EXAMPLE SHEET 4 FOR DIOPHANTINE ANALYSIS, MICHAELMAS 2024

PÉTER VARJÚ

About this example sheet:

- Please send comments and corrections to pv270@dpmms.cam.ac.uk.
- Please submit your solutions of Questions 2 and 4, by Monday 20 January at 13:00.
- The purpose of this example sheet is to complement the material of the lectures. The level of difficulty of the problems varies considerably (in a non-monotone fashion), and they are not intended to be mock exam questions.

1. The goal of this question is to prove a special case of Theorem 47 with an effective constant following an argument of Runge.

Let $d \in \mathbf{Z}_{\geq 3}$, and let $F \in \mathbf{Z}[X, Y]$ be a homogeneous polynomial of degree d without repeated factors, which is *reducible* in $\mathbf{Z}[X, Y]$. Let $G \in \mathbf{Z}[X, Y]$ be a polynomial of degree at most $d - 1$. Suppose that $P = F - G$ is irreducible in $\mathbf{Z}[X, Y]$, and for simplicity $Y \nmid F$.

In what follows you may use without proof the following statement. With

$$F(X, Y) = (X - \alpha_1 Y) \cdots (X - \alpha_d Y)$$

there is some number $R \in \mathbf{R}_{>0}$ and for each $j = 1, \ldots, d$, there is a series

$$\varphi_j(Y) = \alpha_j Y + \beta_{j,0} + \beta_{j,1} Y^{-1} + \ldots$$

such that the following holds. We have $\beta_{j,n} \in \mathbf{Q}(\alpha_j)$ for all $j = 1, \ldots, d$ and $n \in \mathbf{Z}_{\geq 0}$. The series φ_j converges for $Y \in \mathbf{C}$, $|Y| > R$. Furthermore, for all $x, y \in \mathbf{C}$ with $|y| > R$ and $P(x, y) = 0$, we have $x = \varphi_j(y)$ for some j. (That is, $(\varphi_j(y), y)$ for $j = 1, \ldots, d$ are parametrizations of the complex solutions of $P(x, y) = 0$ for large $|y|$.)

(a) Prove that for each $j = 1, \ldots, d$, there is a polynomial $Q(X, Y) \in \mathbf{Z}[X, Y]$ of degree at most $[\mathbf{Q}(\alpha_j) : \mathbf{Q}]$ in the X variable such that

$$Q_j(\varphi_j(Y), Y) = \gamma_{j,1} Y^{-1} + \gamma_{j,2} Y^{-2} + \ldots$$

for some $\gamma_{j,n} \in \mathbf{C}$.

(b) Prove that for all $x, y \in \mathbf{Z}$ with $|y| > R$ and $P(x, y) = 0$, we have $Q_j(x, y) = 0$ for some $j = 1, \ldots, d$.

(c) Prove that for all $x, y \in \mathbf{Z}$ with $P(x, y) = 0$, we have $|x| + |y| < C$ for some effective constant depending only on P.

Comment: The fact that

$$X = \alpha_j Y + \beta_{j,0} + \beta_{j,1} Y^{-1} + \ldots + \beta_{j,n} Y^{-n} + O(Y^{-n-1})$$

for large solutions of $P(X, Y) = 0$ for some j and $\beta_{j,k} \in \mathbf{Q}(\alpha_j)$, where n can be taken arbitrarily large may be proved following the proof of Lemma 49. This weaker fact is, in fact, enough to solve the question. The convergence of the resulting series requires another proof. One way to do this is contained in Lemmata 4.4 and 4.5 in the book of Masser (reference [9] in the notes.)

2. In this question, you may use the following version of the subspace theorem.

Theorem. *Let K be a number field and let $n \geq 2$. Let $S \subset M_K$ be finite containing all the infinite places. For each $v \in S$, let $L_1^{(v)}, \ldots, L_n^{(v)}$ be linearly independent linear forms in n variables with algebraic coefficients. Fix an extension of $|\cdot|_v$ to a number field that contains all the coefficients of the linear forms for each $v \in S$. Then for all $\varepsilon > 0$, the solutions $(x_1, \ldots, x_n) \in \mathcal{O}_K^n$ of*

$$\prod_{v \in S} \prod_{j=1}^{n} |L_j^{(v)}(x_1, \ldots, x_n)|_v^{d_v} \leq H(x_1, \ldots, x_n)^{-\varepsilon}$$

are contained in a finite union of proper subspaces of K^n.

Here $H(x_1, \ldots, x_n)^{[K:\mathbf{Q}]} = \prod_{v \in M_K} \max(|x_1|_v, \ldots, |x_n|_v)^{d_v}$, but it could also be replaced by $\max(H(x_1), \ldots, H(x_n))$ if you prefer that.

(a) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero algebraic numbers, and let K be a number field. Prove that there is a finite set $A \subset \mathcal{O}_K^\times$ depending only on $\alpha_1, \ldots, \alpha_n$ and K such that all solutions $(x_1, \ldots, x_n) \in (\mathcal{O}_K^\times)^n$ of the generalized unit equation

$$\alpha_1 x_1 + \ldots + \alpha_n x_n = 0$$

satisfies that $x_i/x_j \in A$ for some $i \neq j$.

(b) In the setting of the previous part, prove that there is a finite set $A \subset \mathcal{O}_K^\times$ depending only on $\alpha_1, \ldots, \alpha_n$ and K such that for all solutions $(x_1, \ldots, x_n) \in (\mathcal{O}_K^\times)^n$ of the above equation, the index set $\{1, \ldots, n\}$ may be partitioned as $I_1 \cup \cdots \cup I_k$ such that $x_i/x_j \in A$ whenever $i, j \in I_l$ for some $l = 1, \ldots, k$ and

$$\sum_{j \in I_l} \alpha_j x_j = 0$$

for all $l = 1, \ldots, k$.

3. Fix some $n \in \mathbf{Z}_{>1}$ and consider the vector space

$$V = \mathbf{Q}^{n^2}/\{(y,\dots,y) : y \in \mathbf{Q}\}.$$

If $a, b \in \mathbf{Z}_{>1}$, write $e_1(a,b),\dots,e_{n^2}(a,b)$ for an enumeration of the numbers $a^i b^j$ for $i, j = 0,\dots,n-1$. (The enumeration is independent of a, b.)

Let $d = \gcd(a-1, b-1)$, and let $\Lambda_1,\dots,\Lambda_{n^2-1}$ be a basis of V^* and let $v \in \{2, 3, \infty\}$. Prove that there are infinitely many choices for multiplicatively independent $a, b \in \{2^i 3^j : i, j \in \mathbf{Z}_{\geq 1}\}$ such that

$$\prod_{j=1}^{n^2-1} |\Lambda_j(e_1/d,\dots,e_{n^2}/d)|_v > c|a|_v^{n^2(n-1)/2-n+1}|b|_v^{n^2(n-1)/2-n+1}$$

if $v = 2$ or 3 and

$$\prod_{j=1}^{n^2-1} |\Lambda_j(e_1/d,\dots,e_{n^2}/d)|_v > c|a|_v^{n^2(n-1)/2}|b|_v^{n^2(n-1)/2}d^{-n^2+1}$$

if $v = \infty$. Here c is a constant that depends only on the functionals and v.

Conclude that the choice of the forms used in the proof of Theorem 56 is optimal up to a multiplicative constant.

4. Let K be a number field. Prove that there is an effective constant depending only on K such that any solution $\alpha, \beta \in \mathcal{O}_K^\times$ of

$$\alpha + \beta = 1$$

satisfies $H(\alpha), H(\beta) \leq C$.

5. Let $m \in \mathbf{Z}_{\geq 1}$ and let $\alpha_1,\dots,\alpha_m \in \overline{\mathbf{Q}}_{\neq 0}$ be such that α_i/α_j is not a root of unity for any $1 \leq i \neq j \leq m$. Let $P_1,\dots,P_m \in \overline{\mathbf{Q}}[X]$ be non-zero polynomials. Use the version of the subspace theorem in Question 2 to prove that there are finitely many $n \in \mathbf{Z}$ such that

$$P_1(n)\alpha_1^n + \ldots + P_m(n)\alpha_m^n = 0.$$

6. In the setting of the previous question, assume that $|\alpha_1| > |\alpha_j|$ for all $j \geq 3$, but we permit $|\alpha_1| = |\alpha_2|$. (We fix an embedding into $\mathbf{C}$.) Use lower bounds for linear forms in logarithms to prove that there is an effective constant C depending only on $\alpha_1,\dots,\alpha_m$ and $P_1,\dots,P_m$ such that

$$P_1(n)\alpha_1^n + \ldots + P_m(n)\alpha_m^n \neq 0.$$

for all $|n| > C$.

7. Let $x \in (0,1)$ be a number with base 10 digit expansion $0.x_1x_2x_3\ldots$. We say that the expansion has long repetitions if there is some $\varepsilon > 0$ and infinitely many $n \in \mathbf{Z}_{>0}$ such that $x_1x_2\ldots x_n$ contains two identical substrings of length $\lceil \varepsilon n \rceil$. That is to say, there are $1 \le k < m \le n-l+1$ and $l = \lceil \varepsilon n \rceil$ such that $\alpha_{k+j} = \alpha_{m+j}$ for $j = 0, \ldots, l-1$.

(a) Suppose that $x \in (0,1)$ has long repetitions in its base 10 digit expansion. Prove that there is $\varepsilon > 0$ and infinitely many $n \in \mathbf{Z}_{>0}$ such that there are integers $0 \le k \ne m \le n$ and $A \in \mathbf{Z}$ with

$$|10^m x - 10^k x - A| \le 10^{-\varepsilon n}.$$

(b) Use the p-adic subspace theorem (E.g. Theorem 59 should suffice) to show that if $x \in (0,1)$ has long repetition in its base 10 digit expansion, then x is either rational or transcendental.

Comment: This is a result of Adamczewski, Bugeaud and Luca.