EXAMPLE SHEET 3 FOR DIOPHANTINE ANALYSIS, MICHAELMAS 2024

PÉTER VARJÚ

About this example sheet:

- Please send comments and corrections to pv270@dpmms.cam.ac.uk.
- Please submit your solutions of Problems 1 and 3, by Monday 25 November, 13:00.
- The purpose of this example sheet is to complement the material of the lectures. The level of difficulty of the problems varies considerably (in a non-monotone fashion), and they are not intended to be mock exam questions.

1. Let $\alpha \neq 0$ be an algebraic number of degree d and $\beta \neq 0$ an algebraic number of degree D. Suppose $D > 2d$.

Give a lower bound on $|\alpha - \beta|$ by computing the height of $\alpha - \beta$ and using the Liouville bound. The purpose of this question is to improve this bound. The actual values of the numerical constants are not important. If you obtain larger values, that is acceptable.

(a) Let $M \in \mathbf{Z}_{\geq 1}$ be such that $D > 2Md$. Prove that there is a polynomial $P \in \mathbf{Z}[X]$ of degree $D - 1$ that vanishes at α to order M with

$$H(P) \leq D^{2M^2 d/D} H(\alpha)^{2Md}.$$

(b) Prove that

$$|P(\beta)| \geq H(P)^{-D} \cdot D^{-D} \cdot H(\beta)^{-D^2}.$$

(c) Prove that

$$|P(\beta)| \leq D^M H(P) H(\alpha)^{dD} H(\beta)^{D^2} |\alpha - \beta|^M$$

(d) Prove that there is an absolute constant C such that

$$|\alpha - \beta| > D^{-C(Dd)^{1/2}} M(\alpha)^{-CD} M(\beta)^{-C(Dd)^{1/2}}.$$

2. Let $w_1, \ldots, w_n$ be distinct real numbers, let $d_1, \ldots, d_n$ be non-negative rational integers, and let $u_1, \ldots, u_N$ be distinct real numbers for $N = d_1 + \ldots + d_n + n - 1$. Show that there exist polynomials $a_1, \ldots, a_n \in \mathbf{R}[X]$ of degrees $d_1, \ldots, d_n$ respectively, such that the function

$$F(X) = \sum_{j=1}^{n} a_j(X) \exp(w_j X)$$

has a simple zero at each u_i, and no more zeros.

3. Let $w_1, \dots, w_k$ be $\mathbf{Q}$ linearly independent elements of $\mathbf{C}^n$. Show that the functions

$$\mathbf{C}^n \to \mathbf{C} : x = (x_1, \dots, x_n) \mapsto \exp(w_j \cdot x)$$

for $j = 1, \dots, k$ are algebraically independent over the field $\mathbf{Q}(x_1, \dots, x_n)$.

4. Let $x_1, x_2, x_3 \in \mathbf{R}$ and $y_1, y_2 \in \mathbf{R}$ be two sets of $\mathbf{Q}$ linearly independent numbers. The goal of the question is to prove that one at least of the six numbers $\alpha_{i,j} = \exp(x_i y_j)$ is transcendental. (Compare this with Q8 on the first example sheet.)

We suppose henceforth to the contrary that each $\alpha_{i,j}$ is algebraic.

Pick a large integer N and let $T = N^2$, $S = N^3$, $L = T^3 = S^2$. Consider the sets

$$\mathcal{X} = \{t_1 x_1 + t_2 x_2 + t_3 x_3 : t_j = 0, \dots, N-1, j = 1, 2, 3\},$$
$$\mathcal{Y} = \{s_1 y_1 + s_2 y_2 : s_j = 0, \dots, N-1, j = 1, 2\},$$

and the determinant

$$\Delta = [\exp(uw)]_{\substack{u \in \mathcal{X} \\ w \in \mathcal{Y}}}.$$

(a) Prove that $\log |\Delta| \le -cL^2$ for some absolute constant $c > 0$ provided N is sufficiently large (depending on $\alpha_{i,j}$).
(b) Obtain an upper bound for $H(\Delta)$.
(c) Conclude $\Delta = 0$.
(d) Prove that $\Delta \neq 0$, a contradiction.
(e) Let p_1, p_2, p_3 be distinct rational primes. Prove that if p_1^y, p_2^y and p_3^y are simultaneously algebraic for some $y \in \mathbf{R}$ then $y \in \mathbf{Q}$.

Remark: the reason why this proof works is that $3 \cdot 2 > 3 + 2$, and the reason why the second statement in Q8 of example sheet 1 is still a conjecture is that $2 \cdot 2 \not> 2 + 2$.

5. The goal of this question is to give an alternative proof of the real case of the Gelfond Schneider theorem based on Gelfond's proof. Now let $\lambda_1, \lambda_2 \in \mathbf{R}_{\neq 0}$ and suppose $\alpha_1 = e^{\lambda_1}, \alpha_2 = e^{\lambda_2}, \beta = \lambda_2/\lambda_1$ are all algebraic. We aim to derive a contradiction.

(a) Let $L \in \mathbf{Z}_{\geq 1}$ and let $f_1, \dots, f_L : \mathbf{C} \to \mathbf{C}$ be entire functions. Let $S_0, S_1 \in \mathbf{Z}_{\geq 0}$ with $L = (S_0 + 1)S_1$, and let $\xi_1, \dots, \xi_{S_1} \in \mathbf{C}$. Let $r \in \mathbf{R}_{>0}$ with $|\xi_s| \le r$ for $s = 1, \dots, S_1$. Let $E \in \mathbf{R}_{\geq 1}$. Prove that

$$\det[(d^\sigma/dz^\sigma) f_t(\xi_s)]_{\substack{t \\ \sigma,s}} \le E^{-L(L-1)/2 + S_0(S_0+1)S_1/2} \cdot L! \cdot \prod_{t=1}^{L} \cdot \max_{\sigma = 0, \dots, S_0} |(d^\sigma/dz^\sigma) f_t|_{Er}.$$

The indices in the determinant run through the ranges $t = 1, \dots, L$, $\sigma = 0, \dots, S_0$ and $s = 1, \dots, S_1$.

(b) Now let $T, S_0, S_1, L \in \mathbf{Z}_{\geq 0}$ be such that

$$L = (2T+1)^2 = (S_0+1)(2S_1+1),$$

and consider the determinant

$$\Delta = \det[(d^\sigma/dz^\sigma)\exp((t_1+\beta t_2)z)|_{z=\lambda_1 s}]_{\substack{t_1,t_2,\\ \sigma,s}}$$

where the indices run through the ranges $t_1, t_2 = -T, \dots, T$, $\sigma = 0, \dots, S_0$ and $s = -S_1, \dots, S_1$.

Use (a) to give an upper bound on $|\Delta|$.

(c) Use Proposition 41 to show that $\Delta \neq 0$.

(d) Show that Δ is algebraic, estimate its height and find a contradiction with an appropriate choice of the parameters.

(e) Compare Δ with the determinant in Schneider's proof.

6. The goal of this question is to give a proof of Dyson's diophantine exponent without using Siegel's lemma. Let $\alpha \in \mathbf{R} \cap \overline{\mathbf{Q}} \backslash \mathbf{Q}$, and let $\varepsilon > 0$. Suppose to the contrary that there are $p_1/q_1, p_2/q_2 \in \mathbf{Q}$ with $|\alpha - p_j/q_j| < q_j^{-\sqrt{2d}-\varepsilon}$ for $j = 1, 2$ and such that q_1 and $\log q_2/\log q_1$ are both large in terms of α and ε.

Consider the matrix

$$M = [\partial_{\sigma_1,\sigma_2} X^{t_1} Y^{t_2}|_{(X,Y)=\xi_s}]_{\substack{t_1,t_2\\ \sigma_1,\sigma_2,s}},$$

where

$$\partial_{\sigma_1,\sigma_2} = \frac{\partial^{\sigma_1+\sigma_2}}{\sigma_1!\sigma_2!\partial X^{\sigma_1}\partial Y^{\sigma_2}}.$$

The indices t_1 and t_2 run through the ranges $0, \dots, n_1$ and $0, \dots, n_2$, where $n_1, n_2 \in \mathbf{Z}_{\geq 0}$ are large and such that $n_1 \log q_1$ and $n_2 \log q_2$ are close to each other. We take $\xi_1 = (\alpha, \alpha)$, and let $\xi_2, \dots, \xi_d$ be the Galois conjugates of (α, α). We take $\xi_{s+1} = (p_1/q_1, p_2/q_2)$. The indices s, σ_1, σ_2 run through

$$\{(s,\sigma_1,\sigma_2) : s = 1, \dots d, \sigma_1/n_1 + \sigma_2/n_2 \leq \frac{2}{\sqrt{2d}} - \delta_1\} \cup \{(d+1,\sigma_1,\sigma_2) : \sigma_1/n_1 + \sigma_2/n_2 \leq \delta_2\}$$

for appropriate parameters $\delta_1, \delta_2 > 0$, which will be chosen in terms of ε and α.

(a) Use Dyson's lemma to show that M has rank $L := (n_1+1) \times (n_2+1)$.

(b) Show that you can find an $L \times L$ submatrix with nonzero determinant Δ of M that includes a maximal linearly independent subfamily of the columns that correspond to $s = 1$.

(c) Consider the Taylor expansion of all entries in the columns that correspond to $s = d+1$ around $(X,Y) = (\alpha, \alpha)$, and use this to give an upper bound on $|\Delta|$.

(d) Estimate $H(\Delta)$ and find a contradiction for an appropriate choice of the parameters.

7. The purpose of this question is to give a version of the argument in Question 6 for the Gelfond Schneider theorem using an auxiliary polynomial instead of an interpolation determinant. This time we do not need to assume that the logarithms are real.

Let $\lambda_1, \lambda_2 \in \mathbf{C}_{\neq 0}$. Suppose to the contrary that $\alpha_1 = \exp(\lambda_1), \alpha_2 = \exp(\lambda_2), \beta = \lambda_2/\lambda_1 \in \overline{\mathbf{Q}}$ but $\beta \notin \mathbf{Q}$. We will derive a contradiction.

Let $d = [\mathbf{Q}(\alpha_1, \alpha_2, \beta) : \mathbf{Q}]$ and we fix some positive integers T_0, T_1, S. In what follows, c and C are some constants that depend only on α_1, α_2 and β and they may be a different one at each occurrence.

(a) Under the assumption $(2T+1)^2 > 2d(S_0+1)(2S_1+1)$ find some $a_{t_1,t_2} \in \mathbf{Z}$ not all 0 for $t_1, t_2 = -T, \ldots, T$ such that

$$\max_{t_1,t_2} |a_{t_1,t_2}| \leq \exp(CS_0 \log T + CS_1 T)$$

and the function

$$F(x) = \sum_{|t_1|,|t_2| \leq T} a_{t_1,t_2} \exp((t_1 + \beta t_2)x)$$

satisfies

$$\frac{\partial^\sigma}{\partial x^\sigma} F(\lambda_1 s) = 0 \qquad (1)$$

for $\sigma = 0, \ldots, S_0$ and $s = -S_1, \ldots, S_1$.

(b) Fix a number $E \geq 10$. Let F be the function in part (a). Prove that

$$|F(x)| \leq \exp(CS_0 \log T + CTES_1)$$

for all complex $|x| \leq ES_1|\lambda_1|$.

(c) Suppose F satisfies (1) for $\sigma = 0, \ldots, S$ and $s = -S_1, \ldots, S_1$ with some $S \geq S_0$. Prove

$$|F(x)| \leq \exp(-cSS_1 \log E + CS_0 \log T + CTES_1)$$

for all complex $|x| \leq 2S_1|\lambda_1|$.

(d) Under the same assumptions prove that

$$\left|\frac{\partial^{S+1}}{\partial x^{S+1}} F(\lambda_1 s)\right| \leq \exp(-cSS_1 \log E + CS_0 \log T + CTES_1)$$

for $s = -S_1, \ldots, S_1$.

(e) Under the same assumptions prove that

$$\frac{\partial^{S+1}}{\partial x^{S+1}} F(\lambda_1 s) = 0$$

for $s = -S_1, \ldots, S_1$.

(f) Conclude $F = 0$, a contradiction.