EXAMPLE SHEET 1 FOR DIOPHANTINE ANALYSIS, MICHAELMAS 2024

PÉTER VARJÚ

About this example sheet:

- Please send comments and corrections to pv270@dpmms.cam.ac.uk.
- Please submit your solutions of Problems 6 and 9, by Monday 28 October, 13:00.
- The purpose of this example sheet is to complement the material of the lectures. The level of difficulty of the problems varies considerably (in a non-monotone fashion), and they are not intended to be mock exam questions.

1. Let $P \in \mathbf{Z}[X,Y]$ be a non-zero homogeneous polynomial, and let $m \in \mathbf{Z}_{\neq 0}$. Suppose the equation

$$P(X,Y) = m$$

has infinitely many solutions in $\mathbf{Z}^2$. Prove that $P = aQ^k$ for some $Q \in \mathbf{Z}[X,Y]$ homogeneous polynomial of degree at most 2 and $a \in \mathbf{Z}$.

Hint: If $R \in \mathbf{Z}[X,Y]$ with $R|P$ show that $R(X,Y) = m_0$ has infinitely many solutions in $\mathbf{Z}^2$ for some $m_0|m$.

2. Let $n \in \mathbf{Z}_{\geq 2}$ and S a finite set of places of $\mathbf{Q}$ containing ∞. For each $v \in S$, let $L_1^{(v)}, \ldots, L_n^{(v)}$ be a linearly independent collection of linear forms with coefficients in $\mathbf{Q}$ in n variables. Let $l \subset \mathbf{Q}^n$ be a 1-dimensional linear subspace. Suppose none of the $L_j^{(v)}$ vanishes on l. Prove that there is $c = c(l, L_j^{(v)}) > 0$ such that

$$\prod_{v \in S} \prod_{j=1}^{n} |L_j^{(v)}(x_1, \ldots, x_n)|_v > c$$

for all non-zero $(x_1, \ldots, x_n) \in l \cap \mathbf{Z}^n$.

3. Let L_1, L_2, L_3 be linearly independent linear forms in three variables with algebraic coefficients, and let $\varepsilon \in (0,1)$. Let $V \subset \mathbf{Q}^3$ be a 2 dimensional subspace that contains infinitely many solutions of

$$\prod_{j=1}^{3} |L_j(x_1, x_2, x_3)| \leq H(x_1, x_2, x_3)^{-\varepsilon} \tag{1}$$

that cannot be covered by finitely many lines.

Prove that there is some $\alpha \in \overline{\mathbf{Q}}$ and indices $i, j \in \{1, 2, 3\}$ such that $V = \mathrm{Ker}(L_i + \alpha L_j)$.

Conversely show that if $V \subset \mathbf{Q}^3$ is a subspace of the above form, then it contains infinitely many integral solutions of (1).

4. Prove that the $n = 2$ case of the subspace theorem in its Archimedean form (Theorem 6) is equivalent to Roth's theorem.

5. Let $L_1, \ldots, L_n$ be linearly independent linear forms in n variables with real algebraic coefficients. Let $(x_1, \ldots, x_n) \in \mathbf{R}^n$ be non-zero with

$$L_1(x_1, \ldots, x_n) = \ldots = L_{n-1}(x_1, \ldots, x_n) = 0,$$

and suppose the numbers $x_1, \ldots, x_n$ are linearly independent over $\mathbf{Q}$.

Prove that there is a constant $C = C(L_1, \ldots, L_n)$ such that the following holds. Let $V_1, \ldots, V_k \subset \mathbf{Q}^n$ be proper linear subspaces. Then there is $(y_1, \ldots, y_n) \in \mathbf{Z}^n \backslash (V_1 \cup \ldots \cup V_k)$ such that

$$\prod_{j=1}^{n} |L_j(y_1, \ldots, y_n)| \leq C.$$

6. Using the p-adic subspace theorem (Theorem 7) or Roth's theorem, prove that for all $n, a_1, \ldots, a_n \in \mathbf{Z}_{>0}$ and $\varepsilon > 0$, there is a constant $c > 0$ such that the following holds. Let $b_1, \ldots, b_n \in \mathbf{Z}$ with $\max(|b_j|) = B$. Then

$$|b_1 \log a_1 + \ldots + b_n \log a_n| \geq c \exp(-\varepsilon B)$$

provided the linear form in logarithms on the left does not vanish.

7. Let $n \in \mathbf{Z}_{\geq 2}$, and let $\lambda_1, \ldots, \lambda_n \in \mathbf{C}$, (which are not necessarily logarithms of algebraic numbers). Let $B \in \mathbf{Z}_{>1}$. Prove that there are $b_1, \ldots, b_n \in \mathbf{Z}$ not all 0 with $|b_j| \leq B$ such that

$$|b_1 \lambda_1 + \ldots + b_n \lambda_n| \leq C \exp(-(n/2 - 1) \log B),$$

where $C > 0$ is a constant that may depend on $n, \lambda_1, \ldots, \lambda_n$.

8. Prove that Schanuel's conjecture implies the following two statements, which are known as the four exponentials conjecture.

- Let $\lambda_{1,1}, \lambda_{1,2}, \lambda_{2,1}, \lambda_{2,2}$ be logarithms of algebraic numbers. Suppose $\lambda_{1,1} \neq 0$ and $\lambda_{1,2}/\lambda_{1,1}$ and $\lambda_{2,1}/\lambda_{1,1}$ are irrational. Then $\lambda_{1,1}\lambda_{2,2} - \lambda_{1,2}\lambda_{2,1} \neq 0$.
- Let $x_1, x_2, y_1, y_2 \in \mathbf{C}_{\neq 0}$ such that $x_1/x_2, y_1/y_2 \notin \mathbf{Q}$. Then at least one of the four numbers

$$\exp(x_1 y_1), \exp(x_1 y_2), \exp(x_2 y_1), \exp(x_2 y_2)$$

are transcendental.

9. Prove that there is an effective absolute constant $C > 0$ such that the following holds. Let $a_1/a_2 \in \mathbf{Q}_{\neq 0}$ and let $n \in \mathbf{Z}_{\geq 2}$. Suppose $\sqrt[n]{a_1/a_2} \in [1/10, 10]$. Then for all $p/q \in \mathbf{Q}$ with $p/q \neq \sqrt[n]{a_1/a_2}$, we have

$$|\sqrt[n]{a_1/a_2} - p/q| > q^{-C \log A \log n},$$

where $A = \max(|a_1|, |a_2|, 2)$.

10. Prove that the largest prime factor of $n(n+1)$ goes to infinity as $n \in \mathbf{Z}_{>0}$ goes to infinity.

11. Let α be an algebraic number of degree $d \in \mathbf{Z}_{\geq 1}$.

- Prove that there is a proper subspace $V \subset \mathbf{Q}^{d+1}$ and a constant $c > 0$ depending on α such that for all $(q, p_1, \ldots, p_d) \in \mathbf{Z}^{d+1}$ with
$$|\alpha^j - p_j/q| < c/q \qquad \text{for all } j = 1, \ldots, d$$
we have $(q, p_1, \ldots, p_d) \in V$.
- Let $P \in \mathbf{R}[X]$ be a polynomial of degree n. Prove that for all $t \in \mathbf{R}_{\geq 0}$, we have
$$\int_0^t e^{t-X} P(X) dX = e^t \sum_{j=0}^{n} \frac{d^j}{dX^j} P(0) - \sum_{j=0}^{n} \frac{d^j}{dX^j} P(t).$$
Conclude that
$$\left| e^t \sum_{j=0}^{n} \frac{d^j}{dX^j} P(0) - \sum_{j=0}^{n} \frac{d^j}{dX^j} P(t) \right| \leq C \max_{X \in [0,t]} |P(X)|$$
for some constant $C = C(t)$.
- Let $P \in \mathbf{Z}[X]$ be a polynomial that vanishes to order m at some $a \in \mathbf{Z}$. Prove
$$m! \Big| \sum_{j=0}^{n} \frac{d^j}{dX^j} P(a).$$
- Prove e is transcendental.

12. Compute the Mahler measure of $p/q \in \mathbf{Q}$ and the roots of $ax^2 + bx + c = 0$ for $a, b, c \in \mathbf{Z}$ in the case $b^2 < 4ac$.

13. Let α be a non-zero algebraic number. Prove that $M(\alpha) \geq 1$. Determine the set of all numbers for which equality is attained.

14. A Perron number is a real algebraic integer α all of whose Galois conjugates have absolute value strictly less than α. Prove that $M(\alpha)$ is a Perron number for all non-zero algebraic numbers α.