \in A\}, \ A \subseteq \mathbb{R} \text{ Borel}.$$

Show that the corresponding distributions T_{λ_k} converge in $\mathcal{S}'(\mathbf{R})$ as $k \to \infty$ and identify the limit.

(3) Show that T_{λ_k} defines a sequence in $H^{-s}(\mathbf{R})$ whenever $s > 1/2$ and that it converges in this space. What if $s \leq 1/2$?