EXAMPLE SHEET 3 FOR DIOPHANTINE ANALYSIS, MICHAELMAS 2024

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About this example sheet:

- Please send comments and corrections to pv270@dpmms.cam.ac.uk.
- Please submit your solutions of Problems 1 and 3, by Monday 25 November, 13:00.
- The purpose of this example sheet is to complement the material of the lectures. The level of difficulty of the problems varies considerably (in a non-monotone fashion), and they are not intended to be mock exam questions.

1. Let $\alpha \neq 0$ be an algebraic number of degree d and $\beta \neq 0$ an algebraic number of degree D. Suppose $D > 2d$.

Give a lower bound on $|\alpha - \beta|$ by computing the height of $\alpha - \beta$ and using the Liouville bound. The purpose of this question is to improve this bound. The actual values of the numerical constants are not important. If you obtain larger values, that is acceptable.

(a) Let $M \in \mathbf{Z}_{\geq 1}$ be such that $D > 2Md$. Prove that there is a polynomial $P \in \mathbf{Z}[X]$ of degree $D-1$ that vanishes at α to order M with

$$H(P) \leq D^{2M^2d/D} H(\alpha)^{2Md}.$$

(b) Prove that

$$|P(\beta)| \geq H(P)^{-D} \cdot D^{-D} \cdot H(\beta)^{-D^2}.$$

(c) Prove that

$$|P(\beta)| \leq D^M H(P) H(\alpha)^{dD} H(\beta)^{D^2} |\alpha - \beta|^M$$

(d) Prove that there is an absolute constant C such that

$$|\alpha - \beta| > D^{-C(Dd)^{1/2}} M(\alpha)^{-CD} M(\beta)^{-C(Dd)^{1/2}}.$$

Hint: You should get $|\alpha - \beta| > 2^{-dD} M(\alpha)^{-D} M(\beta)^{-d}$ from the Liouville bound. (a): Use Siegel's lemma and $D - Md > D/2$. (b): Estimate the height and use the Liouville bound. You also need to show $P(\beta) \neq 0$. (c): Use Taylor expansion around α to order M with remainder term. Use $\max(|\alpha|, |\beta|) < H(\alpha)^d H(\beta)^D$.

2. Let $w_1, \ldots, w_n$ be distinct real numbers, let $d_1, \ldots, d_n$ be non-negative rational integers, and let $u_1, \ldots, u_N$ be distinct real numbers for $N = d_1 + \ldots + d_n + n - 1$. Show that there exist polynomials $a_1, \ldots, a_n \in \mathbf{R}[X]$ of degrees $d_1, \ldots, d_n$ respectively, such that the function

$$F(X) = \sum_{j=1}^{n} a_j(X) \exp(w_j X)$$

has a simple zero at each u_i, and no more zeros.

Hint: Use linear algebra to find a non-zero F that vanishes at the prescribed points. Use the zero estimate to show it has no more zeroes.

3. Let $w_1, \ldots, w_k$ be $\mathbf{Q}$ linearly independent elements of $\mathbf{C}^n$. Show that the functions

$$\mathbf{C}^n \to \mathbf{C} : x = (x_1, \ldots, x_n) \mapsto \exp(w_j \cdot x)$$

for $j = 1, \ldots, k$ are algebraically independent over the field $\mathbf{Q}(x_1, \ldots, x_n)$.

Hint: You need to show that

$$\sum_{l_1, \ldots, l_k} a_{l_1, \ldots, l_k}(x) \exp((l_1 w_1 + \ldots + l_k w_k) \cdot x) \neq 0$$

for any choice of non-zero $a_{l_1, \ldots, l_k} \in \mathbf{Q}[x_1, \ldots, x_n]$, where the indices $l_1, \ldots, l_k$ run through some finite range. Prove this by induction on the sum of the total degrees of the $a's$ by a similar argument to Proposition 41.

4. Let $x_1, x_2, x_3 \in \mathbf{R}$ and $y_1, y_2 \in \mathbf{R}$ be two sets of $\mathbf{Q}$ linearly independent numbers. The goal of the question is to prove that one at least of the six numbers $\alpha_{i,j} = \exp(x_i y_j)$ is transcendental. (Compare this with Q8 on the first example sheet.)

We suppose henceforth to the contrary that each $\alpha_{i,j}$ is algebraic.

Pick a large integer N and let $T = N^2$, $S = N^3$, $L = T^3 = S^2$. Consider the sets

$$\mathcal{X} = \{t_1 x_1 + t_2 x_2 + t_3 x_3 : t_j = 0, \ldots, N-1, j = 1, 2, 3\},$$
$$\mathcal{Y} = \{s_1 y_1 + s_2 y_2 : s_j = 0, \ldots, N-1, j = 1, 2\},$$

and the determinant

$$\Delta = [\exp(uw)]_{\substack{u \in \mathcal{X} \\ w \in \mathcal{Y}}}.$$

(a) Prove that $\log |\Delta| \leq -cL^2$ for some absolute constant $c > 0$ provided N is sufficiently large (depending on $\alpha_{i,j}$).
(b) Obtain an upper bound for $H(\Delta)$.
(c) Conclude $\Delta = 0$.
(d) Prove that $\Delta \neq 0$, a contradiction.
(e) Let p_1, p_2, p_3 be distinct rational primes. Prove that if p_1^y, p_2^y and p_3^y are simultaneously algebraic for some $y \in \mathbf{R}$ then $y \in \mathbf{Q}$.

Remark: the reason why this proof works is that $3 \cdot 2 > 3 + 2$, and the reason why the second statement in Q8 of example sheet 1 is still a conjecture is that $2 \cdot 2 \not> 2 + 2$.

Hint: (a): use Proposition 37 for the functions $f_u(z) = \exp(uz)$. (b): Express Δ as a polynomial in $\alpha_{i,j}$ and use Proposition 28. (d): Use Proposition 40. (e): Take $x_j = \log p_j$.

5. The goal of this question is to give an alternative proof of the real case of the Gelfond Schneider theorem based on Gelfond's proof. Now let $\lambda_1, \lambda_2 \in \mathbf{R}_{\neq 0}$ and suppose $\alpha_1 = e^{\lambda_1}, \alpha_2 = e^{\lambda_2}, \beta = \lambda_2/\lambda_1$ are all algebraic. We aim to derive a contradiction.

(a) Let $L \in \mathbf{Z}_{\geq 1}$ and let $f_1, \ldots, f_L : \mathbf{C} \to \mathbf{C}$ be entire functions. Let $S_0, S_1 \in \mathbf{Z}_{\geq 0}$ with $L = (S_0 + 1)S_1$, and let $\xi_1, \ldots, \xi_{S_1} \in \mathbf{C}$. Let $r \in \mathbf{R}_{>0}$ with $|\xi_s| \leq r$ for $s = 1, \ldots, S_1$. Let $E \in \mathbf{R}_{\geq 1}$. Prove that

$$\det[(d^\sigma/dz^\sigma)f_t(\xi_s)]_{\substack{t\\ \sigma,s}} \leq E^{-L(L-1)/2+S_0(S_0+1)S_1/2} \cdot L! \cdot \prod_{t=1}^{L} \cdot \max_{\sigma=0,\ldots,S_0} |(d^\sigma/dz^\sigma)f_t|_{Er}.$$

The indices in the determinant run through the ranges $t = 1, \ldots, L$, $\sigma = 0, \ldots, S_0$ and $s = 1, \ldots, S_1$.

(b) Now let $T, S_0, S_1, L \in \mathbf{Z}_{\geq 0}$ be such that

$$L = (2T + 1)^2 = (S_0 + 1)(2S_1 + 1),$$

and consider the determinant

$$\Delta = \det[(d^\sigma/dz^\sigma)\exp((t_1 + \beta t_2)z)|_{z=\lambda_1 s}]_{\substack{t_1,t_2\\ \sigma,s}},$$

where the indices run through the ranges $t_1, t_2 = -T, \ldots, T$, $\sigma = 0, \ldots, S_0$ and $s = -S_1, \ldots, S_1$.

Use (a) to give an upper bound on $|\Delta|$.

(c) Use Proposition 41 to show that $\Delta \neq 0$.

(d) Show that Δ is algebraic, estimate its height and find a contradiction with an appropriate choice of the parameters.

(e) Compare Δ with the determinant in Schneider's proof.

Hint: (a): Try the same proof as in Proposition 37, but now you cannot pull out factors of $z^{\deg f_t}$ from the rows. Instead try to expand the determinant and estimate the degree of each term. (c): Same as the proof of Proposition 40, but now you need to count zeros with multiplicities. (e): Gelfond's Δ is the transpose of Schneider's.

6. The goal of this question is to give a proof of Dyson's diophantine exponent without using Siegel's lemma. Let $\alpha \in \mathbf{R} \cap \overline{\mathbf{Q}} \backslash \mathbf{Q}$, and let $\varepsilon > 0$. Suppose to the contrary that there are $p_1/q_1, p_2/q_2 \in \mathbf{Q}$ with $|\alpha - p_j/q_j| < q_j^{-\sqrt{2d}-\varepsilon}$ for $j = 1, 2$ and such that q_1 and $\log q_2/\log q_1$ are both large in terms of α and ε.

Consider the matrix

$$M = [\partial_{\sigma_1,\sigma_2} X^{t_1} Y^{t_2}|_{(X,Y)=\xi_s}]_{\substack{t_1,t_2 \\ \sigma_1,\sigma_2,s}},$$

where

$$\partial_{\sigma_1,\sigma_2} = \frac{\partial^{\sigma_1+\sigma_2}}{\sigma_1!\sigma_2!\partial X^{\sigma_1}\partial Y^{\sigma_2}}.$$

The indices t_1 and t_2 run through the ranges $0,\ldots,n_1$ and $0,\ldots,n_2$, where $n_1, n_2 \in \mathbf{Z}_{\geq 0}$ are large and such that $n_1 \log q_1$ and $n_2 \log q_2$ are close to each other. We take $\xi_1 = (\alpha,\alpha)$, and let $\xi_2,\ldots,\xi_d$ be the Galois conjugates of (α,α). We take $\xi_{s+1} = (p_1/q_1, p_2/q_2)$. The indices s, σ_1, σ_2 run through

$$\{(s,\sigma_1,\sigma_2) : s = 1,\ldots d, \sigma_1/n_1+\sigma_2/n_2 \leq \frac{2}{\sqrt{2d}}-\delta_1\}\cup\{(d+1,\sigma_1,\sigma_2) : \sigma_1/n_1+\sigma_2/n_2 \leq \delta_2\}$$

for appropriate parameters $\delta_1, \delta_2 > 0$, which will be chosen in terms of ε and α.

(a) Use Dyson's lemma to show that M has rank $L := (n_1+1)\times(n_2+1)$.
(b) Show that you can find an $L \times L$ submatrix with nonzero determinant Δ of M that includes a maximal linearly independent subfamily of the columns that correspond to $s = 1$.
(c) Consider the Taylor expansion of all entries in the columns that correspond to $s = d+1$ around $(X,Y) = (\alpha,\alpha)$, and use this to give an upper bound on $|\Delta|$.
(d) Estimate $H(\Delta)$ and find a contradiction for an appropriate choice of the parameters.

Hint: (a): Consider a linear combination of the rows of M and show that the corresponding polynomial cannot have so much vanishing to make the linear combination all 0 if δ_1, δ_2 are appropriately chosen and n_2/n_1 is small enough, which you may force by requiring that $\log q_2/\log q_1$ is large. (c): Use column operations on the determinant to remove low degree terms from the Taylor expansions. You need to use that you have enough columns in your determinant that span the space generated by all columns of M corresponding to $s = 1$.

7. The purpose of this question is to give a version of the argument in Question 6 for the Gelfond Schneider theorem using an auxiliary polynomial instead of an interpolation determinant. This time we do not need to assume that the logarithms are real.

Let $\lambda_1, \lambda_2 \in \mathbf{C}_{\neq 0}$. Suppose to the contrary that $\alpha_1 = \exp(\lambda_1), \alpha_2 = \exp(\lambda_2), \beta = \lambda_2/\lambda_1 \in \overline{\mathbf{Q}}$ but $\beta \notin \mathbf{Q}$. We will derive a contradiction.

Let $d = [\mathbf{Q}(\alpha_1,\alpha_2,\beta) : \mathbf{Q}]$ and we fix some positive integers T_0, T_1, S. In what follows, c and C are some constants that depend only on α_1, α_2 and β and they may be a different one at each occurrence.

(a) Under the assumption $(2T+1)^2 > 2d(S_0+1)(2S_1+1)$ find some $a_{t_1,t_2} \in \mathbf{Z}$ not all 0 for $t_1, t_2 = -T, \ldots, T$ such that

$$\max_{t_1,t_2} |a_{t_1,t_2}| \le \exp(CS_0 \log T + CS_1 T)$$

and the function

$$F(x) = \sum_{|t_1|,|t_2| \le T} a_{t_1,t_2} \exp((t_1 + \beta t_2)x)$$

satisfies

(1) $$\frac{\partial^\sigma}{\partial x^\sigma} F(\lambda_1 s) = 0$$

for $\sigma = 0, \ldots, S_0$ and $s = -S_1, \ldots, S_1$.

(b) Fix a number $E \ge 10$. Let F be the function in part (a). Prove that

$$|F(x)| \le \exp(CS_0 \log T + CTES_1)$$

for all complex $|x| \le ES_1|\lambda_1|$.

(c) Suppose F satisfies (1) for $\sigma = 0, \ldots, S$ and $s = -S_1, \ldots, S_1$ with some $S \ge S_0$. Prove

$$|F(x)| \le \exp(-cSS_1 \log E + CS_0 \log T + CTES_1)$$

for all complex $|x| \le 2S_1|\lambda_1|$.

(d) Under the same assumptions prove that

$$\Big|\frac{\partial^{S+1}}{\partial x^{S+1}} F(\lambda_1 s)\Big| \le \exp(-cSS_1 \log E + CS_0 \log T + CTES_1)$$

for $s = -S_1, \ldots, S_1$.

(e) Under the same assumptions prove that

$$\frac{\partial^{S+1}}{\partial x^{S+1}} F(\lambda_1 s) = 0$$

for $s = -S_1, \ldots, S_1$.

(f) Conclude $F = 0$, a contradiction.

Hint: (a): Use Siegel's lemma. (c): Use the maximum modulus principle for the function

$$\frac{F(x)}{(x - S_1\lambda_1)^{S+1} \cdots (x + S_1\lambda_1)^{S+1}}.$$

(d): Use Cauchy's formula to express the derivative as a contour integral on the circle $|x| = 2S_1|\lambda_1|$. (e): Estimate the height of the number in question and compare the Liouville bound with the upper bound in the previous part. At this point you need to choose the parameters. Take S_1 to be a large constant depending on $\alpha_1, \alpha_2, \beta$, $E = S^{1/10}$ and choose T and S_0 to be sufficiently large satisfying the condition in part (a). (f) Run the arguments in parts (c)-(d) repeatedly to show that all derivatives of F vanish at 0 to infinite order (and at all the other points we worked with).